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# ENUMERATION OF DIGRAPH EMBEDDINGS

YICHAO CHEN, JONATHAN L. GROSS, AND XIAODONG HU

Abstract. A cellular embedding of an Eulerian digraph $D$ into a closed surface is said to be *directed* if the boundary of each face is a directed closed walk in $D$. The ***directed genus polynomial*** of an Eulerian digraph $D$ is the polynomial

$$\Gamma_D(x) = \sum_{h \geq 0} g_h(D) x^h$$

where $g_h(D)$ is the number of directed embeddings into the orientable surface $S_h$, of genus $h$, for $h = 0, 1, \ldots$ . The sequence $\{g_h(D) | h \geq 0\}$, which is called the ***directed genus distribution*** of the digraph $D$, is known for very few classes of graphs, compared to the genus distribution of a graph. This paper introduces a variety of methods for calculating the directed genus distributions of Eulerian digraphs. We use them to derive an explicit formula for the directed genus distribution of any 4-regular outerplanar digraph. We show that the directed genus distribution of such a digraph is determined by the *red-blue star decompositions* of the *characteristic tree* for an outerplanar embedding. The directed genus distribution of a 4-regular outerplanar digraph is proved to be *log-concave*, which is consistent with an affirmative answer to a question of Bonnington, et al. [2]. Indeed, the corresponding genus polynomial is real-rooted. We introduce *Eulerian splitting* at a vertex of a digraph, and we prove a splitting theorem for digraph embedding distributions that is analogous to the splitting theorem for (undirected) graph embedding distributions. This new splitting theorem allows conversion of the enumeration of embeddings of a digraph with vertex degrees larger than 4 into a problem of enumerating the embeddings of some 4-regular digraphs.

## 1. Introduction

Whereas the study of graph embeddings dates back to the $18^{\text{th}}$ century and has been studied extensively over the past 40 years, the systematic study of *directed embeddings* of a Eulerian digraph begins with Bonnington, Conder, Morton, and McKenna [2]. Bonnington, Hartsfield, and Širáň [3] have studied obstructions to

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directed embeddings of Eulerian digraphs in the plane and proved Kuratowski-type theorems for directed embeddings in the plane. Stahl [16] has offered a visualization of a *permutation-partition*, which is an abstract combinatorial generalization of a graph embedding, by means of a kind of Eulerian digraph called the *transition digraph.* Building on that, Cerè, Donati and Ferri [5] have defined a new kind of *genus of a group* $G$ to be the minimum genus of a surface into which the transition digraph of a presentation of $G$ embeds. Hao and Liu [12] have calculated the directed genus distribution for a class of cross-ladder digraphs. Hales and Hartsfield [11] have investigated the directed genus of the de Bruijn graph. Further information is available, including [1, 13].

Analogous to a well-known conjecture [10], that the genus distribution of a graph is strongly unimodal (or, equivalently, log-concave), Bonnington, et al. [2] asked whether the directed genus distribution of a digraph is (strongly) unimodal. Hao, Liu, Zhang, and Xu [14] have calculated the directed genus distributions for two classes of 4-regular digraphs, one of which they also proved to be strongly unimodal. One purpose of this paper is to respond to the question of Bonnington, et al. The other is to begin to provide methods that may prove useful in the development of a theory of directed genus distributions.

In this paper, we introduce the concept of an *Eulerian splitting of a vertex* and augment the emerging theory of directed genus distributions, by proving a splitting theorem for directed embeddings. This splitting theorem implies that the directed genus distribution of a digraph is a linear combination of the directed genus distributions of some 4-regular digraphs. Furthermore, we derive an explicit formula for the directed genus distribution of any 4-regular outerplanar digraph and we prove its strong unimodality. (By a 4-regular digraph, we mean that every vertex has *indegree* 2 and *outdegree* 2.)

