LOG-CONCAVITY OF GENUS DISTRIBUTIONS FOR CIRCULAR LADDERS

YICHAO CHEN, JONATHAN L. GROSS, AND TOUFIK MANSOUR

Abstract. A well-known conjecture in topological graph theory says that the genus distribution of every graph is log-concave. In this paper, the genus distribution of the circular ladder $CL_n$ is re-derived, using overlap matrices and Chebyshev polynomials, which facilitates proof that this genus distribution is log-concave.

1. Introduction

A real polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

is log-concave (or, equivalently, strongly unimodal) if $a_j^2 \geq a_{j-1}a_{j+1}$, for any index $j$ such that $1 \leq j \leq n-1$. Applications of log-concavity arise in algebra, combinatorics, geometry, and computer science, as described by Stanley [15] and Brenti [1]. The study of genus distributions problems was inaugurated by Gross and Furst [8] and has become a frequently discussed topic in topological graph theory.

A circular ladder $CL_n$ is given by the graphical Cartesian product $CL_n = C_n \times K_2$, where $K_2$ is the complete graph on two vertices, and where $C_n$ is the cycle graph on $n$ vertices. Another kind of ladder graph, called a closed-end ladder, is obtained from the cartesian product $P_n \times K_2$ by doubling both of the end edges. Figure 1.1 portrays the circular ladder $CL_4$ and the closed-end ladder $L_4$.

![Figure 1.1. The circular ladder $CL_4$ and the closed-end ladder $L_4$.](image)

Our proofs involve algebraic objects associated with a third class of ladder graph. The Ringel ladder $RL_n$ is a graph obtained from the closed-end ladder $L_n$ by placing a subdivision vertex

\[2000\] Mathematics Subject Classification. Primary: 05C10; Secondary: 30B70, 42C05.

Key words and phrases. Log-concavity; genus distribution; circular ladders.

The work of J. L. Gross is supported by Simons Foundation Grant #315001.
in the interior of each of the two opposite end edges, and then joining those two new vertices. A Ringel ladder is illustrated in Figure 1.2.

![Figure 1.2. The Ringel ladder RL₄.](image)

Furst, Gross, and Statman [7] showed that the genus distributions of the closed-end ladders are log-concave. Gross, Robbins, and Tucker [11] calculated the genus distribution of the bouquet \(B_n\) (which has a single vertex and \(n\) self-loops) and proved its log-concavity. They also posed the following conjecture.

**Conjecture 1.1.** The genus distribution sequence of a graph is log-concave.

Stahl [14] posed a stronger conjecture, namely, that the zeros of every genus polynomial are real. However, Chen and Liu [3] found counterexamples among the circular ladders. Recently, Gross, Mansour, Tucker, and Wang [9] proved that the genus distribution of every Ringel ladder is log-concave. The objective of this paper is to prove the log-concavity of the genus distributions of circular ladders.

1.1. **Genus polynomial.** It is assumed that the reader is reasonably familiar with the basics of topological graph theory, as found in Gross and Tucker [10]. A graph \(G = (V(G), E(G))\) is permitted to have both loops and multiple edges. A **surface** is a compact closed 2-dimensional manifold without boundary. In topology, surfaces are classified into two sequences: \(S_m\) denotes the **orientable surface** with \(m\) (\(m \geq 0\)) handles, and \(N_n\) denotes the **nonorientable surface** with \(n\) (\(n > 0\)) crosscaps. A graph embedding into a surface means a cellular embedding.

For any spanning tree of \(G\), the number of co-tree edges is called the **Betti number** of \(G\) and is denoted by \(\beta(G)\). A **rotation** at a vertex \(v\) of a graph \(G\) is a cyclic order of all edges incident with \(v\). A **pure rotation system** \(P\) of a graph \(G\) is a collection of the rotations at all vertices of \(G\). An embedding of \(G\) into an orientable surface \(S\) induces a pure rotation system as follows: the rotation at \(v\) is the cyclic permutation corresponding to the order in which the edge-ends are traversed in an orientation-preserving tour around \(v\). Conversely, by the Heffter-Edmonds principle, every rotation system induces a unique embedding (up to homeomorphism) of \(G\) into some orientable surface \(S\). The bijectivity of this correspondence implies that the total number of orientable embeddings is \(\prod_{v \in G} (d_v - 1)!\).

A **general rotation system** is a pair \((P, \lambda)\), where \(P\) is a pure rotation system and \(\lambda\) is a mapping \(E(G) \rightarrow \{0, 1\}\). The edge \(e\) is said to be **twisted** (respectively, **untwisted**) if \(\lambda(e) = 1\) (respectively, \(\lambda(e) = 0\)). It is well known that every orientable embedding of a graph \(G\) can be described by a general rotation system \((P, \lambda)\) with \(\lambda(e) = 0\) for all \(e \in E(G)\). By allowing \(\lambda\) to take the non-zero value, we can describe nonorientable embeddings of \(G\). A **T-rotation system** \((P, \lambda)\) of \(G\) is a general rotation system \((P, \lambda)\) such that \(\lambda(e) = 0\), for all \(e \in E(T)\).
The genus polynomial of a graph $G$ is the polynomial $g_G(x) = \sum_{i=0}^{\infty} g_i x^i$ where the number $g_i$, for $i = 0, 1, \ldots$, counts the embeddings into the orientable surface $S_i$.

1.2. Overlap matrices. Mohar [12] introduced an invariant that has subsequently been used numerous times in the calculation of distributions of graph embeddings, including non-orientable embeddings. Let $T$ be a spanning tree of a graph $G$, and let $(\rho, \lambda)$ be a $T$-rotation system. Let $e_1, e_2, \ldots, e_{\beta(G)}$ be the cotree edges of $T$, where $\beta(G)$ is the cycle rank of $G$. The overlap matrix of $(\rho, \lambda)$ is the $\beta(G) \times \beta(G)$ matrix $M = [m_{ij}]$ over $\mathbb{Z}_2$ such that

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } e_i \text{ is twisted;} \\ 1, & \text{if } i \neq j \text{ and the restriction of the underlying pure rotation system to the subgraph } T + e_i + e_j \text{ is nonplanar;} \\ 0, & \text{otherwise.} \end{cases}$$

When the restriction of the underlying pure rotation system to the subgraph $T + e_i + e_j$ is nonplanar, we say that edges $e_i$ and $e_j$ overlap. The importance of overlap matrices is indicated by this theorem of Mohar [12]:

**Theorem 1.2.** Let $(\rho, \lambda)$ be a general rotation system for a graph, and let $M$ be the overlap matrix. Then the rank of $M$ equals twice the genus of the corresponding embedding surface, if that surface is orientable, and it equals the crosscap number otherwise. It is independent of the choice of a spanning tree.

It is known that the overlap matrix of the circular ladder $CL_n$ has the following form. Further details are given by [6].

$$M_{n+2}^{c,x,y,X,Y,Z} = \begin{pmatrix} x_c & c & x & z_2 & z_3 & \cdots & z_{n-1} & y \\ c & x_f & z_1 & z_2 & z_3 & \cdots & z_{n-1} & z_n \\ x & z_1 & x_1 & y_1 & \cdots \\ z_2 & z_2 & y_1 & x_2 & y_2 \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ z_{n-1} & z_{n-1} & 0 & y_{n-2} & x_{n-1} & y_{n-1} \\ y & z_n & \cdots & y_{n-2} & x_n & \cdots & y_n \\ z_{n-1} & z_{n-1} & 0 & y_{n-2} & x_{n-1} & y_{n-1} \end{pmatrix}$$

To facilitate the readability of this paper, we have assigned labels to most of our unavoidably many definitions, and we make back-reference to them when the use of a definition appears a few pages of more below the definition itself.

