VALENCE-PARTITIONED GENUS POLYNOMIALS AND THEIR APPLICATION TO GENERALIZED DIPOLES

JONATHAN L. GROSS, TOUFIK MANSOUR, AND

THOMAS W. TUCKER

To the memory of Dan Archdeacon, our friend and colleague.

Abstract. Calculations of *genus polynomials* are given for three kinds of dipoles: with no loops; with a loop at one vertex; or with a loop at both vertices. We include a very concise, elementary derivation of the genus polynomial of a loopless dipole. To describe the general effect on the face-count and genus polynomials of the operation of adding a loop at a vertex, we introduce imbedding types that are partitions of integers, specifically, partitions of the valences of the vertices at which loops are to be added. Adding a loop at a root-vertex changes the possible number of imbedding types from the number of partitions of the valence prior to adding the loop to the number of partitions of the valence afterward.

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1. Introduction

Given a finite, connected graph G, possibly with multiple edges and loops, its ***genus polynomial***

$$\Gamma_G(z) = \sum_{g=\gamma_{\min}(G)}^{\gamma_{\max}(G)} a_g z^g$$

is the generating function for the number a_g of cellular imbeddings of G in the oriented surface S_g of genus g. Not much is known about the coefficients of genus polynomials, except that they are non-negative, and

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that between the first nonzero coefficient and the last, there are no zeros. (This "interpolation theorem" has an easy topological explanation [GT01].) There are closed formulas for the genus polynomials of bouquets of circles (one-vertex graphs) [GRT89] and dipoles (two-vertex graphs having no loops) [Rie87, KL93]. There are also recursions or closed formulas for a small number of "linear" and "ring-like" families of graphs, and a few others. Some relatively recent additions to the list of known genus polynomials are [CGM12, Gr11a, Gr11b, Gr13, GKP10, GKP14, GMTW14, KPG12, PKG11, PKG14].

To this day, the genus polynomial for the complete graphs K_n with $n \geq 8$ are not known; this polynomial lies beyond present capacity for brute-force calculation, since K_n has $[(n-2)!]^n$ possible embeddings. It has been conjectured [GRT89] that the sequence of coefficients of any genus polynomial is log-concave; this conjecture, called the ***LCGD conjecture***, has been confirmed for bouquets, dipoles, and certain linear families [CGM15, GMTW15, GMTW16].

In this paper, we consider a graph imbedding locally, from the perspective of a single root-vertex, and we study how the imbedding changes as we add a proper edge or a loop. This leads us to a reconsideration of dipoles. It also leads us to a general formula for adding a loop at a vertex, which entails a detailed analysis of the partition of the set of edges incident to that vertex, with parts corresponding to faces incident to that vertex. In particular, we generalize the notion of a dipole to allow a loop at one or both vertices.

Given a graph G with vertex set V and edge set E, we assign each edge e two directions, e^+ and e^-, to give a set D of directed edges. Let λ be the involution on D that interchanges the direction of each edge. An imbedding of G in an oriented surface defines a permutation ρ with $|V|$ cycles, each of which specifies the cyclic order of the directed edges beginning at the corresponding vertex, as determined by the orientation of the surface. The permutation ρ is called the ***rotation system*** for the imbedding, and the cycle it assigns at a given vertex v is called the ***rotation*** at that vertex.

The boundary walks of the oriented faces of the imbedding are then given by the cycles of the permutation $\lambda\rho$, viewed as cyclic lists of directed edges treated as incoming to a vertex. We begin with an incoming edge at vertex u, reverse its direction (i.e., apply λ) so that it is outgoing at u, rotate by ρ to the next outgoing edge from u, follow that directed edge to its end vertex v where it is incoming; and then repeat to trace out the directed face boundary. Ringel pioneered the use of rotation systems in the 1950s to study minimum-genus graph imbedding of families of complete graphs, which led to the solution [RY68] to the Heawood map-color problem.

The vertex-edge-face incidence structures of the imbeddings of a graph are in bijective correspondence with possible rotation systems for the graph. The genus polynomial $\Gamma_G(z)$ can therefore be obtained from the cycle-count generating function for $\lambda\rho$, where ρ varies over all rotations for the graph G: if there are c cycles, then $g = (|E| - |V| - c)/2$ by the Euler polyhedral formula, since c is the number of faces.

> Thus, any question regarding enumeration by genus of the imbeddings of a given graph G can be interpreted as a question about the enumeration of cycle-counts in a product of a fixed (free) involution on a set D with a set of permutations of D, where the set is determined by the adjacencies in G.

The organization of this paper is as follows. Section 2 describes *inner and outer permutations* at a root vertex, introduced in [GMTW16]. Section 3 transforms the problem of counting faces in an imbedding of a dipole D_n to counting cycles in a product of two arbitrary n-cycles. Section 4 describes the relationship of the Hultman numbers to the face-count polynomial for a digraph and revisits the genus polynomial of D_n [Rie87, KL93] from this viewpoint. Section 5 introduces *valence-partitioned genus polynomials* for a graph G rooted at a vertex v and reviews the representation of topological operations by *productions*. Section 6 gives explicit formulae for valence-partitioned productions for adding a proper edge to a dipole, a loop at the non-root vertex of a dipole, and a loop at the root of an arbitrary graph G, which lead to calculation of genus polynomials for generalized dipoles. Section 7 suggests a possible approach to the LCGD conjecture using [GMTW15], an observation by Féray [Fe15] about a result by Stanley [St11], and ideas from this paper.

2. Inner and Outer Permutations at a Vertex

We abbreviate "face-boundary walk" as ***fb-walk***.

Relative to a given vertex v and to a given rotation system ρ, or to the corresponding imbedding, we call the faces and fb-walks that are incident at v ***inner faces*** and ***inner fb-walks***, and we call the other faces and fb-walks ***outer faces*** and ***outer fb-walks***.

A ***semi-trail*** in a graph is an oriented walk in which no oriented edge appears more than once.