1.1. **Directed graphs.** A ***directed graph*** or ***digraph*** $D$ consists of a finite nonempty set $V(D)$ of vertices together with a set $A(D)$ called ***arcs*** or ***directed edges***. Associated to each arc is either a pair of vertices, called its ***tail*** and its ***head***, or a single vertex, in which case the arc is called a ***self-arc***. Two or more arcs with the same head and the same tail are said to be ***parallel arcs***. The digraph is called ***simple*** if it has no self-arcs and no instance of parallel arcs.

For an arc $a$ with tail $u$ and head $v$, we may write $a = \overrightarrow{uv}$. In a simple digraph, the notation $\overrightarrow{uv}$ can be used as an unambiguous way to designate the arc. Then arc $a$ is said to join $u$ and $v$. We further say that arc $a$ is an ***out-arc*** at $u$ and an ***in-arc*** at $v$. Moreover, $u$ is said to be ***adjacent to*** $v$, and $v$ is said to be ***adjacent from*** $u$. The ***outdegree*** $out(v)$ of a vertex $v$ of a digraph $D$ is the number of out-arcs at $v$. The ***indegree*** $in(v)$ of $v$ is the number of in-arcs at $v$. The ***degree*** $d(v)$ of a vertex $v$ of $D$ is defined by $d(v) = out(v) + in(v)$.

The ***underlying graph*** of a digraph $D$ is the graph $G$ obtained from $D$ by deleting all directions from the arcs of $D$, which means eliminating the distinction between head and tail. The ***vertex-connectivity*** $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal from $G$ results in a disconnected or trivial graph. The ***edge-connectivity*** $\kappa_1(G)$ of $G$ is the minimum number of edges whose removal from $G$ results in a disconnected or trivial graph. A digraph $D$ is said to be $\kappa$***-connected*** ($\kappa_1$-edge connected) if its underlying graph $G$ is $\kappa$-connected ($\kappa_1$-edge connected), or ***strongly connected*** if for every ordered pair of vertices $u$ and $v$, there there is a directed path from $u$ to $v$. A digraph $D$ is called an ***Eulerian digraph*** if $D$ contains a directed Eulerian circuit. It is known that a digraph $D$ is Eulerian if and only if $in(v) = out(v)$ for each vertex $v$ of $D$. In this paper all digraphs considered are Eulerian.

1.2. **Directed embeddings.** A ***surface*** is a compact 2-manifold without boundary. Topologists classify surfaces into the ***orientable surfaces*** $S_g$, with $g$ handles ($g \geq 0$), and the ***nonorientable surfaces*** $N_k$, with $k$ crosscaps ($k > 0$). A ***directed embedding*** of an Eulerian directed graph $D$ into an orientable surface $S_g$ is a ***cellular embedding***, i.e., the interior of every face is homeomorphic to an open disc, of $D$ into $S_g$ such that every face is bounded by a directed circuit in $D$. An "embedding" here is taken to be cellular unless it is explicitly declared to be otherwise.

The ***(minimum)directed genus*** of a digraph $D$, denoted $\gamma_{min}(D)$, and the ***maximum directed genus*** of $D$, denoted $\gamma_{max}(D)$, are the minimum value and maximum value of $p$, respectively, for which the digraph $D$ has a directed embedding into a surface of genus $p$. There is an analogue [2] to Duke's interpolation theorem, which asserts that for any integer $k$ such that $\gamma_{min}(D) \leq k \leq \gamma_{max}(D)$, there exists a directed embedding of $D$ into the surface $S_k$.

We denote the number of cellular directed embeddings of $D$ on the surface $S_i$ by $g_i(D)$, where, by the ***number of embeddings***, we mean the number of equivalence classes under ambient isotopy. The ***directed genus distribution*** of the digraph $D$ is the sequence

$$g_0(D),\ g_1(D),\ g_2(D),\ \ldots,$$

The ***directed genus polynomial*** of $D$ is the polynomial

$$\Gamma_D(x) \ = \ \sum_{h\geq 0} g_h(D)x^h.$$

As with undirected embeddings, a directed embedding has a combinatorial representation. An ***alternating rotation at a vertex*** $v$ of a Eulerian digraph $D$ is a cyclic ordering of all the arcs incident with $v$ such that the in-arcs and out-arcs at $v$ alternate. An ***alternating rotation system*** $\rho$ of a graph $D$ is an assignment of an alternating rotation at every vertex of $D$. Two directed embeddings of a digraph $D$ are ***equivalent*** (or, informally, ***the same***) if they have the same alternating rotation system.