2. Rank-Distribution Polynomials

Our calculation of the genus distribution of a circular ladder is somewhat more complicated than the original derivation of that genus distribution by McGeoch [13]. Our approach here, using rank-distribution polynomials, enables us to prove the log-concavity. Toward this objective,
we define two sets of matrices and two polynomials:

\[ A_{n+2} = \text{the set of all matrices over } \mathbb{Z}_2 \text{ of the form } M_{n+2}^{c,x,y,X,Y,Z} \]

\[ A_{n+2}(z) = \sum_{j=0}^{n+2} A_{n+2}(j) z^j \text{ is the rank-distribution polynomial of the set } A_{n+2}, \]

in which \( A_{n+2}(j) \) denotes the number of matrices in \( A_{n+2} \) of rank \( j \), for general rotation systems for the circular ladder \( CL_n \);

\[ B_{n+2} = \text{the set of all matrices of the form } M_{n+2}^{c,x,y,0,Y,Z}; \]

\[ B_{n+2}(z) = \sum_{j=0}^{n+2} B_{n+2}(j) z^j \text{ is the rank-distribution polynomial of the set } B_{n+2}, \]

in which \( B_{n+2}(j) \) denotes the number of matrices in \( B_{n+2} \) that are of rank \( j \), for pure rotation systems for the circular ladder \( CL_n \).

In a matrix of the form \( M_{n+2}^{c,x,y,X,Y,Z} \), suppose that we first add the second row to the first row and next add the second column to the first column. Without changing the rank of the matrix, this produces a matrix of the following form:

\[
\begin{pmatrix}
  x_e & c + x_f & x & 0 & \cdots & 0 & 0 & y \\
  c + x_f & x_f & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} & z_n \\
  x & z_1 & x_1 & y_1 & & & & \\
  0 & z_2 & y_1 & x_2 & \cdots & 0 & & \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \\
  0 & z_{n-2} & 0 & y_{n-2} & x_{n-1} & y_{n-1} & & \\
  y & z_n & & & & & y_{n-1} & x_n
\end{pmatrix}
\]

For \( xy \in \{00, 01, 10, 11\} \), we define two more sets and two more polynomials:

\[ A_{n+2}^{xy} = \text{the set of all matrices over } \mathbb{Z}_2 \text{ of the form } M_{n+2}^{c,x,y,X,Y,Z}; \]

\[ A_{n+2}^{xy}(z) = \sum_{j=0}^{n+2} A_{n+2}^{xy}(j) z^j \text{ as the rank-distribution polynomial of the set } A_{n+2}^{xy}; \]

\[ B_{n+2}^{xy} = \text{the set of all matrices of the form } M_{n+2}^{c,x,y,0,Y,Z}; \]

\[ B_{n+2}^{xy}(z) = \sum_{j=0}^{n+2} B_{n+2}^{xy}(j) z^j \text{ as the rank-distribution polynomial of the set } B_{n+2}^{xy}. \]

The following property is self-explanatory.

**Property 2.1.** For all \( n \geq 1 \),

\[ A_{n+2}(z) = \sum_{xy=00,01,10,11} A_{n+2}^{xy}(z) \text{ and } B_{n+2}(z) = \sum_{xy=00,01,10,11} B_{n+2}^{xy}(z). \]
As in [6], for the vectors \( X = (x_0, x_1, \ldots, x_n) \), \( Y = (y_1, y_2, \ldots, y_{n-1}) \), and \( Z = (z_1, z_2, \ldots, z_n) \), with \( x_i, y_j, z_k \in \mathbb{Z}_2 \), we define the matrices

\[
M_{n}^{X,Y} = \begin{pmatrix}
x_1 & y_1 & 0 \\
y_1 & x_2 & y_2 \\
 & y_2 & x_3 & y_3 \\
 & & \ddots & \ddots & \ddots \\
0 & y_{n-2} & x_{n-1} & y_{n-1} & x_n
\end{pmatrix}
\]

and

\[
M_{n+1}^{X,Y,Z} = \begin{pmatrix}
x_0 & z_1 & z_2 & z_3 & \ldots & z_{n-1} & z_n \\
z_1 & x_1 & y_1 & 0 \\
z_2 & y_1 & x_2 & y_2 \\
z_3 & y_2 & x_3 & \ddots \\
 & \ddots & \ddots & \ddots & \ddots \\
z_{n-1} & 0 & y_{n-2} & x_{n-1} & y_{n-1} \\
z_n & y_{n-1} & x_n
\end{pmatrix}.
\]

As previously described by [2] and [5], every overlap matrix of the closed-end ladder \( L_{n-1} \) has the form \( M_{n+1}^{X,Y} \), and every overlap matrix of the Ringel ladder \( R_{n-1} \) has the form \( M_{n+1}^{X,Y,Z} \) (We observe that the subscripts of \( R_{n-1} \) and \( M_{n+1}^{X,Y,Z} \) differ by two.) Here we further define

\[
\mathcal{P}_n = \text{as the set of all matrices over } \mathbb{Z}_2 \text{ of the form } M_{n}^{0,Y,Z};
\]

\[
C_n(j) = \text{as the number of overlap matrices for the Ringel ladder } R_n \text{ of rank } j;
\]

\[
P_n(z) = \sum_{j=0}^{n+1} C_n(j) z^j \text{ as the rank-distribution polynomial of the set } \mathcal{P}_n; \text{ and}
\]

\[
O_n(z) = \text{as the rank-distribution polynomial of the overlap matrices of the closed-end ladder } L_{n-1} \text{ over the set of matrices of the form } M_{n}^{0,Y}.
\]

**Theorem 2.2.** (from [4]) The rank-distribution polynomial \( O_n(z) \) of the overlap matrix for the closed-end ladder \( L_{n-1} \) satisfies the recurrence relation

\[
O_n(z) = O_{n-1}(z) + 2z^2O_{n-2}(z)
\]

with the initial conditions \( O_1(z) = 1 \) and \( O_2(z) = z^2 + 1 \). Moreover, the generating function \( O(t; z) = \sum_{n \geq 1} O_n(z) t^n \) is given by

\[
O(t; z) = \frac{t + z^2t^2}{1 - t - 2z^2t^2}.
\]
Theorem 2.3. (from [5]) Let \( O_{n-1}(z) \) be the rank-distribution polynomial of the closed-end ladder \( L_{n-2} \). Then the polynomial \( P_n(z) \) \((n \geq 3)\) satisfies the recurrence relation
\[
P_{n+1}(z) = P_n(z) + 8z^2P_{n-1}(z) + 2^{n-1}z^2O_{n-1}(z),
\]
with the initial condition \( P_2(z) = z^2+1, P_3(z) = 7z^2+1 \) and \( P_4(z) = 12z^4+19z^2+1 \). Moreover, the generating function \( P(t; z) = \sum_{n \geq 2} P_n(z)t^n \) is given by
\[
P(t; z) = \frac{t^2(1+z^2 - 2(1-2z^2)t - 4z^2(4+z^2)t^2 - 32z^4t^3)}{(1-2t - 8z^2t^2)(1-t - 8z^2t^2)}.
\]

2.1. Rank-distribution polynomial for circular ladders. One of the computational complications is the need to split the count of some matrices and to split the rank-distribution polynomial \( P_n(z) \) (def. (2.13)) into cases according to parity. Substituting “0” for “X” in a superscript means that in the matrix, each coordinate \( x_i \) is set to zero.

\[
\begin{align*}
M_{n+1}^{0,Y,Z_{even}} &= \text{the number of matrices of the form } M_{n+1}^{0,Y,Z} \text{ (def. (2.10)) with evenly many elements of } Z \text{ equal to 1}; \\
M_{n+1}^{0,Y,Z_{odd}} &= \text{the number of matrices of the form } M_{n+1}^{0,Y,Z} \text{ with oddly many elements of } Z \text{ equal to 1}; \\
P_{n+1}^{0}(z) &= \text{the rank distribution polynomial over the set } M_{n+1}^{0,Y,Z_{odd}}; \\
P_{n+1}^{E}(z) &= \text{the rank distribution polynomial over the set } M_{n+1}^{0,Y,Z_{even}}.
\end{align*}
\]

By their definitions, we have \( P_n(z) = P_n^{0}(z) + P_n^{E}(z) \).