For any imbedding of a graph G, each inner fb-walk at a vertex v can be visualized as having been formed by the union of two kinds of oriented semi-trails:

- a ***corner strand at*** v is a semi-trail (of length 2) comprising a directed edge inward to v and the directed edge outward from v that follows the inward edge immediately on whatever inner fb-walk contains the inward edge;
- an ***outer strand at*** v is any semi-trail of an inner fb-walk that remains after deleting all the corner strands at v from all the fb-walks.

We now define two permutations, ζ and π, on the *undirected edges* incident at v. Each loop at v is subdivided by adding a 2-valent vertex. Thus, no edge is twice incident at v. The ***inner permutation*** ζ at vertex v is simply the rotation at v with directions ignored. The permutation ζ at v takes the first edge (incoming, but undirected) in a corner strand to the next edge (outgoing, but undirected). The ***outer permutation*** π at v takes the undirected edge (but outgoing from v) at the beginning of an outer strand to the undirected edge (but incoming at v) at the end of that outer strand. The composition $\zeta\pi$ takes an undirected (but incoming) edge at v to the next undirected (but incoming) edge at v in a fb-walk. Thus, each cycle of $\zeta\pi$ gives the cyclic order of the undirected (but incoming) edges incident at v that are encountered in the traversal of that fb-walk.

Accordingly, the outer permutation π is obtained from the cycles of $\lambda\rho$ by deleting all directed edges except those incoming to v, and then ignoring the directions of those edges. We observe that if we change the rotation only at v, but leave the rotation the same at all other vertices, then the outer permutation π remains unchanged.

This can be viewed solely in terms of permutation as follows. Let ρ_v be rotation at v, and let ρ' be all the other cycles of ρ, so that $\rho = \rho_v\rho'$. If we rewrite $\lambda\rho$ as $(\lambda\rho_v\lambda)(\lambda\rho')$, then the first factor $(\lambda\rho_v\lambda)$ is just the rotation at v, changing the sign of each directed edge from outgoing to incoming. For the outer permutation, we take the second factor $(\lambda\rho')$ and delete all directed edges except those incoming at v. Thus both $\lambda\rho_v\lambda$ and $\lambda\rho'$ are permutations only on the incoming edges at v.

See Example 2.2 of [GMTW16] for an illustration of inner and outer strands and Example 2.3 for an illustration of inner and outer permutations. We can summarize this discussion as follows:

Proposition 2.1 (Proposition 3.1 of [GMTW16]). *Let ζ and π be the inner and outer permutations at a vertex v of an imbedded graph G with rotation ρ. Then each cycle of the composition $\zeta\pi$ is the list of the edges incident at v that occur on an inner fb-walk, that is, on an fb-walk incident at v. In particular, the number of cycles of $\zeta\pi$ is the number of the inner faces at v of the imbedding. If we change the rotation ρ only at v, then π stays the same.*

3. Counting Faces of a Dipole Rotation System

For the case of a loopless dipole D_n with vertices u and v, imbedded in an oriented surface, the outer permutation at v is simply the rotation at u with directions ignored. Thus we make the following observation.

Proposition 3.1. *Let ρ be the rotation for an imbedded dipole D_n with vertices v and u, and let ζ and π, respectively, be the rotations at v and u with signs ignored. Then each cycle of $\zeta\pi$ gives the incoming edges at v, with signs ignored, in a fb-walk of the imbedding.* □

Example 3.1. We consider the dipole D_4 with vertices u and v and edges 1, 2, 3, and 4. The directed edges incoming edges to vertex v are 1^+, 2^+, 3^+, and 4^+; the directed edges incoming to v are 1^-, 2^-, 3^-, and 4^-. We assign rotations $(1^+\,3^+\,4^+\,2^+)$ at u and $(1^-\,2^-\,3^-\,4^-)$ at v. Figure 3.1 illustrates the projection of this rotation system for the dipole D_4 and the two fb-walks that are obtained by face-tracing.

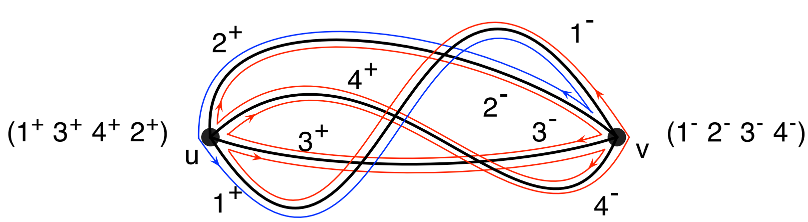

Figure 3.1. Projection of a rotation system for the dipole D_4.

We now compute (where $\lambda = (1^+\,1^-)(2^+\,2^-)(3^+\,3^-)(4^+\,4^-)$)

$$\begin{aligned}\lambda\rho &= \lambda(1^-\,2^-\,3^-\,4^-)(1^+\,3^+\,4^+\,2^+)\\ &= (1^+\,2^-)(3^+\,4^-\,2^+\,3^-\,4^+\,1^-).\end{aligned}$$

The first cycle, $(1^+\,2^-)$, is the fb-walk for the blue face and the second cycle is the fb-walk for the red face. The directed edges incoming to v

are 1^+, 2^+, 3^+, and 4^+. If we keep track only of those four directed edges in $\lambda\rho$, we have $(1^+)(3^+\ 2^+\ 4^+)$; ignoring signs we get $(1)(3\ 2\ 4)$. The inner permutation ζ at v is just the rotation at v with directions ignored, $(1\ 2\ 3\ 4)$. The outer permutation π at v is just the rotation at u with directions ignored, $(1\ 3\ 4\ 2)$. Then:

$$\zeta\pi \ = \ (1\ 2\ 3\ 4)(1\ 3\ 4\ 2) \ = \ (1)(3\ 2\ 4).$$

We note that if we had kept track in $\lambda\rho$ of outgoing edges at v (those labeled $-$), instead of incoming edges, we would get a different permutation, $(2)(4\ 3\ 1)$, which is $\pi\zeta$ instead of $\zeta\pi$.