Analogous to the undirected case, there is a bijection between the set of alternating rotation systems and the set of directed embeddings of an Eulerian digraph $D$. This implies the following property.

**Proposition 1.1.** *For any Eulerian digraph $D$, the number of directed embeddings of $D$ equals*

$$\sum_{h\geq 0} g_h(D) \;=\; \prod_{v\in V(D)} \left(\frac{d(v)}{2}-1\right)! \left(\frac{d(v)}{2}\right)!. \qquad \square$$

## 2. Eulerian Splitting and the Splitting Theorem

Gross [6] obtained a splitting theorem for the genus distribution of graphs with maximum degree 4. Chen, et al. [4] proved a more general splitting theorem for the genus distribution of a graph and used this result to derive the genus distributions of some small diameter graphs. In this section, we derive a splitting theorem for digraph embeddings that is analogous to the splitting theorem of Chen, et al.

### 2.1. **Eulerian splitting.**

In order to describe all the possible ways to split an Eulerian digraph $D$ at a given vertex so as to obtain an Eulerian digraph $\overline{D}$, we provide some definitions.

**Definition 2.1.** Let $e = \overrightarrow{uw}$ be an arc of an Eulerian digraph $D$, such that the head vertex $w$ has valence $2n \geq 4$, and let $k$ be a number such that $2 \leq k \leq n-1$. We let

$$e_1 = \overrightarrow{u_1w},\ e_2 = \overrightarrow{u_2w},\ \ldots,\ e_{n-1} = \overrightarrow{u_{n-1}w},$$

be the other in-arcs to $w$, besides $e = \overrightarrow{uw}$, and we let

$$f_1 = \overrightarrow{wv_1},\ f_2 = \overrightarrow{wv_2},\ \ldots,\ f_n = \overrightarrow{wv_n},$$

be the out-arcs from $w$. The operation called a ***$2k$-degree Eulerian splitting of a digraph $D$ at vertex $w$ with designated in-arc*** $e = \overrightarrow{uw}$ has two cases.

Case (a): We choose a $(k-2)$-subset of the in-arcs to $w$

$$e_{i_1} = \overrightarrow{u_{i_1}w},\ e_{i_2} = \overrightarrow{u_{i_2}w},\ \ldots,\ e_{i_{k-2}} = \overrightarrow{u_{i_{k-2}}w},$$

and a $k$-subset of the out-arcs from $w$.

$$f_{j_1} = \overrightarrow{wv_{j_1}},\ f_{j_2} = \overrightarrow{wv_{j_2}},\ \ldots,\ f_{j_k} = \overrightarrow{wv_{j_k}},$$

The Eulerian digraph

$$D_{u_{i_1},\ldots,u_{i_{k-2}},v_{j_1},\ldots,v_{j_k}}$$

is obtained from the digraph $D$ in six steps:

(1) Insert new vertex $x$ and new arc $\overrightarrow{ux}$, and delete arc $e = \overrightarrow{uw}$.
(2) Insert new vertex $y$ and new arc $\overrightarrow{yx}$.
(3) For $t = 1, 2, \ldots, k-2$, insert new arc $\overrightarrow{u_{i_t}x}$, and delete arc $e_{i_t}$.
(4) Delete every other in-arc at $w$, and join its tail to vertex $y$.
(5) For $t = 1, 2, \ldots, k$, insert new arc $\overrightarrow{xv_{i_t}}$, and delete arc $f_{i_t}$.
(6) Delete every other out-arc from $w$, and join $y$ to its head.