Lemma 2.4. The polynomial \( P_{n}^{E}(z) \) \((n \geq 4)\) satisfies the recurrence relation
\[
P_{n+1}^{E}(z) = P_{n}^{E}(z) + 4z^2P_{n-1}(z) + 2^{n-2}z^2O_{n-1}(z),
\]
with the initial condition \( P_{2}^{E}(z) = 3z^2+1 \) and \( P_{4}^{E}(z) = 6z^4+9z^2+1 \), where \( O_{n-1}(z) \) and \( P_{n}(z) \) are the rank-distribution polynomials of the closed-end ladder \( L_{n-2} \) and the Ringel ladder \( R_{n-2} \), respectively.

Proof. The proof has two cases.

- Case 1: \( y_{n-1} = 0 \). If \( z_n = 0 \), then the matrix contributes a term \( P_{n}^{E}(z) \); otherwise, if \( z_n = 1 \), it contributes a term \( 2^{n-2}z^2O_{n-1}(z) \) (def. (2.14)).
- Case 2: \( y_{n-1} = 1 \). If \( z_n = 0 \), then we add the \( n+1 \)-st column and row to the first and \( n-1 \)-st column and row, which transforms the matrix (2.10) above into the following form:

\[
\begin{pmatrix}
0 & z_1 & z_2 & \cdots & z_{n-2} & 0 & 0 \\
z_1 & 0 & y_1 & & & & \\
z_2 & y_1 & 0 & \ddots & & & \\
& \ddots & \ddots & \ddots & & & \\
z_{n-2} & & & \ddots & \ddots & \ddots & \\
0 & 0 & & & 0 & 1 & 0 \\
0 & 1 & 0 & & & &
\end{pmatrix}
\]
Depending whether $z_{n-1} = 0$ or 1, this matrix contributes either $2z^2p_{n-1}^E(z)$ or $2z^2p_{n-1}^0(z)$ to $P_{n+1}^E(z)$. Otherwise $z_n = 1$, and we add the last column and row to the first column and row, and we thereby obtain the following matrix.

$$
\begin{pmatrix}
0 & z_1 & z_2 & \cdots & z_{n-2} & 0 & 1 \\
z_1 & 0 & y_1 & & & & \\
z_2 & y_1 & 0 & \ddots & & & 0 \\
& & \ddots & \ddots & \ddots & & \\
z_{n-2} & 0 & 0 & \cdots & y_{n-2} & 1 & 1 \\
0 & 1 & y_1 & 0 & \cdots & 0 & y_{n-2} \\
1 & 1 & 0 & \cdots & \cdots & \cdots & y_{n-1} \\
\end{pmatrix},
$$

(1) $y_{n-2} = 0$. In this subcase, according to the values of $z_{n-1}$, it contributes to $P_{n+1}^E(z)$ by a term $z^2p_{n-1}^0(z)$ or $z^2p_{n-1}^E(z)$

(2) $y_{n-2} = 1$. We add the last row to the $n-1$-st row in this subcase. Depending on the value of $z_{n-1}$, it contributes either $z^2p_{n-1}^0(z)$ or $z^2p_{n-1}^E(z)$:

$$
\begin{pmatrix}
0 & z_1 & z_2 & \cdots & z_{n-2} + 1 & 0 & 1 \\
z_1 & 0 & y_1 & & & & \\
z_2 & y_1 & 0 & \ddots & & & 0 \\
& & \ddots & \ddots & \ddots & & \\
z_{n-2} + 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 1 & y_1 & 0 & \cdots & 0 & y_{n-2} \\
1 & 1 & 0 & \cdots & \cdots & \cdots & y_{n-1} \\
\end{pmatrix}.
$$

Proposition 2.5. For all $n \geq 3$, we have

(2.21) $P_n^E(z) = \frac{1}{2}(P_n(z) + 1 - z^2)$.

Moreover, the generating function $P^E(t; z) = \sum_{n \geq 3} P_n^E(z)t^n$ is given by

$$
t^3(1 + 3z^2 - 3(1 - z^2)(1 + 2z^2)t + 2(1 - 10z^2 - 13z^4)t^2 + 8z^2(3 - 2z^2 - 4z^4)t^3 + 64z^4t^4)/(1 - 2t - 8z^2t^2)(1 - t - 8z^2t^2)(1 - t).
$$

Proof. Using Lemma 2.4 together with induction on $n$, we have

(2.22) $P_n^E(z) = P_4^E(z) + \sum_{j=3}^{n-2}(4z^2P_j(z) + 2j^{-1}z^2O_j(z))$.

Thus, by (2.15) we obtain

(2.23) $P_n^E(z) = P_4^E(z) + \frac{1}{2} \sum_{j=3}^{n-2}(P_{j+2}(z) - P_{j+1}(z)) = P_4^E(z) + \frac{1}{2}(P_n(z) - P_4(z))$

(2.24) $= \frac{1}{2}(P_n(z) + 1 - z^2)$,

as claimed. The rest follows immediately from Theorem 2.3.

\[\square\]
Lemma 2.6. The rank-distribution polynomial $B_{00}^n(z)$ ($n \geq 4$) (in def. (2.8)) is given by

$$B_{00}^n(z) = B_{n+2}(z) = P_{E}^{n+1}(z) + 2^{n-1}z^2O_n(z) = \frac{1}{2}(P_{n+1}(z) + 1 - z^2) + 2^{n-1}z^2O_n(z).$$

where $O_{n-1}(z)$ is the rank-distribution polynomial of closed-end ladders $L_{n-2}$ and $P_{E}^n(z)$ is the rank-distribution polynomial over the set $M_{0,Y,Z}^{0,1,0,0,Y,Z}.$

Proof. The case $c = 0$ contributes the term $P_{n+1}(z)$ to $B_{00}^n(z).$ Otherwise $c = 1,$ and since there are $2^{n-1}$ possible choices of $z_1, z_2, \ldots, z_n$ such that the number of variables in $\{z_1, z_2, \ldots, z_n\}$ equal to 1 is odd, yielding a contribution of $2^{n-1}z^2O_n(z)$ to $B_{00}^n(z).$ The rest follows immediately by Proposition 2.5. □

Lemma 2.7. The rank-distribution polynomial $B_{10}^n(z)$ ($n \geq 4$) is given by the formula

$$B_{10}^n(z) = 4z^2P_n(z),$$

where $P_n(z)$ is the rank-distribution polynomial of the Ringel ladder $R_{n-2}.$

Proof. Multiply row 3 by the constant $c,$ and add to the second row. Similarly, multiply row 3 by the constant $c,$ and add the result to the second row. If $z_1 = 1,$ then we add the first row to the third row and next add the first column to the third column. There is a similar situation for $y_1.$ We interchange row 2 with row 3 and column 2 with column 3, and we thereby transform the matrix $M_{n+2}^{0,1,0,0,Y,Z}$ to the following form:

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & z_{n-1} & z_n \\
1 & 0 & z_2 & z_3 & \cdots & z_{n-1} & z_n \\
0 & z_2 & 0 & y_2 & \cdots & 0 \\
0 & z_3 & y_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & z_{n-1} & 0 & 0 & \cdots & y_{n-1} \\
z_n & 0 & 0 & \cdots & \cdots & y_{n-1} & 0
\end{pmatrix}.
$$

Note that the lower-right $n \times n$ submatrix has the form $M_{n}^{0,Y,Z}.$ There are $2^2$ different assignments of the variables $y_1$ and $z_1$ in this case, so it contributes $4z^2P_n(z)$ to the polynomial $B_{10}^n(z).$ □

By symmetry, we have the following lemma.