We denote the set of cyclic permutations of n objects by $\mathcal{Q}_n$, and we infer the following from Proposition 3.1.

Corollary 3.2. *The face-count distribution polynomial for the dipole D_n equals the cycle-count distribution polynomial taken over all products of two permutations in $\mathcal{Q}_n$.* □

Remark For the time being, we are still envisioning distinct polynomials for enumerating face-counts and the genus of imbeddings. We combine the two in Section 5.

The following theorem is the fruit of our new perspective on dipole imbeddings.

Theorem 3.3. *The face-count distribution polynomial for the dipole D_n equals $(n-1)!$ times the cycle-count distribution polynomial taken over all products of a fixed $\pi \in \mathcal{Q}_n$ with each possible $\zeta \in \mathcal{Q}_n$.*

Proof. We envision an $(n-1)! \times (n-1)!$ array whose rows are labeled by the possible inner permutations ζ at v and whose columns are labeled by the outer permutations π at v. The entries of the matrix are just the number of cycles in the product $\zeta\pi$. We claim that the cycle-count distribution for columns π and π' are the same. From the graph theoretic viewpoint, we can change from π to π' by a graph isomorphism relabeling edges. Since the entries in each column give all possible cycle counts over all possible ζ the cycle-count distribution remains unchanged. From a permutation viewpoint, $\pi' = \sigma^{-1}\pi\sigma$. Thus for any cycle ζ, we have

$$\zeta\pi' \ = \ \zeta(\sigma^{-1}\pi\sigma) \ = \ \sigma^{-1}(\sigma\zeta\sigma^{-1})(\pi)\sigma.$$

Since conjugate permutations have the same cycle count, we infer that $\zeta\pi'$ has the same cycle count as $(\sigma\zeta\sigma^{-1})\pi$. The row labels are all possible cycles ζ, which is the same as all conjugates of a fixed cycle ζ. Thus the cycle-count distribution for each column is the same. □

4. Hultman Numbers

In this section, we give a very short calculation of the genus polynomials for the loopless dipoles.

Bona and Flynn [BF09] give the closed formula

$$H_{n,k} \;=\; \begin{cases} \left[{n+2 \atop k}\right] / \binom{n+2}{2} & \text{if } n-k \text{ is odd} \\ 0 & \text{if } n-k \text{ is even} \end{cases}$$

for what has been called the ***Hultman number*** $H_{n,k}$. Hultman [H99] conceived of them while studying enumerative problems concerned with distances between genomes. Here we use the Karamata notation $\left[{n \atop k}\right]$ (for which we say ***"n-cycle-k"***) for the unsigned Stirling number of the first kind. For the first few values of n, the Hultman numbers are shown in the following array:

	1	2	3	4	5	6	7
1	0	1					
2	1	0	1				
3	0	5	0	1			
4	8	0	15	0	1		
5	0	84	0	35	0	1	

Topological graph theorists may observe that the numbers $H_{n,k}$ appear quite explicitly in the work of Kwak and Lee [KL93], some years before Hultman [H99]. Accordingly, here we call them the ***HKL numbers***.

Theorem 4.1. *The face-count distribution sequence for the product of a fixed permutation in $\mathcal{Q}_n$ with all the permutations in $\mathcal{Q}_n$ is the HKL sequence $H_{n-1,k}$ for $k \in \mathbb{Z}^+$.*

Proof. This follows immediately from Corollary 1 of [BF09], which is based on work of Doignon and Labarre [DL07]. The polynomial itself was first obtained by Rieper [Rie87]. □

Corollary 4.2. *The face-count polynomial for the set of imbeddings of the dipole D_n equals $(n-1)!$ times the generating function for the HKL sequence*

$$H_{n-1,1},\; H_{n-1,2},\; \ldots,\; H_{n-1,n}.$$

Proof. This follows from Theorem 3.3 and Theorem 4.1. □

Corollary 4.3. *The genus polynomial of the dipole D_n is given by*

$$\Gamma_{D_n}(z) \;=\; \frac{2(n-1)!}{n(n+1)} \sum_{k=0+1}^{\lfloor \frac{n-1}{2} \rfloor} \begin{bmatrix} n+1 \\ n-2k \end{bmatrix} z^k. \qquad \square$$

5. Valence-Partitioning and Productions

Consider an imbedding of a graph G with a d-valent root-vertex v. The inner faces at v give a partition of its set of incoming edges, which induces a partition of the valence d, called the ***face-incidence partition*** of d. For different imbeddings of G, we may get different partitions of the valence d. The collection of partitions of d arising from all imbeddings of G forms a complete set of ***imbedding types*** for (G, v), in the sense of [GKMT15]. The ***partial genus polynomial*** for any of these partitions is the genus polynomial for the set of imbeddings within the corresponding imbedding type.

We use the notation $1^{i_1}2^{i_2}\ldots$ for the partition having i_1 parts of size 1, i_2 parts of size 2 etc. To avoid subscripts, we write the partial genus polynomial $f(z)$ corresponding to the partition $1^{i_1}2^{i_2}...$ as $f(z){\cdot}1^{i_1}2^{i_2}...$. We call this ***valence-partitioned notation***. The following example illustrates how this notation works.

Example 5.1. The complete graph (K_4, v), rooted at any vertex v, has 16 imbeddings. Of these, there are two imbeddings of genus 0 in which three distinct faces are incident at v; and there are 12 imbeddings of genus 1 in which one face is twice incident at v and another once incident; there are two imbeddings of genus 1 in which one face is thrice incident at v. We then can write the genus polynomial for K_4 rooted at v in valence-partitioned notation as:

$$\Gamma_{(K_4,v)}(z) \;=\; 2z^0{\cdot}1^3 + 12z^1{\cdot}1^1 2^1 + 2z^1{\cdot}3^1. \tag{5.1}$$

We call such a polynomial a ***valence-partitioned genus polynomial*** of G for root-vertex v.