We observe that the digraph $D_{u_{i_1},\ldots,u_{i_{k-2}},v_{j_1},\ldots,v_{j_k}}$ has in-degree and out-degree $k$ at vertex $x$, in-degree and out-degree $n-k+1$ at vertex $y$, and in-degree and out-degree $n$ elsewhere.

Case (b): We choose a $(k-1)$-subset of the in-arcs to $w$

$$e_{i_1} = \overrightarrow{u_{i_1}w},\ e_{i_2} = \overrightarrow{u_{i_2}w},\ \ldots,\ e_{i_{k-1}} = \overrightarrow{u_{i_{k-1}}w},$$

and a $(k-1)$-subset of the out-arcs from $w$.

$$f_{j_1} = \overrightarrow{wv_{j_1}},\ f_{j_2} = \overrightarrow{wv_{j_2}},\ \ldots,\ f_{j_{k-1}} = \overrightarrow{wv_{j_{k-1}}},$$

The Eulerian digraph

$$D_{u_{i_1},\ldots,u_{i_{k-1}},v_{j_1},\ldots,v_{j_{k-1}}}$$

is obtained from the digraph $D$ in six similar steps. There are differences in three of the steps.

(2) Insert new vertex $y$ and new arc $\overrightarrow{xy}$.
(3) For $t = 1, 2, \ldots, k-1$, insert new arc $\overrightarrow{u_{i_t}x}$, and delete arc $e_{i_t}$.
(5) For $t = 1, 2, \ldots, k-1$, insert new arc $\overrightarrow{yv_{i_t}}$, and delete arc $f_{i_t}$.

As in Case 1, the digraph $D_{u_{i_1},\ldots,u_{i_{k-1}},v_{j_1},\ldots,v_{j_{k-1}}}$ has in-degree and out-degree $k$ at vertex $x$, in-degree and out-degree $n-k+1$ at vertex $y$, and in-degree and out-degree $n$ elsewhere. Any digraph that can result from this operation is called a ***$2k$-degree Eulerian split of the digraph $D$ at $w$***. Figure 2.1 illustrates all such digraphs.

**Definition 2.2.** The operation called a ***$2k$-degree Eulerian splitting of a digraph $D$ at vertex $w$ with designated out-arc*** $e = \overrightarrow{wu}$ is defined similarly to the case with designated in-arc.