Lemma 2.8. The polynomial $B_{01}^n(z)$ ($n \geq 4$) satisfies this formula:

$$B_{01}^n(z) = 4z^2P_n(z),$$

where $P_n(z)$ is the rank-distribution polynomial of the Ringel ladder $R_{n-2}.$
Let
\[
M_{n+1}^{X,Y,Z,1} = \begin{pmatrix}
    x_0 & z_1 & z_2 & z_3 & \ldots & z_{n-1} & z_n \\
    z_1 & x_1 & y_1 & 0 & \ldots & 0 & 1 \\
    z_2 & y_1 & x_2 & y_2 & & & \\
    z_3 & 0 & y_2 & x_3 & & \ddots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    z_{n-1} & 0 & y_{n-2} & x_{n-1} & y_{n-1} & & \\
    z_n & 1 & y_{n-1} & x_n & & &
\end{pmatrix},
\]
where
\[
X = (x_0, x_1, \ldots, x_n) \in \mathbb{Z}_2^{n+1}, \ Y = (y_1, y_2, \ldots, y_{n-1}) \in \mathbb{Z}_2^{n-1}, \text{ and } Z = (z_1, z_2, \ldots, z_n) \in \mathbb{Z}_2^n.
\]

We let \(Q_{n+1}\) be the set of all matrices over \(\mathbb{Z}_2\) of the form \(M_{n+1}^{0,Y,Z,1}\), and we let
\[
(2.28) \quad Q_{n+1}(z) = \sum_{j=0}^{n+1} Q_{n+1}(j) z^j
\]
be the rank-distribution polynomial of the set \(Q_{n+1}\).

We again split cases according to parity:
\[
\begin{align*}
(2.29) \quad M_{n+1}^{0,Y,Z,\text{even},1} &= \text{the number of matrices of the form } M_{n+1}^{0,Y,Z,1} \\
& \text{with evenly many elements of Z equal to 1;} \\
(2.30) \quad M_{n+1}^{0,Y,Z,\text{odd},1} &= \text{the number of matrices of the form } M_{n+1}^{0,Y,Z,1} \\
& \text{with oddly many elements of Z equal to 1;} \\
(2.31) \quad Q_{n+1}^E(z) &= \text{the rank-distribution polynomial over the set } M_{n+1}^{0,Y,Z,\text{odd},1}; \\
(2.32) \quad Q_{n+1}^O(z) &= \text{the rank-distribution polynomial over the set } M_{n+1}^{0,Y,Z,\text{even},1}.
\end{align*}
\]

It is clear that \(Q_{n+1}(z) = Q_{n+1}^E(z) + Q_{n+1}^O(z)\).

Similarly, as in [6], we have these additional notations:
\[
\begin{align*}
(2.33) \quad H_{n+1} &= \text{the set of all matrices over } \mathbb{Z}_2 \text{ of the form } M_{n+1}^{X,Y,Z,1}; \\
(2.34) \quad H_{n+1}(z) &= \sum_{j=0}^{n+1} H_{n+1}(j) z^j \text{ is the rank-distribution polynomial of the set } Q_{n+1}; \\
(2.35) \quad M_{n+1}^{X,Y,Z,\text{even},1} &= \text{the number of matrices of the form } M_{n+1}^{X,Y,Z,1} \\
& \text{with evenly many elements of Z equal to 1;} \\
(2.36) \quad Q_{n+1}^E(z) &= \text{the rank-distribution polynomial of the set } M_{n+1}^{X,Y,Z,\text{even},1}; \\
(2.37) \quad M_{n+1}^{X,Y,Z,\text{odd},1} &= \text{the number of matrices of the form } M_{n+1}^{0,Y,Z,1} \\
& \text{with oddly many elements of Z equal to 1;} \\
(2.38) \quad H_{n+1}(z) &= \text{the rank-distribution polynomial of the set } M_{n+1}^{X,Y,Z,\text{odd}}.
\end{align*}
\]

Lemma 2.9. The polynomial \(Q_{n+1}^E(z) (n \geq 4)\) satisfies the recurrence relation
\[
(2.39) \quad Q_{n+1}^E(z) = 6z^2 p_{n-1}(z) + 4z^2 Q_{n-1}^E(z)
\]
with the initial conditions \(Q_2^E(z) = 1\) and \(Q_3^E(z) = 1 + 3z^2\), where \(p_{n-1}(z)\) is the rank-distribution polynomial of the Ringel ladder \(R_{n-3}\).
Proof. There are four cases.

Case 1: \( z_n = 0, y_{n-1} = 0 \). If \( z_1 = 1 \), we first add the last row to the first row, and then add the last column to the first column. Invoking similar operations for \( y_1 \), we can transform the matrix \( M_{n+1}^{0,Y,Z,1} \) (def. (2.29)) into the following forms:

\[
\begin{pmatrix}
0 & z_1 & z_2 & z_3 & \ldots & z_{n-1} & 0 \\
z_1 & 0 & y_1 & 0 & \ldots & 0 & 1 \\
z_2 & y_1 & 0 & y_2 & \\
z_3 & y_2 & 0 & \ddots & \ddots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & y_{n-2} \\
z_{n-1} & y_{n-2} & 0 & 0 & \\
0 & 1 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 & 0 & z_2 & z_3 & \ldots & z_{n-1} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
z_2 & 0 & 0 & y_2 & \\
z_3 & y_2 & 0 & \ddots & \ddots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & y_{n-2} \\
z_{n-1} & y_{n-2} & 0 & 0 & \\
0 & 1 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}
\]

Note that there are four different ways to assign the variables \( y_1 \) and \( z_1 \) in the matrix \( M_{n+1}^{0,Y,Z,1} \). If \( z_1 = 0 \), the matrix contributes \( 2z^2pE_{n-1}(z) \) (def. (2.19)) to the polynomial \( Q^{E}_{n+1}(z) \) (def. (2.36)). Otherwise \( z_1 = 1 \), and it contributes \( 2z^2p^0_{n-1}(z) \) (def. (2.18)). Since \( p_{n-1}(z) = pE_{n-1}(z) + p^0_{n-1}(z) \), it contributes \( 2z^2p_{n-1}(z) \).

Case 2: \( z_n = 1, y_{n-1} = 0 \). We first add the second row to the first row, and then add the second column to the first column. This transforms \( M_{n+1}^{0,Y,Z,1} \) into the following form:

\[
\begin{pmatrix}
0 & z_1 & z_2 + y_1 & z_3 & \ldots & z_{n-1} & 0 \\
z_1 & 0 & y_1 & 0 & \ldots & 0 & 1 \\
z_2 + y_1 & y_1 & 0 & y_2 & \\
z_3 & y_2 & 0 & \ddots & \ddots & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & y_{n-2} \\
z_{n-1} & y_{n-2} & 0 & 0 & \\
0 & 1 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}
\]

If \( z_1 = 1 \), we first add the last row to the first row, and then add the last column to the first column. By similar discussion to that regarding \( y_1 \), we can transform the matrix
above into the following form:

\[
\begin{pmatrix}
0 & 0 & z_2 + y_1 & z_3 & \ldots & z_{n-1} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
z_2 + y_1 & 0 & 0 & y_2 & \vdots & \vdots & \vdots \\
z_3 & y_2 & 0 & \ddots & 0 & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
z_{n-1} & y_{n-2} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}
\]

Accordingly, this case contributes \(2z^2p_{n-1}(z)\) to the polynomial \(Q_{n+1}(z)\).