A ***production*** is a rule that describes the effect that a given topological operation on a graph, such as the addition of one or more edges, has on an imbedding type. A ***valence-partitioned production*** is one where the imbedding types are partitions at a root vertex v. To illustrate our notation, we extend Example 5.1.

Example 5.1. (continued) We consider how each of three valence partitions for (K_4, v) is affected by addition of a loop at v, by which we obtain the graph (G, v). In the following three productions, a counter

z^g keeps track of the effect of the loop addition on genus (either $g \to g$, or $g \to g{+}1$)) of the imbeddings in the resulting new partitions.

$$z^g{\cdot}1^3 \;\to\; 6z^g{\cdot}1^3 2^1 + 6z^{g+1}{\cdot}1^1 4^1 \tag{5.2}$$

$$z^g{\cdot}1^1 2^1 \;\to\; 4z^g{\cdot}1^2 3^1 + 4z^g{\cdot}1^1 2^2 + 4z^{g+1}{\cdot}5^1 \tag{5.3}$$

$$z^g{\cdot}3^1 \;\to\; 6z^g{\cdot}1^1 4^1 + 6z^g{\cdot}2^1 3^1 \tag{5.4}$$

For instance, Production (5.2) means that a loop can be added at a root-vertex of a graph whose imbedding is of type 1^3 in 12 possible ways. There are three corners at v into which both ends of the loop can be inserted, and there are two possible orientations of the loop, leading to six imbeddings of type $1^3 2^1$, all of the same genus as of the original imbedding. There are also six ways (counting the two possible orientations) for inserting the edge-ends of the loop into two different corners at v, each of which reduces the total number of faces by one and increases the genus by one, and results in an imbedding of type $1^1 4^1$.

We then extend these rules linearly to the valence-partitioned genus polynomial (5.1) for K_4, with appropriate values of the exponent g, to get the valence-partitioned genus polynomial for G:

$$\begin{aligned}\Gamma_{(G,v)}(z_\pi) \;=\; & 12z^0{\cdot}1^3 2^1 \;+\; 48z^1{\cdot}1^2 3^1 \;+\; 48z^1{\cdot}1^1 2^2 \\ & + 24z^1{\cdot}1^1 4^2 \;+\; 12z^1{\cdot}2^1 3^1 \;+\; 48z^2{\cdot}5^1\end{aligned} \tag{5.5}$$

Suppressing the partitioning, we get the genus polynomial for G:

$$\Gamma_G(z) = 12 + 132z + 48z^2$$

We may recognize the three partitions of 3 as imbedding types of (K_4, v) and the seven partitions of 5 as imbedding types of (G, v), and we may represent the production system comprising (5.2), (5.3), and (5.4) as a *non-square* production matrix. This leads to the following matrix equation:

$$\begin{bmatrix} 0 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 4 & 0 \\ 6z & 0 & 6 \\ 0 & 0 & 6 \\ 0 & 4z & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 12z \\ 2z \end{bmatrix} \begin{matrix} 1^3 \\ 1^1 2^1 \\ 3^1 \end{matrix} \;=\; \begin{bmatrix} 0 \\ 12 \\ 48z \\ 48z \\ 24z \\ 12z \\ 48z^2 \end{bmatrix} \begin{matrix} 1^5 \\ 1^3 2^1 \\ 1^2 3^1 \\ 1^1 2^2 \\ 1^1 4^1 \\ 2^1 3^1 \\ 5^1 \end{matrix} \tag{5.6}$$

The columns of partitions beside the corresponding partitioned genus distribution vectors serve as an aid to reconciling this matrix equation with the partitions of the valences of the root-vertices.

6. Valence-Partitioned Productions for Adding an Edge to a Dipole

As an application of the use of valence-partitioned production, we now examine the valence-partitioned productions obtained when we add an edge to a rooted dipole. We consider three cases: adding a proper edge, adding a loop at the non-root vertex, and adding a loop at the root vertex. We first obtain a valence-partitioned genus polynomial for the loopless dipole. Next we add a loop at one vertex, while keeping track of the effect on the valence-partitioned genus polynomial at the second vertex. Then we add a loop at the second vertex.

Valence-Partitions for a Loopless Dipole

In a loopless dipole imbedding, occurrences of the two vertices alternate on every fb-walk and the two vertices have the same valence-partition. Accordingly, that partition gives the global face-count and the distribution of face-sizes as well as the genus. We have the following initial valence-partitioned genus polynomial:

$$\Gamma_{D_1}(z_\pi) \;=\; z^0 \cdot 1^1. \tag{6.1}$$

This means that there is only one imbedding of D_1, in which the only face is 2-sided, and in which the only imbedding is on the sphere.

In general, the number of parts is the number of faces. The minimum number of parts for D_n is 1, if n is odd, or 2, if n is even. The maximum number of parts is n if n is odd and $n-1$ if n is even. The sum of the parts is n. We easily calculate (by ad hoc methods) that

$$\Gamma_{D_2}(z_\pi) \;=\; z^0 \cdot 1^2 \quad \text{and} \tag{6.2}$$

$$\Gamma_{D_3}(z_\pi) \;=\; 2z^0 \cdot 1^3 + 2z^1 \cdot 3^1. \tag{6.3}$$

Inserting One More Proper Edge in a Dipole

When a proper edge is added to a dipole imbedding, it can either split a face into two faces or it can join two faces. As a preliminary to giving a general production, we consider this example that illustrates the possible outcomes. For a loopless dipole, the size of each face is twice the number of occurrences of the root-vertex.

Remark Of course. we already know the genus polynomial for a loopless dipole D_n. However, we need a valence-partitioned genus polynomial for D_n in order to derive genus polynomials for the generalized dipoles with a possible loop at either or both vertices.