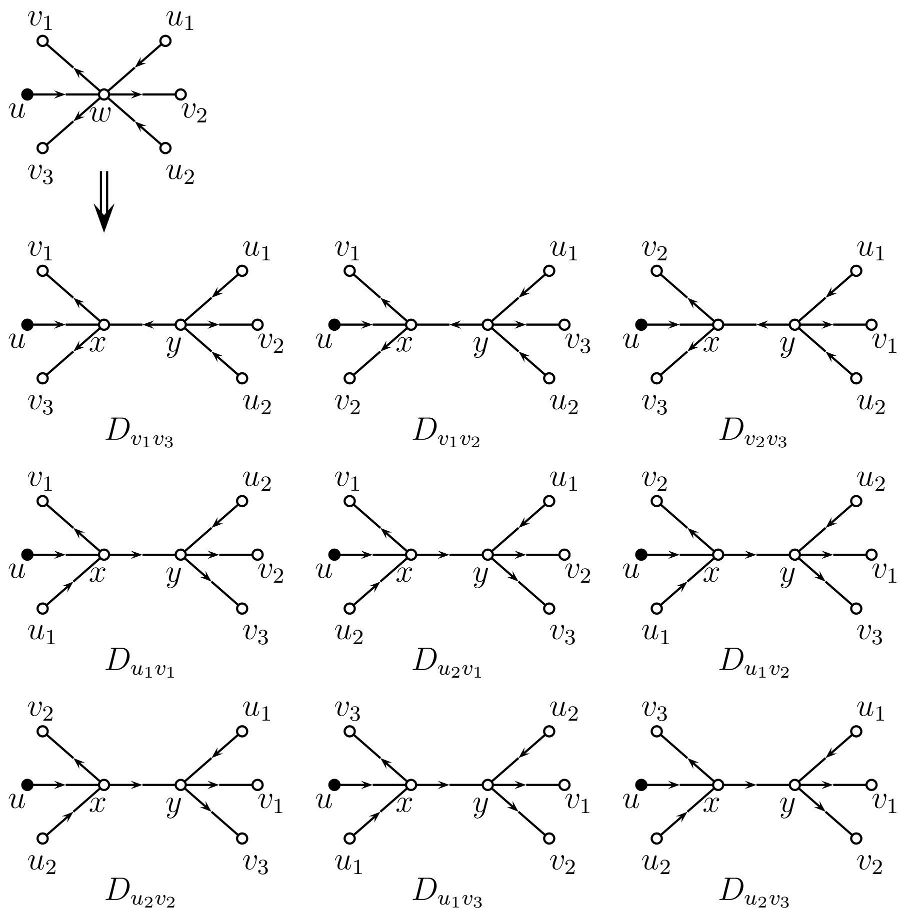


FIGURE 2.1. The 2-degree Eulerian splits of a digraph $D$ at $w$ with a designated in-arc $uw$.

**Proposition 2.1.** *Let $w$ be a vertex of in-degree and out-degree $n$ in an Eulerian digraph $D$. Then the number of $2k$-degree Eulerian splits at $w$ of types $D_{u_{i_1},\dots,u_{i_{k-2}},v_{j_1},\dots,v_{j_k}}$ and $D_{u_{i_1},\dots,u_{i_{k-1}},v_{j_1},\dots,v_{j_{k-1}}}$ are*

$$\binom{n-1}{k-2}\binom{n}{k} \quad \textit{and} \quad \binom{n-1}{k-1}\binom{n}{k-1},$$

*respectively.* □

**Proposition 2.2.** *Let $w$ be a vertex of in-degree and out-degree $n$ in an Eulerian digraph $D$, and let $\overline{D}$ be a $2k$-degree Eulerian split at $w$. Then the ratio of the number of directed embeddings of $D$ to the number of directed embeddings of $\overline{D}$ is*

$$\frac{n!(n-1)!}{k!(k-1)!(n-k+1)!(n-k)!}.$$ □

**Proposition 2.3.** *Let $n_0 = \Gamma_D(1)$ be the number of embeddings of the Eulerian digraph $D$, and let $n_1$ be the sum of the numbers of embeddings of its $2k$-degree Eulerian splits at a vertex $w$ with a designated in-arc (or, alternatively, with a designated out-arc). Then the ratio of the those two numbers is $1 : 2k-1$.*

*Proof.* Using Propositions 2.1 and 2.2, we calculate

$$\begin{aligned} \frac{n_0}{n_1} &= \frac{n!(n-1)!}{\left(\binom{n-1}{k-2}\binom{n}{k} + \binom{n-1}{k-1}\binom{n}{k-1}\right)(k!(k-1)!(n-k)!(n-k+1)!)} \\ &= \frac{1}{2k-1} \end{aligned}$$

□

**Example 2.1.** We observe in Figure 2.1, for instance, that the number of directed embeddings of the digraph $D$ would be $2!3!N = 12N$, where $N$ depends on the degrees at the vertices other than $w$. We observe that each of the 9 Eulerian splittings has $2!1!2!1!N = 4N$ directed embeddings. Thus, the ratio is $12 : 9 \cdot 4$, that is, $1 : 3$. Since $k = 2$, this corroborates Proposition 2.3.

2.2. **Splitting theorem.** In this subsection, we derive a splitting theorem for an Eulerian digraph. We present only the case of a vertex with a designated in-arc. The case with a designated out-arc is similar.