**Case 3:** \(z_n = 0, y_{n-1} = 1\).

\[
\begin{pmatrix}
0 & z_1 & z_2 & z_3 & \ldots & z_{n-2} & z_{n-1} & 0 \\
z_1 & 0 & y_1 & 0 & \ldots & 0 & 0 & 1 \\
z_2 & y_1 & 0 & y_2 & \vdots & \vdots & \vdots & \vdots \\
z_3 & 0 & y_2 & 0 & \ddots & 0 & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
z_{n-2} & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
z_{n-1} & 0 & 0 & 0 & y_{n-2} & 0 & 0 & 1 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0
\end{pmatrix}
\]

After we add the \(n\)-th row to the second row, then add the \(n\)-th column to the second column, the resulting matrix has the following form:

\[
\begin{pmatrix}
0 & z_1 + z_{n-1} & z_2 & z_3 & \ldots & z_{n-2} & z_{n-1} & 0 \\
z_1 + z_{n-1} & 0 & y_1 & 0 & \ldots & y_{n-2} & 0 & 0 \\
z_2 & y_1 & 0 & y_2 & \vdots & \vdots & \vdots & \vdots \\
z_3 & 0 & y_2 & 0 & \ddots & 0 & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
z_{n-2} & y_{n-2} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
z_{n-1} & 0 & 0 & 0 & y_{n-2} & 0 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0
\end{pmatrix}
\]

No matter what the assignments of values to \(z_{n-1}\) and \(y_{n-2}\), we can transform the above matrix to the following form:

\[
\begin{pmatrix}
0 & z_1 + z_{n-1} & z_2 & z_3 & \ldots & z_{n-2} & 0 & 0 \\
z_1 + z_{n-1} & 0 & y_1 & 0 & \ldots & y_{n-2} & 0 & 0 \\
z_2 & y_1 & 0 & y_2 & \vdots & \vdots & \vdots & \vdots \\
z_3 & 0 & y_2 & 0 & \ddots & 0 & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
z_{n-2} & y_{n-2} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1
\end{pmatrix}
\]
If $y_{n-2} = 0$, it contributes $2z^2\mathcal{P}_{n-1}^E(z)$ to the polynomial $Q_{n+1}^E(z)$. Otherwise $y_{n-2} = 1$, and it contributes $2z^2Q_n^E(z)$.

**Case 4:** $z_n = 1, y_{n-1} = 1$.

$$
\begin{pmatrix}
0 & 1 & z_1 & z_2 & z_3 & \ldots & z_{n-1} & 1 \\
z_1 & 0 & y_1 & 0 & \ldots & 0 & 1 \\
z_2 & y_1 & 0 & y_2 & & & & \\
z_3 & y_2 & 0 & \ddots & 0 & & & \\
\vdots & 0 & \ddots & \ddots & \ddots & & & \\
z_{n-1} & \ddots & \ddots & \ddots & \ddots & \ddots & y_{n-2} \\
1 & 1 & 0 & \ldots & 0 & 1 & 0 \\
\end{pmatrix}.
$$

We first add the $n-$th row to the first and second row, then add the $n-$th column to the first and second column, the resulting matrix has the following form:

$$
\begin{pmatrix}
0 & z_1 + z_{n-1} & z_2 & z_3 & \ldots & z_{n-2} + y_{n-2} & z_{n-1} & 0 \\
z_1 + z_{n-1} & 0 & y_1 & 0 & \ldots & y_{n-2} & 0 & 0 \\
z_2 & y_1 & 0 & y_2 & & & & 0 \\
z_3 & y_2 & 0 & \ddots & 0 & & & \\
\vdots & \ddots & \ddots & \ddots & & & & \\
z_{n-2} + y_{n-2} & y_{n-2} & 0 & \ldots & 0 & y_{n-2} & 0 \\
z_{n-1} & 0 & 0 & \ldots & 0 & y_{n-2} & 0 & 1 \\
0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\
\end{pmatrix}.
$$

If $y_{n-2} = 0$, it contributes $2z^2\mathcal{P}_{n-1}^E(z)$ to the polynomial $Q_{n+1}^E(z)$. Otherwise $y_{n-2} = 1$, and it contributes $2z^2Q_n^E(z)$. \hfill \Box

**Proposition 2.10.** The generating function $Q^E(t; z) = \sum_{n \geq 2} Q_n^E(z)t^n$ is given by

$$
\frac{t^2(1 - (2 - 3z^2)t - (1 + 19z^2 - 6z^4)t^2 + 2(1 + 4z^2)(1 - 3z^2)t^3 + 8z^2(3 + 5z^2 - 3z^4)t^4 + 64z^4t^5)}{(1 - 2t - 8z^2t^2)(1 - t - 8z^2t^2)(1 - 4z^2t^2)}.
$$

**Proof.** Multiplying the recurrence relation in the statement of Lemma 2.9 by $t^{n+1}$ and summing over all $n \geq 3$, and with the aid of Theorem 2.3, we obtain the specified result. \hfill \Box

**Lemma 2.11.** The polynomial $\mathcal{B}_{n}^{11}(z)$ (def. (2.8)), for $n \geq 4$, is given by

$$
(2.40) \quad \mathcal{B}_{n+2}^{11}(z) = 2z^2\mathcal{P}_n(z) + 4z^2Q_n^E(z),
$$

where $\mathcal{P}_n(z)$ is the rank-distribution polynomial of the Ringel ladder $R_{n-2}$, and where $Q_n^E(z)$ is the rank distribution polynomial over the set $M_n^{0,Y,Z_{even}}$ (def. (2.29)).
Proof. In this case, the matrix has the following form:

\[
\begin{pmatrix}
0 & c & 1 & 0 & \ldots & 1 \\
c & 0 & z_1 & z_2 & \ldots & z_{n-1} & z_n \\
1 & z_1 & 0 & y_1 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
z_{n-2} & 0 & \ldots & 0 & y_{n-2} \\
1 & z_{n-1} & 0 & y_{n-2} & 0 & y_{n-1} \\
1 & z_n & 0 & y_{n-2} & 0 & y_{n-1}
\end{pmatrix}
\]

Multiply the last row by the constant \(c\) and add to the second row. Similarly, multiply the last column by the constant \(c\) and add to the second column. We do the same with the third row and column. The matrix above is transformed into this:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & 0 & z_1 + z_n & z_2 & \ldots & z_{n-2} & z_{n-1} + cy_{n-1} & z_n \\
0 & z_1 + z_n & 0 & y_1 & \ldots & 0 & y_{n-1} & 0 \\
0 & z_2 & y_1 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & z_{n-2} & 0 & \ldots & 0 & y_{n-2} \\
0 & z_{n-1} + cy_{n-1} & y_{n-1} & 0 & y_{n-2} & 0 & y_{n-1} \\
1 & z_n & 0 & y_{n-2} & 0 & y_{n-1} & 0
\end{pmatrix}
\]

Case 1: \(c = 0\). There are four subcases.

- **Subcase 1**: \(z_n = 0, y_{n-1} = 0\). Note that the down-right submatrix is the matrix the form \(M^0_{n,Y,Z_{even}}\) (def. (2.16)). This case contributes \(z^2P^E_{n}(z)\) to the polynomial \(B_{n+2}^{11}(z)\) (def. (2.8)).
- **Subcase 2**: \(z_n = 1, y_{n-1} = 0\). We first add the first row to the second row, then add the first column to the second column. Note that the down-right submatrix is the matrix the form \(M^0_{n,Y,Z_{even}}\). This case contributes \(z^2P^E_{n}(z)\).
- **Subcase 3**: \(z_n = 0, y_{n-1} = 1\). We first delete the first row and the last row, then delete the first column and the last column. As the result, we obtain a matrix of the form \(M^0_{n-1,Y,Z_{even}}\). This case contributes \(z^2Q^E_{n-1}(z)\) (def. (2.36)).
Subcase 4: \( z_n = 1, y_{n-1} = 1 \). We first delete the first row and the last row, then delete the first column and the last column. The result here is a matrix of the form \( M_{n-1}^{0,Y,Z,even,1} \). This case contributes \( z^2Q_{n-1}^E(z) \).