Example 6.1. For an imbedding $(D_7, v) \to S_2$ with two digons and a 10-gon, we have the particularized production

$$(6.4)\ z^2 \cdot 1^2 5^1 \ \to\ 2z^2 \cdot 1^3 5^1 + 10z^2 \cdot 1^3 5^1 + 10z^2 \cdot 1^2 2^1 4^1 + 5z^2 \cdot 1^2 3^2 + 2z^3 \cdot 3^1 5^1 + 20z^3 \cdot 1^1 7^1.$$

We call the non-root-vertex u. The first line of Production (6.4) counts the ways to insert a new u-v edge inside either of the two digons or inside the 10-gon. The terms $2 \cdot 1^3 5^1$ and $10 \cdot 1^3 5^1$ count the ways to insert the additional edge along a side of a face, as in Figure 6.1, so as to create an additional digon. The terms $10z^2 \cdot 1^2 2^1 4^1$ and $5z^2 \cdot 1^2 3^2$ count the ways to insert a u-v chord so as to split the 10-gon into a digon and a 4-gon or into two hexagons, respectively.

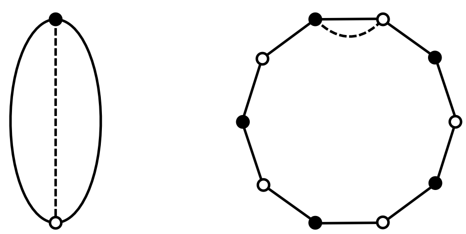

Figure 6.1. Adding a digon along a side of a face.

The term $2z^3 \cdot 3^1 5^1$ counts the ways to join the two digons into a hexagon.

The term $20z^3 \cdot 1^1 7^1$ counts ways to insert the additional edge so that it joins one of the two digons to the 10-gon, which is 20 ways when both digons are considered.

Note that the sum of the coefficients on the right side of the production is 49, since there are seven different corners at each vertex of any imbedding of D_7, which implies there are 7^2 places to insert the ends of the new proper edge.

Notation We will sometimes modify our notation for partitions (e.g., in (6.5)) to allow separate parts of equal size or have parts not in the order of ascending size. Furthermore, we allow a negative exponent on a part to show that a part of that size has been removed from the partition. These notational conventions considerable simplify the formulas for the three main theorems regarding productions.

Theorem 6.1. *The valence-partitioned production for adding a proper edge to an imbedding of a loopless digon D_n with vertices u and v is*

$$z^g\cdot 1^{j_1}2^{j_2}\cdots n^{j_n} \to \sum_{i=1}^{n} ij_iz^g\cdot\sum_{k=1}^{i} i^{-1}k^1(i-k+1)^1 1^{j_1}2^{j_2}\cdots n^{j_n} \tag{6.5}$$

$$+ \sum_{i=1}^{n} 2i^2\binom{j_i}{2}z^{g+1}\cdot i^{-2}(2i+1)^1 1^{j_1}2^{j_2}\cdots n^{j_n} \tag{6.6}$$

$$+ \sum_{i=1}^{n}\sum_{k=i+1}^{n} 2ikj_ij_kz^{g+1}\cdot i^{-1}k^{-1}(i+k+1)^1 1^{j_1}2^{j_2}\cdots n^{j_n}. \tag{6.7}$$

Proof. This production follows the same lines as the the particularized discussion of Example 6.1. Let's suppose that vertex u is white and vertex v is black. Then these two colors alternate around each fb-walk. Line (6.5) enumerates ways to insert the additional edge as a chord inside a $(2i)$-gon, so as to join a pair of corners of opposite colors, taken over all possible face-sizes 2, 4, ..., $2n$. We observe that the sum of the coefficients of this first double sum is $\sum_{i=1}^{n} j_i i^2$, which corresponds to the total number of ways to insert such a chord. Line (6.6) enumerates ways to insert the additional edge so that it joins two faces of the same size. Line (6.7) counts the ways to insert the additional edge so that it joins two faces of different sizes. We observe that the sum of all the coefficients in the consequent is

$$\sum_{i=1}^{n} j_i(i^2+i) + \sum_{i=1}^{n}\binom{j_i}{2}2i^2 + \sum_{i=1}^{n}\sum_{k=i+1}^{n} j_ij_k2ik$$
$$= \sum_{i=1}^{n} j_i^2 i^2 + \sum_{i=1}^{n}\sum_{k=i+1}^{n} j_ij_k2ik \;=\; \left(\sum_{i=1} ij_i\right)^2 \;=\; n^2.$$

This is exactly the factor by which we anticipate that the sum of the coefficients of the genus polynomial will increase, when we insert an additional proper edge, because the valences at the two vertices of the dipole D_n are both n. $\square$

Using Theorem 6.1, we can now calculate the productions

$$z^0\cdot 1^3 \;\to\; 3z^0\cdot 1^4 + 6z^1\cdot 1^1 3^1$$
$$z^1\cdot 3^1 \;\to\; 3z^1\cdot 2^2 + 6z^1\cdot 1^1 3^1$$

and apply them to the valence-partitioned genus polynomial (from (6.3))

$$\Gamma_{D_3}(z_\pi) \;=\; 2z^0{\cdot}1^3 + 2z^1{\cdot}3^1,$$

which yields the valence-partitioned genus polynomial

$$\Gamma_{D_4}(z_\pi) \;=\; 6z^0{\cdot}1^4 + 24z^1{\cdot}1^1 3^1 + 6z^1{\cdot}2^2. \tag{6.8}$$

To continue one step further to a valence-partitioned genus polynomial for the dipole D_5, we calculate the productions

$$\begin{aligned} z^0{\cdot}1^4 &\;\rightarrow\; 4z^0{\cdot}1^5 + 12z^1{\cdot}1^2 3^1 \\ z^1{\cdot}1^1 3^1 &\;\rightarrow\; 7z^1{\cdot}1^2 3^1 + 3z^1{\cdot}1^1 2^2 + 6z^2{\cdot}5^1 \\ z^1{\cdot}2^2 &\;\rightarrow\; 8z^1{\cdot}1^1 2^2 + 8z^2{\cdot}5^1 \end{aligned}$$

Applying them to the valence-partitioned genus polynomial (6.8) yields the valence-partitioned genus polynomial

$$\Gamma_{D_5}(z_\pi) \;=\; 24z^0{\cdot}1^5 + 240z^1{\cdot}1^2 3^1 + 120z^1{\cdot}1^1 2^2 + 192z^2{\cdot}5^1. \tag{6.9}$$

Adding a Loop at the Non-Root Vertex of a Dipole

Our next task is to develop productions for the effect on the valence-partition at the second vertex of adding a loop at the first vertex. For purposes of exposition, it seems helpful to begin with an example.