Suppose that the vertex $w$ of the Eulerian digraph $D$ has degree $2n$, with in-arcs $e$ and $e_1, e_2, \ldots, e_{n-1}$ and out-arcs $f_1, f_2, \ldots, f_n$. Consider an alternating rotation system $\rho$ on $D$, with rotation $\rho(w)$ as follows:

$$w.\ f_{i_1} e_{j_1} f_{i_2} e_{j_2} \ldots f_{i_{n-1}} e_{j_{n-1}} f_{i_n} e$$

where $i_l \in \{1, 2, \ldots, n\}$, for $l = 1, 2, \ldots, n$, and where $j_m \in \{1, 2, \ldots, n-1\}$, for $m = 1, 2, \ldots, k-1$ and $e = \overrightarrow{uw}$, as in Figure 2.2.

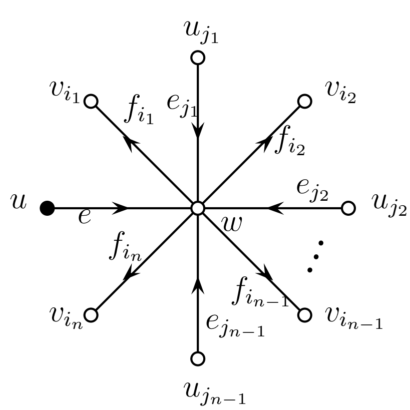


FIGURE 2.2. The rotation $\rho(w)$ at vertex $w$ of digraph $D$.

Now suppose that vertices $x$ and $y$ of the digraph $D_{u_{i_1},\ldots,u_{i_{k-2}},v_{j_1},\ldots,v_{j_k}}$, obtained by a $2k$-degree splitting at $w$, with designated in-arc $e = \overrightarrow{uw}$, are given the rotations

$$\begin{array}{ll} x. & f_{i_1}e_{j_1}f_{i_2}e_{j_2}\ldots f_{i_{k-m}}\overrightarrow{yx}f_{i_{n-m}}e_{j_{n-m}}\ldots f_{i_{n-1}}e_{j_{n-1}}f_{i_n}e \\ y. & e_{j_{k-m}}f_{i_{k-m+1}}e_{j_{k-m+1}}f_{i_{k-m+2}}e_{j_{k-m+2}}\ldots f_{i_{n-m}}e_{j_{n-m}}\overrightarrow{yx} \end{array} \tag{2.1}$$

Suppose that all other rotations are whatever $\rho$ has assigned to the other vertices of the digraph $D$. We call this an ***induced rotation system*** or a $2k$***-degree Eulerian split of the rotation system*** $\rho$, and we denote it by $\rho_{(\overrightarrow{yx},m)}$. We observe that the rotation at vertex $w$ has been split so that there are $2k$ consecutive alternating arc incidences at new vertex $x$ and $2n-2k+2$ alternating arc incidences at new vertex $y$. We observe that each of the split embeddings contracts to the embedding of $D$ corresponding to $\rho$; topologically, under a ***contraction***, we envision the arc $\overrightarrow{yx}$ shrinking down to a single point, thus merging vertices $y$ and $x$. Figure 2.3 illustrates the rotations of $\rho_{(\overrightarrow{yx},m)}$ at vertices $x$ and $y$.

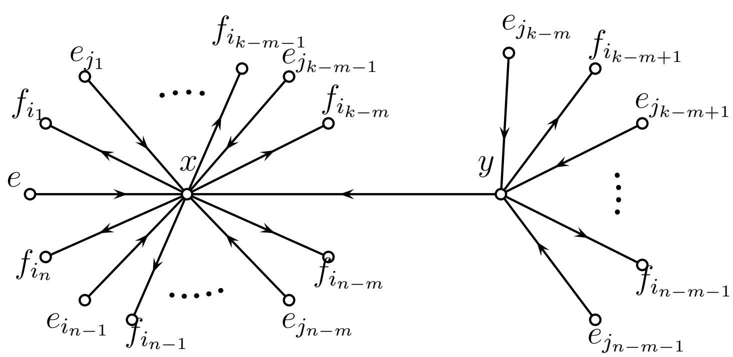


FIGURE 2.3. The induced rotation at the vertices $x$ and $y$.