**Case 2:** \( c = 1 \). In this case also, we have four subcases:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} + cy_{n-1} & z_n \\
0 & z_1 + z_n & 0 & y_1 & \cdots & 0 & y_{n-1} & 0 \\
0 & z_2 & y_1 & 0 & \cdots & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & z_{n-2} & 0 & 0 & \cdots & y_{n-2} & 0 & \\
0 & z_{n-1} + cy_{n-1} & y_{n-1} & 0 & y_{n-2} & 0 & y_{n-1} & 0 \\
1 & z_n & 0 & 0 & \cdots & y_{n-1} & 0 &
\end{pmatrix}
\]

Subcase 1: \( z_n = 0, y_{n-1} = 0 \). Note that the lower right submatrix has the form \( M_{n-1}^{0,Y,Z,odd} \). This case contributes \( z^2P_{n-1}^0(z) \) (def. (2.18)) to the polynomial \( B_{11}^{n+2}(z) \).

Subcase 2: \( z_n = 1, y_{n-1} = 0 \). We first add the first row to the second row, then add the first column to the second column. Again the lower right submatrix has the form \( M_{n-1}^{0,Y,Z,odd} \). This case contributes \( z^2P_{n-1}^0(z) \).

Subcase 3: \( z_n = 0, y_{n-1} = 1 \). We first delete the first row and the last row, then delete the first column and the last column. The resulting matrix has the form \( M_{n-1}^{0,Y,Z,odd,1} \). This case contributes \( z^2Q_{n-1}^E(z) \):

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & z_1 & z_2 & \cdots & z_{n-2} & z_{n-1} + 1 & 0 \\
0 & z_1 & 0 & y_1 & \cdots & 0 & 1 & 0 \\
0 & z_2 & y_1 & 0 & \cdots & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & z_{n-2} & 0 & 0 & \cdots & y_{n-2} & 0 & \\
0 & z_{n-1} + 1 & 1 & 0 & y_{n-2} & 0 & 1 & 0 \\
1 & z_n & 0 & 0 & \cdots & y_{n-1} & 0 &
\end{pmatrix}
\]

Subcase 4: \( z_n = 1, y_{n-1} = 1 \). We first delete the first row and the last row, then delete the first column and the last column. The resulting matrix has the form \( M_{n-1}^{0,Y,Z,even,1} \). This case contributes \( z^2Q_{n-1}^E(z) \):

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & z_1 + 1 & z_2 & \cdots & z_{n-2} & z_{n-1} + 1 & 1 \\
0 & z_1 + 1 & 0 & y_1 & \cdots & 0 & 1 & 0 \\
0 & z_2 & y_1 & 0 & \cdots & 0 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & z_{n-2} & 0 & 0 & \cdots & y_{n-2} & 0 & \\
0 & z_{n-1} + 1 & 1 & 0 & y_{n-2} & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & \cdots & y_{n-1} & 0 &
\end{pmatrix}
\]
Theorem 2.12. For all \( n \geq 5 \),
\[
(2.41) \quad B_n(z) = \frac{1}{2}(P_n(z) + 1 - z^2) + \Omega_n^E(z).
\]
Moreover, the generating function \( B(t; z) = \sum_{n \geq 5} B_n(z) t^n \) is given by
\[
(1 - 2t - 8zt^2)(1 - t - 8t^2)(1 - t)(1 - 4zt^2),
\]
where
\[
f(t; z) = 100z^4 + 27z^2 + 1 + (z^2 - 1)(160z^4 + 76z^2 + 3)t + (30z^2 - 640z^4 + 2 - 1312z^6)t^2
\]
\[-4z^2(-213z^2 - 7 + 60z^4 + 352z^6)t^3 + 8z^2(-15z^2 + 448z^4 + 528z^6 - 1)t^4
\]+32z^4(96z^6 - 3 + 68z^2 - 65z^2)t^5 - 256z^6(1 + 19z^2 + 12z^4)t^6.
\]

Proof. By Property 2.1 and Lemmas 2.6, 2.7, 2.8 and 2.11, we obtain \( B_n(z) = \frac{1}{2}(P_n(z) + 1 - z^2) + \Omega_n^E(z) \), and by Proposition 2.5, we obtain \( B_n(z) = \frac{1}{2}(P_n(z) + 1 - z^2) + \Omega_n^E(z) \). Multiplying by \( t^n \), next summing over all \( n \geq 5 \), and then using Theorem 2.3 and Proposition 2.10, we complete the proof. \( \Box \)

3. The Genus Polynomials of a Circular Ladder

We now derive the genus polynomial for a circular ladder.

Theorem 3.1. We have \( \sum_{n \geq 4} g_{\text{CL}_n}(x) t^n = \frac{2}{t}B(t; \sqrt{x}) \). Moreover, for all \( n \geq 4 \), the number of distinct cellular imbeddings of \( \text{CL}_n \) in a surface of genus \( j \) is
\[
(3.1) \quad \left\{ \begin{array}{ll}
\frac{7n+2j}{2}3^{j-3}(n-j-1) + \frac{n-4j+2j+3}{2}3^{j-3}(n-j-1) + \frac{n-j-1}{2}2^{n+j-1}(n-j-1) \\
+2^{n-1}n_{2j} + 2^{n}n_{2j+1} - 3 \cdot 2^{n-1}n_{2j} & j \geq 2, \\
2^n + 8n - 2 + 8n_{2j} & j = 1, \\
2 & j = 0.
\end{array} \right.
\]

Proof. Let \( GCL(t; x) = \sum_{n \geq 4} g_{\text{CL}_n}(x) t^n \). Then by Theorem 2.12 we have
\[
GCL(t; x) = \frac{2t^4f(t; x)}{(1 - 2t - 8xt^2)(1 - t - 8t^2)(1 - t)(1 - 4xt^2)},
\]
where
\[
f(t; z) = 100x^2 + 27x + 1 + (x-1)(160x^2 + 76x + 3)t + (30x - 640x^2 + 2 - 1312x^3)t^2
\]
\[-4x(-213x - 7 + 60x^2 + 352x^3) + 8x(-15x + 448x^2 + 528x^3 - 1)t^4
\]+32x^3(96x^6 - 3 + 68x^2 - 65x)t^5 - 256x^6(1 + 19x^2 + 12x^4)t^6.
\]

By several algebraic operations, we calculate that the coefficient of \( x^j \) in \( GCL(t; x) \) is given by
\[
(3.2) \quad \left\{ \begin{array}{ll}
-\frac{8t^4(1-t)^2(1-t)^2(1-t)^2(1-t)^2)}{(1-t)^2} + \frac{1}{(1-t)^2} + (2t^2 + 2t + 3/2)2j(2j) & j \geq 2, \\
\frac{2t^4(27+50t-8t^2-73t^3)}{(1-t)^2} & j = 1, \\
\frac{2t^4}{1-t} & j = 0.
\end{array} \right.
\]
Hence, the coefficient of \(x^jt^n\) in \(GCL(t; x)\) is given by
\[
2^{3j-3} \left[ -8\binom{n-j-1}{j} + 15\binom{n-j}{j} + 3\binom{n-j+1}{j} - 2\binom{n-j+2}{j} \right] + 2^{n+j+1} \left[ 2\binom{n-j+2}{j} - 3\binom{n-j+1}{j} + \binom{n-j}{j} \right] + 2^{n-1}\delta_n,2j + 2^n\delta_n,2j+1 - 3 \cdot 2^{n-1}\delta_n,2j
\]
for \(j \geq 2\),
\[
\frac{2^n + 8n - 2 + 8\delta_n,4}{2}
\]
for \(j = 1\),
\[
\text{for } j = 0.
\]
which is equivalent to
\[
2^{n+j+3}j^{-1}\binom{n-j-1}{j-1} + \frac{n-3j+2}{n-j}\binom{n-j+1}{j-1} + \frac{n}{j-1}2^{n+j-1}\binom{n-2}{j-2}
\]
for \(j \geq 2\),
\[
\frac{2^n + 8n - 2 + 8\delta_n,4}{2}
\]
for \(j = 1\),
\[
\text{for } j = 0.
\]
as claimed. \(\square\)

Note that it is not hard to check that our formula in the above theorem is equivalent to the formula in [13, Theorem 3.10].