Example 6.2. We consider the rooted loopless dipole (D_3, v). For adding a loop at the other vertex, u, the productions for the effect on the valence-partitioned genus polynomial at root-vertex v are as follows:

$$\begin{aligned} z^0{\cdot}1^3 &\;\rightarrow\; 6z^0{\cdot}1^3 + 6z^1{\cdot}1^1 2^1 \\ z^1{\cdot}3^1 &\;\rightarrow\; 6z^1{\cdot}3^1 + 6z^1{\cdot}1^1 2^1. \end{aligned}$$

Production (6.10) holds because there are six ways to insert a loop in one of the three corners at vertex u, none of which changes the valence partition at v, and there are six ways to insert a loop that joins faces by joining two corners at u, each of which changes the partition at v from 111 to 12. Production (6.10) holds because there are six ways, in all, to insert a loop at one of the three occurrences of u on the fb-walk of a hexagon, and six ways to join two different u-corners. Thus, using (6.3), we obtain

$$\Gamma_{(D_3 * B_1, v)}(z_\pi) \;=\; 12z^0{\cdot}1^3 + 12z^1{\cdot}3^1 + 24z^1{\cdot}1^1 2^1. \tag{6.10}$$

Theorem 6.2. *The valence-partitioned production for adding a loop at the non-root vertex u of a rooted loopless digon (D_n, v) is*

$$z^g\cdot 1^{j_1}2^{j_2}\cdots n^{j_n} \;\to\; \sum_{i=1}^{n} i j_i z^g\cdot\sum_{k=0}^{i} i^{-1}k^1(i-k)^1 1^{j_1}2^{j_2}\cdots n^{j_n} \tag{6.11}$$

$$+\sum_{i=1}^{n} 2i^2\binom{j_i}{2} z^{g+1}\cdot i^{-2}(2i)^1 1^{j_1}2^{j_2}\cdots n^{j_n} \tag{6.12}$$

$$+\sum_{i=1}^{n}\sum_{k=i+1}^{n} 2ikj_ij_kz^{g+1}\cdot i^{-1}k^{-1}(i+k)^1 1^{j_1}2^{j_2}\cdots n^{j_n}. \tag{6.13}$$

Proof. The double sum at line (6.11) counts the number of ways to add a loop with both ends in the same u-corner of a $2i$-gon. A u-u chord in an $2i$-sided face splits it into two faces, one of which has k occurrences of vertex v with the other having $i-k$ occurrences of v. The inner sum takes care of both orientations of any chord. (Parts of size zero in a partition are simply ignored.) The sum of the coefficients for the inner sum is $i+1$. Thus, the sum of the coefficients in the double sum equals $\sum_{i=1}^{n}(i^2+i)j_i = n + \sum_{i=1}^{n} i^2 j_i$.

Line (6.12) counts the ways to join two faces of the same size. Line (6.13) counts the ways to join two faces of unequal size. The inner sum is empty when $k=n$, because if there are n instances of vertex u on a face, then it is the only face.

The sum of the coefficients taken over all three sums in the consequent of the production is

$$\begin{aligned}
&\sum_{i=1}^{n}(i^2+i)j_i \;+\; \sum_{i=1}^{n} i^2(j_i^2-j_i) \;+\; \sum_{i=1}^{n}\sum_{k=i+1}^{n} 2ikj_ij_k \\
=&\; n \;+\; \sum_{i=1}^{n} j_i^2 i^2 + \sum_{i=1}^{n}\sum_{k=i+1}^{n} j_ij_k 2ik \\
=&\; n \;+\; \left(\sum_{i=1} ij_i\right)^2 \;=\; n\;+\;n^2.
\end{aligned}$$

This coincides with our expectation that the factor by which the number of imbeddings increases when a loop is added at one vertex of D_n is n^2+n. □

Example 6.3. Now we add a loop to one vertex of D_4, starting with the valence-partitioned genus polynomial (6.8).

$$\Gamma_{D_4}(z_\pi) \;=\; 6z^0\cdot 1^4 + 24z^1\cdot 1^1 3^1 + 6z^1\cdot 2^2$$

Using Theorem 6.2, we calculate specific productions.

$$\begin{aligned} z^0\cdot 1^4 &\;\to\; 8z^0\cdot 1^4 + 12z^1\cdot 1^2 2^1 \\ z^1\cdot 1^1 3^1 &\;\to\; 8z^1\cdot 1^1 3^1 + 6z^1\cdot 1^2 2 + 6z^2\cdot 4^1 \\ z^1\cdot 2^2 &\;\to\; 8z^1\cdot 2^2 + 4z^1\cdot 1^2 2^1 + 8z^2\cdot 4^1 \end{aligned}$$

We apply these productions to obtain the result

$$\begin{aligned} \Gamma_{(D_4 * B_1, v)}(z_\pi) \;=\;& 48z^0\cdot 1^4 + 240z^1\cdot 1^2 2 + 192z^1\cdot 1^1 3^1 \\ & +48z^1\cdot 2^2 + 192z^2\cdot 4^1. \end{aligned}$$