**Proposition 2.4.** *Let $\rho$ be a rotation system for an Eulerian digraph $D$ with a $2n$-valent vertex $w$ with designated in-arc. Then there are $k-1$ $2k$-degree Eulerian splits of $\rho$ at vertex $w$ with rotations at new vertices $x$ and $y$ as given by* (2.1)*, each of which contracts to $\rho$.*

*Proof.* There is one such split for each value of $m = 1, 2, \ldots, k-1$. □

Similarly, the split digraphs $D_{u_{i_1},\ldots,u_{i_{k-1}},v_{j_1}\ldots,v_{j_{k-1}}}$ have induced rotation systems, for $l = 0, 1, \ldots, k-1$, in which the vertices $x$ and $y$ are given the rotations

$$\begin{array}{ll} x. & f_{i_1}e_{j_1}f_{i_2}e_{j_2}\ldots f_{i_{k-l}}e_{j_{k-l}}\overrightarrow{xy}\,e_{j_{n-l}}f_{i_{n-l+1}}e_{j_{n-l+1}}\ldots e_{j_{n-1}}f_{i_n}e \\ y. & f_{i_{k-l+1}}e_{j_{k-l+1}}f_{i_{k-l+2}}e_{j_{k-l+2}}\ldots f_{i_{n-l-1}}e_{j_{n-l-1}}f_{i_{n-l}}\overrightarrow{xy} \end{array} \tag{2.2}$$

**Proposition 2.5.** *Let $\rho$ be a rotation system for an Eulerian digraph $D$ with a $2n$-valent vertex $w$ with a designated in-arc. Then there are $k$ Eulerian splits of $\rho$ at vertex $w$ with rotations at new vertices $x$ and $y$ as given by* (2.2)*, each of which contracts to $\rho$.*

*Proof.* There is one such split for each value of $l = 0, 1, \ldots, k-1$. □

**Lemma 2.6.** *Let $w$ be a $2n$-valent vertex* ($2n \geq 6$)*, with designated in-arc $e = \overrightarrow{uw}$, in an Eulerian digraph $D$ with rotation system $\rho$. Then there correspond exactly $2k-1$ $2k$-degree Eulerian splits of $D$ at $w$ with designated in-arc $\overrightarrow{uw}$ that are $2k$-degree Eulerian splits of $\rho$, each of which contracts to $\rho$.*

*Proof.* This follows from Propositions 2.4 and 2.5. □

**Lemma 2.7.** *Let $w$ be a $2n$-valent vertex* ($2n \geq 6$)*, with designated out-arc $\overrightarrow{wu}$, in an Eulerian digraph $D$ with rotation system $\rho$. Then there correspond exactly $2k-1$ $2k$-degree Eulerian splits of $D$ at $w$ with designated out-arc $\overrightarrow{wu}$ that are $2k$-degree Eulerian splits of $\rho$, each of which contracts to $\rho$.*

*Proof.* This follows by the same reasoning as for Lemma 2.6. □

**Theorem 2.8** (Splitting Theorem)**.** *Let $w$ be a $2n$-valent vertex (where $2n \geq 6$), with designated in-arc or out-arc, in a connected Eulerian digraph $D$. Let $\Lambda$ be the set of all graphs $D_{u_{i_1},\ldots,u_{i_{k-2}},v_{j_1},\ldots,v_{j_k}}$ and $D_{u_{i_1},\ldots,u_{i_{k-1}},v_{j_1},\ldots,v_{j_{k-1}}}$ obtainable by $2k$-degree Eulerian splitting at $w$. Then we have*

$$\Gamma_D(x) \;=\; \frac{1}{2k-1}\sum_{\overline{D}\in\Lambda}\Gamma_{\overline{D}}(x).$$