As a corollary of the above theorem we can obtain the following result.

**Corollary 3.2.** For all \(n \geq 4\),
\[
g_{CL,n}(x) = 1 - x + \frac{1 - 3x - 2\sqrt{x}}{4x}(-2\sqrt{x})^n + \frac{1 - 3x + 2\sqrt{x}(2\sqrt{x})^n}{4x} + 2^n x(i\sqrt{2x})^n \left[ U_n \left( \frac{1}{2i\sqrt{2x}} \right) - U_{n-2} \left( \frac{1}{2i\sqrt{2x}} \right) \right] + (1 - x)(2i\sqrt{2x})^n \left[ U_n \left( \frac{1}{4i\sqrt{2x}} \right) - U_{n-2} \left( \frac{1}{4i\sqrt{2x}} \right) \right],
\]
where \(U_s\) is the \(s\)-th Chebyshev polynomial of the second kind and \(i^2 = -1\).

**Proof.** By Theorem 3.1 we infer that the generating function \(f = \sum_{n \geq 4} g_{CL,n}(x)t^n\) can be written as
\[
f = -2(1 + 19x + 12x^2)t^3 - 4(1 + 3x)t^2 - 4t - \frac{1}{2}x(1 + 3x - 2x^2) + \frac{1 - 3x + 4xt}{2x(1 - 4xt^2)} - \frac{2(x(1-t))}{1 - 2t - 8xt^2} + \frac{2 - 2x + xt - t}{1 - t - 8xt^2} + \frac{1 - x}{1 - t}.
\]
Let \(n \geq 4\), so the coefficient \(g_{CL,n}(x)\) of \(t^n\) in the generating function \(f\) is given by
\[
g_{CL,n}(x) = 1 - x + \frac{1 - 3x - 2\sqrt{x}}{4x}(-2\sqrt{x})^n + \frac{1 - 3x + 2\sqrt{x}(2\sqrt{x})^n}{4x} + 2^n x(i\sqrt{2x})^n \left[ U_n \left( \frac{1}{2i\sqrt{2x}} \right) - U_{n-2} \left( \frac{1}{2i\sqrt{2x}} \right) \right] + (1 - x)(2i\sqrt{2x})^n \left[ U_n \left( \frac{1}{4i\sqrt{2x}} \right) - U_{n-2} \left( \frac{1}{4i\sqrt{2x}} \right) \right],
\]
where the first, second and third lines are the coefficients of \(t^n\) in the generating functions
\[
\frac{1 - x}{1 - t} + \frac{1 - 3x + 4xt}{2x(1 - 4xt^2)}, \quad \frac{2(x(1-t))}{1 - 2t - 8xt^2}, \quad \text{and } \frac{2 - 2x + xt - t}{1 - t - 8xt^2},
\]
respectively. \(\square\)
We end the paper by showing that the genus distribution polynomial of every circular ladder is log-concave.

**Theorem 3.3.** The genus distribution \( \{a_j\} \) of every circular ladder \( CL_n \) is log-concave.

**Proof.** A short computation shows that the theorem is true for \( n \leq 8 \). Accordingly, we now suppose that \( n \geq 9 \). From Theorem 3.1, we know that (for all \( n \geq 4 \)) the number of distinct cellular imbeddings of the cellular ladder \( CL_n \) in a surface of genus \( j \) is

\[
a_j = \begin{cases} 
\frac{7n+2j-3}{j} (n-j^{-1}) + \frac{n-4j+2j^2-3}{j} (n-j^{-1}) + \frac{n}{j-1} 2^{n+j-1}(n-j^{-2}) & \text{for } j \geq 2, \\
2^{n-j} + 2^{n-j+2} + 2^n \delta_{n,2j+1} - 3 \cdot 2^{n-j} \delta_{n,2j} & \text{for } j = 1, \\
2^n + 8n - 2 + 8\delta_{n,4} & \text{for } j = 0.
\end{cases}
\]  

(3.5)

Therefore, in order to prove that the genus distribution of every circular ladder is log-concave, we have to show that

\[
a_j^2 - a_{j-1}a_{j+1} \geq 0,
\]

for all \( j \) such that \( 0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \). We do that in two steps, whose details appear in Appendix A. Step 1 is to show that (3.6) holds for \( j = 0, 1, 2 \) and for \( j = \left\lfloor \frac{n}{2} \right\rfloor - 1 \). Step 2 is to show that (3.6) holds for \( j = 3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 2 \). \( \square \)

**Acknowledgement**

The authors would like to thank the referees for their careful reading of our paper and their valuable suggestions for improving it.

**References**


4. Appendix A: Details of the proof of Theorem 3.3

Step 1. We now show that Inequality (3.6) holds for \( j = 0, 1, 2 \) and \( j = \lfloor \frac{n}{2} \rfloor - 1 \). From Theorem 3.1, we have

\[
\begin{align*}
(4.1) & \quad a_0 = 2 \\
(4.2) & \quad a_1 = 2^n + 8n - 2 \\
(4.3) & \quad a_2 = 32n^2 - 104n + n \cdot 2^{n+1} \\
(4.4) & \quad a_3 = \frac{32}{3}(8n^3 - 75n^2 + 169n) + n(n-3)2^{n+1}
\end{align*}
\]

from which we calculate

\[
(4.5) \quad a_1^2 - a_0a_2 = 2^{2n} + (12n - 4) \cdot 2^n + 176n + 4 > 0
\]

and also

\[
(4.6) \quad a_2^2 - a_1a_3 = \frac{2n}{3}(3n(n+3)4^n + 2^{n+1}(n(20n+327)-1361) + 32(4n(n(4n-1)-61)+169)).
\]

Hence, Inequality (3.6) holds for \( j = 0, 1, 2 \). Also, Inequality (3.6) holds for \( j = \lfloor \frac{n}{2} \rfloor - 1 \), since for \( n = 2j + 2 \), we have (by Theorem 3.1)

\[
(4.7) \quad a_j^2 - a_{j-1}a_{j+1} = \frac{1}{15}4^{2j-5}(61440 + 322 + j(1+j)^2(5120 + j(2+j)(817 + 21j(2+j))) + \frac{(1+j)^2}{15}4^{3j-5}(15360 + j(2+j)(19514 + 3j(2+j)(1425 + 83j(2+j)))) \geq 0
\]

and for \( n = 2j + 1 \), we have (also by Theorem 3.1)

\[
(4.8) \quad a_j^2 - a_{j-1}a_{j+1} = \frac{2^{5j-12}(1+j)(2+j)(3+2j)}{4725}(-480j(3+j)(-172 + 127j(3+j)) + 2^3(3+2j)(1075200 + j(3+j)(1060880 + j(3+j)(181982 + 11291j(3+j)))) \geq 0
\]