Example 6.4. As a third example, we add a loop to one vertex of D_5. We begin with the valence-partitioned genus polynomial (6.9).

$$\Gamma_{D_5}(z_\pi) \;=\; 24z^0\cdot 1^5 + 240z^1\cdot 1^2 3^1 + 120z^1\cdot 1^1 2^2 + 192z^2\cdot 5^1$$

As in previous examples, we next construct specific productions.

$$\begin{aligned} z^0\cdot 1^5 &\;\to\; 10z^0\cdot 1^5 + 20z^1\cdot 1^3 2^1 \\ z^1\cdot 1^2 3^1 &\;\to\; 10z^1\cdot 1^2 3^1 + 6z^1\cdot 1^3 2^1 + 2z^2\cdot 2^1 3^1 + 12z^2\cdot 1^1 4^1 \\ z^1\cdot 1^1 2^2 &\;\to\; 10z^1\cdot 1^1 2^2 + 4z^1\cdot 1^3 2^1 + 8z^2\cdot 1^1 4^1 + 8z^2\cdot 2^1 3^1 \\ z^2\cdot 5^1 &\;\to\; 10z^2\cdot 5^1 + 10z^2\cdot 1^1 4^1 + 10z^2\cdot 2^1 3^1 \end{aligned}$$

We apply these productions to obtain the result

$$\begin{aligned} \Gamma_{(D_5 * B_1, v)}(z_\pi) \;=\;& 240z^0\cdot 1^5 + 2400z^1\cdot 1^3 2^1 + \; 2400z^1\cdot 1^2 3^1 \\ & + 1200z^1\cdot 1^1 2^2 + \; 3360z^2\cdot 2^1 3^1 \\ & + 5760z^2\cdot 1^1 4^1 + \; 1920z^2\cdot 5^1. \end{aligned}$$

Adding a Loop at the Root Vertex of an Arbitrary Graph

Our last theorem theorem gives a general production that represents the effect on a valence-partitioned genus polynomial of the operation of adding a loop at the root-vertex of any rooted graph, not just a dipole.

Theorem 6.3. *Let (G, v) be a graph with a n-valent root vertex v, imbedded so that the valence partition at v is*

$$\pi \;=\; 1^{j_1} 2^{j_2} \cdots . n^{j_n}.$$

Then the operation of adding a loop at v has the following production:

$$z^g\cdot\pi \;\to\; \sum_{i=1}^{n} ij_i\cdot\sum_{k=0}^{i} z^g\cdot i^{-1}(k+1)^1(i-k+1)^1\pi \tag{6.14}$$

$$+\sum_{i=1}^{n} ij_i i(j_i-1)z^{g+1}\cdot i^{-2}(2i+2)^1\pi \tag{6.15}$$

$$+\sum_{i=1}^{n} ij_i\cdot\sum_{k=1,k\neq i}^{n} kj_k z^{g+1}\cdot i^{-1}k^{-1}(i+k+2)^1\pi. \tag{6.16}$$

Proof. We recall that the sizes of the parts in our partition correspond to the number of occurrences of the root-vertex on an fb-walk. In this proof, we abuse the phrase "face size", by letting it mean that number of occurrences, rather than the length of the fb-walk.

To add a loop at root-vertex v, we first choose a face corner where the added loop begins. For each of lines (6.14), (6.15),(6.16), the sum on index i counts the number of such possible beginning corners, where i is the size of the face . There are j_i faces of size i at vertex v, and there are i possible corners in each face.

We next choose the corner where the added loop ends. Line (6.14) counts the number of ways to do this when the end corner is in the same face as the beginning corner — that is, when the added loop splits one face into two faces. The index k of the inner sum says how much further this corner is around the face; notice that the sum includes $k = 0$ and $k = i$, so that, if the beginning and end corner are the same, then we count twice, since the loop can be oriented in two ways. The term $i^{-1}(k+1)^1(i-k+1)^1\pi$ means we have removed from the partition π a part of size i and replaced it by parts of size $k+1$ and size $i-k+1$; this corresponds to splitting a face of size i into faces of size $k+1$ and $i-k+1$. (The newly added loop accounts for the increase of two sides to the sum of the face sizes.) Note that the number of faces of the imbedding, and hence the number of parts of the partition, increases by 1, and consequently, the genus counter z^g stays the same, since the number of edges has also increased by 1.

Lines (6.15) and (6.16) count the number of ways where the added loop ends in the corner of a different face. Line (6.15) treats the case where beginning face and the end face have the same size. There are $j_i - 1$ choices for the second face and i possible corners in that face. The new imbedding has two fewer two parts of size i and one additional part of size $2i+2$. Accordingly, the new partition removes two parts of size i

and adds one of size $2i+2$. The genus increases to z^{g+1}, since the number of faces decreases by 1, while the number of edges increases by 1, which together lead to a total decrease of 2 in the value of the Euler formula for the imbedding. Line (6.16) treats the case where the end face has a different size k. There are j_k choices for the end face and k possible corners in that face. The partition removes one part of size i and one of size k and adds one of size $i+k+2$. The genus counter again increases to z^{g+1}.

In line (6.14), the inner sum counts all the ways to insert a v-v-chord, based at a fixed corner of a $2i$-gon to any other corner or itself. Clearly there are $i+1$ ways to do this. We multiply by i since we could start the chord at any of the i corners. Thus, the sum of the coefficients of the double sum is $\sum i = 1^n j_i(i^2+i) = n + \sum_{i=1}^n i^2 j_i$.

Line (6.15) counts the number of ways to join two faces of the same size. The sum of the coefficients is $\sum_{i=1}^n i^2(j_i^2 - j_i)$. Line (6.16) counts the ways to join two faces of different sizes.

The sum of the coefficients taken over all three sums in the consequent of the production is

$$\begin{aligned}\sum_{i=1}^{n}(i^2+i)j_i + \sum_{i=1}^{n} i^2(j_i^2 - j_i) \;+ \sum_{i=1}^{n}\sum_{k=i+1}^{n} 2ikj_ij_k \\ = n \;+ \sum_{i=1}^{n} j_i^2 i^2 + \sum_{i=1}^{n}\sum_{k=i+1}^{n} j_ij_k 2ik \\ = n \;+ \left(\sum_{i=1} ij_i\right)^2 \;=\; n \;+\; n^2.\end{aligned}$$

This again coincides with our expectation that the factor by which the number of imbeddings increases when a loop is added at an n-valent vertex is n^2+n. □

To calculate the genus polynomial of a dipole with loops at both vertices, we would first calculate the valence-partitioned genus polynomial at the root-vertex, as loops are added at the other vertex.

Example 6.5. As per (6.10), the rooted dipole $(D_3 * B_1, v)$ with a loop at u has the valence-partitioned genus polynomial

$$\Gamma_{(D_3*B_1,v)}(z_\pi) \;=\; 12z^0\cdot 1^3 + 24z^1\cdot 1^1 2^1 + 12z^1\cdot 3^1.$$

Using Theorem 6.3, we calculate the productions for adding a loop at vertex v:

$$\begin{aligned} z^0{\cdot}1^3 &\rightarrow 6z^0{\cdot}1^32^1 + 6z^1{\cdot}1^14^1 \\ z^1{\cdot}1^12^1 &\rightarrow 2z^1{\cdot}1^12^2 + 4z^1{\cdot}1^23^1 + 2z^1{\cdot}1^12^2 + 4z^2{\cdot}5^1 \\ z^1{\cdot}3^1 &\rightarrow 6z^1{\cdot}1^14^1 + 6z^1{\cdot}1^12^2. \end{aligned}$$

Substituting these productions into $\Gamma_{(D_3*B_1,v)}(z_\pi)$, we obtain

$$\begin{aligned} \Gamma_{(D_3*2B_1,v)}(z_\pi) &= 72z^0{\cdot}1^32^1 + 168z^1{\cdot}1^12^2 + 96z^1{\cdot}1^23^1 \\ &\quad +144z^1{\cdot}1^14^1 + 96z^2{\cdot}5^1. \\ \therefore \Gamma_{(D_3*2B_1,v)}(z) &= 72 + 408z + 96z^2. \end{aligned}$$

Example 6.6. As per (6.14), the rooted dipole $(D_4 * B_1, v)$ with a loop at the non-root vertex u has the valence-partitioned genus polynomial

$$\begin{aligned} \Gamma_{(D_4*B_1,v)}(z_\pi) &= 48z^0{\cdot}1^4 + 240z^1{\cdot}1^22 + 192z^1{\cdot}1^13^1 \\ &\quad +48z^1{\cdot}2^2 + 192z^2{\cdot}4^1. \end{aligned}$$

Theorem 6.2 yields the following productions:

$$\begin{aligned} z^0{\cdot}1^4 &\rightarrow 8z^0{\cdot}1^42^1 + 12z^1{\cdot}1^24^1 \\ z^1{\cdot}1^22^1 &\rightarrow 6z^1{\cdot}1^22^2 + 4z^1{\cdot}1^33^1 + 8z^2{\cdot}1^15^1 + 2z^2{\cdot}2^14^1 \\ z^1{\cdot}1^13^1 &\rightarrow 8z^1{\cdot}1^12^13^1 + 6z^1{\cdot}1^24^1 + 6z^2{\cdot}6^1 \\ z^1{\cdot}2^2 &\rightarrow 8z^1{\cdot}1^12^13^1 + 4z^1{\cdot}2^3 + 8z^2{\cdot}6^1 \\ z^2{\cdot}4^1 &\rightarrow 8z^2{\cdot}1^15^1 + 8z^2{\cdot}2^14^1 + 4z^2{\cdot}3^2. \end{aligned}$$

By substitution into (6.14), we obtain

$$\begin{aligned} \Gamma_{(D_4*2B_1,v)}(z_\pi) &= 384z^0{\cdot}1^42^1 + 1728z^1{\cdot}1^24^1 + 1440z^1{\cdot}1^22^2 + 960z^1{\cdot}1^33^1 \\ &\quad +1920z^1{\cdot}1^12^13^1 + 192z^1{\cdot}2^3 + 3456z^2{\cdot}1^15^1 \\ &\quad +1536z^2{\cdot}6^1 + 2016z^1{\cdot}2^14^1 + 768z^2{\cdot}3^2. \\ \therefore \Gamma_{(D_4*2B_1,v)}(z) &= 384 + 6240z + 7776z^2. \end{aligned}$$

7. Conclusions

Some special properties of dipoles have been critical to our derivations. In particular, both vertices of a loopless dipole have the same valence-partition. This enabled us to calculate the effect of topological surgery at the non-root vertex on the valence-partitioned genus polynomial, relative to the partition at the root-vertex.

Although our derivations in Section 6 of the productions in Theorem 6.1, Theorem 6.2, and Theorem 6.3 took the intuitive topological approach of drawing edges in polygons, this could have been combinatorialized. At other times, taking a combinatorial approach greatly facilitates the derivation of results that are not easily derived via intuitive topology.

In particular, Féray [Fe15] has noted that the conjecture we have elsewhere [GMTW16] called the *combinatorial local log-concavity conjecture* (CLLC) is settled in the affirmative by a theorem of Stanley [St11]. This affirmation reduces the log-concavity conjecture to a purely combinatorial conjecture that certain lists of cycle-count distribution polynomials can always be ordered so that each polynomial is *synchronous* with the sum of the previous polynomials in the list, as described by [GMTW16].

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Department of Computer Science, Columbia University, New York, NY 10027, USA; email: gross@cs.columbia.edu

Department of Mathematics, University of Haifa, 3498838 Haifa, Israel; email: tmansour@univ.haifa.ac.il

Department of Mathematics, Colgate University, Hamilton, NY 13346, USA; email: ttucker@colgate.edu