Step 2. We finish by proving that (3.6) holds for all \( 3 \leq j \leq \lfloor \frac{n}{2} \rfloor - 2 \). Toward this objective, we write

\[
a_j^2 - a_{j-1}a_{j+1} = \frac{n^2(n-1-j)![(n-j-2)!]}{(n-2j+4)!(n-2j+2)!j!(j+1)!} \cdot \frac{\ell}{\ell'}
\]

with

\[
\ell = \frac{(k_1 2^{3j-3} + j(n-j)2^{n+j-1})(n-2j+3)(n-2j+4)(j+1)(n-j-1) + k_2 2^{3j} + k_3 2^{n+j}}{j(k_2 2^{3j} + k_3 2^{n+j})(k_4 2^{3j-6} + k_5 2^{n+j-2})(n-j)(n-2j+2)(n-2j+1)}
\]

Also, in Theorem 3.1, we have

\[
a_j = \frac{1}{4}(2^{2j-1}((3n(n+3)4^n + 2^{n+1}(n(20n+327)-1361) + 32(4n(n(4n-1)-61)+169)))^{1/2} - 2^{2j-1}((3n(n+3)4^n + 2^{n+1}(n(20n+327)-1361) + 32(4n(n(4n-1)-61)+169)))^{1/2}
\]

and since

\[
a_j^2 - a_{j-1}a_{j+1} = \frac{1}{15}4^{2j-5}(61440 + 322 + j(1+j)^2(5120 + j(2+j)(817 + 21j(2+j))) + \frac{(1+j)^2}{15}4^{3j-5}(15360 + j(2+j)(19514 + 3j(2+j)(1425 + 83j(2+j)))) \geq 0
\]
where
\[
\begin{align*}
k_1 &= 8n^2 - 33nj + 24n + 33j^2 - 48j + 16, & k_2 &= 8n^2 - 33jn + 33j^2 - 9n + 18j + 1, \\
k_3 &= nj - j^2 - 2j + n - 1, & k_4 &= 97 + 33j^2 - 114j - 33nj + 57n + 8n^2, \\
k_5 &= nj - n - j^2 + 2j - 1. &
\end{align*}
\]

We now define
\[
L = \frac{\ell - \ell'}{26j}.
\]

In order to prove (3.6), it is sufficient to show that \(L \geq 0\). Define \(q = 2^z\) with \(z = n - 2j\). Direct calculations give

\[
L = L_0 + L_1q + L_2q^2,
\]

where \(L_0\), \(L_1\) and \(L_2\) are polynomials in \(n\) and \(j\) defined by the following equations:

\[
64L_0 = 2(j - 1)(j + 1)(5j^4 - 95j^2 + 1536) + (12765j^2 + 3262j - 400j^3 - 405j^4 + 30j^5 + 4j^6 - 7936)z \\
+ (12j^5 - 810j^3 + 12765j - 250j^4 + 16985j^2 - 4608)z^2 \\
+ (17185j + 9630j^2 - 550j^3 - 51j^4 + 4416)z^3 + (6912 + 10035j + 2313j^2 - 122j^3)z^4 \\
+ (171j^2 + 3456 + 2593j)z^5 + (240j + 768)z^6 + 64z^7,
\]

\[
\frac{-64}{j(n - j)}L_1 = 2(j^2 - 1)(31j^2 - 655) + (j + 1)(5j^3 + 119j^2 - 3033j + 1661)z \\
+ (10j^3 - 2914j - 439 - 9j^4 - 1944j^2)z^2 + (-1477 - 18j^3 - 291j^2 - 2006j)z^3 \\
+ (63j^2 - 871 - 296j)z^4 + (72j - 184)z^5,
\]

\[
4L_2 = (j + 1)(n - j)(n - j - 1)(2 + 10j^2 + (5 + 4j^2 + 10j)z + 4(j + 1)z^2 + z^3).
\]

In order to show that \(L \geq 0\), we prove these two inequalities:

\[
L \geq L_0 + L_1q + L_2z^2 = L_0 + (L_1 + L_2z^2)q,
\]

\[
L_0 + (L_1 + L_2z^2)q \geq L_0 + (L_1 + L_2z^2)z^2 \geq 0,
\]

whenever \(3 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 2\).

First, we show (4.10). Since \(4^t > 2^tt^2\) for \(t \geq 4\), and since \(L_2 > 0\) whenever \(3 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 2\), we infer that

\[
L \geq L_0 + (L_1 + L_2z^2)q \geq 0,
\]

which gives (4.10).
Second, we show (4.11). We begin by expanding the polynomial $64(L_1 + L_2 z^2)$ around the point $n = 2j + 2$ as
\[
110652 j^2 + 62136 j + 88512 j^3 + 33288 j^4 + 6696 j^5 + 1116 j^6 \\
+ (223363 j^2 + 165330 j + 141344 j^3 + 49822 j^4 + 11982 j^5 + 1439 j^6)(z - 2) \\
+ (178549 j^2 + 172469 j + 96048 j^3 + 34586 j^4 + 7635 j^5 + 553 j^6)(z - 2)^2 \\
+ (75102 j^2 + 93999 j + 37806 j^3 + 13281 j^4 + 2043 j^5 + 64 j^6)(z - 2)^3 \\
+ (18605 j^2 + 28699 j + 9509 j^3 + 264 j^4 + 192 j^5)(z - 2)^4 \\
+ (2912 j^2 + 1433 j^3 + 208 j^4 + 5015 j)(z - 2)^5 \\
+ (456 j + 96 j^3 + 296 j^5)(z - 2)^6 + (16 j^2 + 16 j)(z - 2)^7,
\]
which shows that $L_1 + L_2 z^2 \geq 0$. From the fact that $2^t > t^2$ for $t \geq 4$, we obtain the first part of (4.11), viz., $L \geq L_0 + (L_1 + L_2 z^2)z^2$. Hence, it remains to show $L_0 + (L_1 + L_2 z^2)z^2 \geq 0$. To do that, we expand the polynomial $64L_0 + 64(L_1 + L_2 z^2)z^2$ around the point $z = 1$, that is, at $n = 2j + 1$, as
\[
180 j(j+1)(j^2 + j + 16)^2 \\
+ 3(j+1)(287 j^5 + 754 j^4 + 7351 j^3 + 10724 j^2 + 32352 j + 15360)(z - 1) \\
+ (97536 + 168185 j^2 + 191368 j + 99622 j^3 + 37246 j^4 + 7314 j^5 + 1577 j^6)(z - 1)^2 \\
+ (84224 + 154853 j^2 + 158132 j + 95949 j^3 + 32741 j^4 + 8664 j^5 + 1311 j^6)(z - 1)^3 \\
+ (37952 + 99863 j^2 + 97710 j + 54816 j^3 + 19702 j^4 + 5400 j^5 + 489 j^6)(z - 1)^4 \\
+ (9408 + 41942 j^2 + 46649 j + 20605 j^3 + 8203 j^4 + 1659 j^5 + 64 j^6)(z - 1)^5 \\
+ (1216 + 11053 j^2 + 15718 j + 5594 j^3 + 2019 j^4 + 192 j^5)(z - 1)^6 \\
+ (1049 j^3 + 1856 j^2 + 208 j^4 + 3319 j + 64)(z - 1)^7 + 8 j(27 j + 12 j^2 + 47)(z - 1)^8 \\
+ 16 j(j+1)(z - 1)^9.
\]
This establishes that $L_0 + (L_1 + L_2 z^2)z^2 \geq 0$ for all $3 \leq j \leq \lceil \frac{n}{2} \rceil - 2$.

By combining Inequalities (3.6), (4.10), and (4.11), we complete the proof. \hfill \Box

College of Mathematics and Econometrics, Hunan University, 410082 Changsha, China

E-mail address: ycchen@hnu.edu.cn

Department of Computer Science, Columbia University, New York, NY 10027 USA

E-mail address: gross@cs.columbia.edu

Department of Mathematics, University of Haifa, 31905 Haifa, Israel

E-mail address: toufik@math.haifa.ac.il