*Proof.* This follows from Proposition 2.3 and Lemma 2.6. □

**Example 2.2.** Up to isomorphism, there is only one way to give an Eulerian assignment of directions to the bouquet $B_3$. According to Proposition 1.1, there are 12 embeddings of $B_3$. We proceed to use the Splitting Theorem to calculate the genus polynomial for $B_3$. Corresponding to each Eulerian embedding of $B_3$, there are three split embeddings corresponding to rotations like (2.1) at the new vertices. There are also six split embeddings corresponding to rotations like (2.2). Among these nine split digraphs, the underlying graph of three is isomorphic to the dipole $D_4$, and of the other six like $J_3$. The graphs $B_3$, $D_4$, and $J_3$ are illustrated in Figure 2.4.

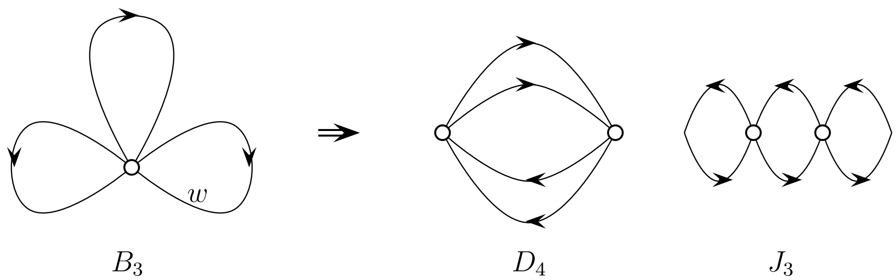


FIGURE 2.4. The directed Eulerian bouquet $B_3$ and the two Eulerian digraphs into which it splits.

By face-tracing, we calculate the directed genus polynomials

$$\Gamma_{D_4}(x) = 2 + 2x \quad \text{and} \quad \Gamma_{J_3}(x) = 4.$$

We now calculate $\Gamma_{B_3}(x)$, using Theorem 2.8, to obtain

$$\begin{aligned}\Gamma_{B_3}(x) &= \frac{1}{3}\left[3\Gamma_{D_4}(x) + 6\Gamma_{J_3}(x)\right] \\ &= \frac{1}{3}\left[3(2+2x) + 6 \times 4\right] \\ &= 10 + 2x.\end{aligned}$$

The result $\Gamma_{B_3}(x) = 10 + 2x$ can be confirmed by routine face-tracing.

## 3. Directed Genus Distributions of 4-Regular Outerplanar Digraphs.

An ***outerplanar graph*** is a graph that can be embedded in the plane so that all of the vertices lie on the boundary walk of the unbounded region $f_\infty$. Alternatively, a graph $G$ is outerplanar if the graph formed from $G$ by adding a new vertex, with edges connecting it to all the other vertices, is a planar graph. An outerplane embedding is said to be ***normalized*** if all self-loops of the graph lie on the face-boundary walk of the unbounded region $f_\infty$. We observe that an outerplanar graph may have inequivalent outerplane embeddings. By an ***outerplane graph***, we mean an outerplanar graph with a fixed outerplane embedding.

The ***weak dual*** of a plane graph $G$ is obtained from the dual by deleting the vertex corresponding to the unbounded region $f_\infty$ of $G$. Suppose that $G$ is a 2-connected outerplane graph; that is, the minimum number of vertices of $G$ whose removal would disconnect $G$ is at least 2. It is easy to see that the weak dual of $G$ is acyclic, since a cycle in the weak dual would represent a set of bounded regions in $G$ that separate a vertex $v$ from the unbounded face. By Theorem 26 of [17], we know that the dual of a 2-connected plane graph is 2-connected. This leads to the following proposition:

**Proposition 3.1.** *The weak dual of a 2-connected graph $G$ is a plane tree.* □

We call this plane tree the ***characteristic tree*** of the outerplane graph $G$. Figure 3.1 illustrates an outerplane graph and its characteristic tree. We shall see that a 2-connected outerplane graph is uniquely characterized by its characteristic tree.

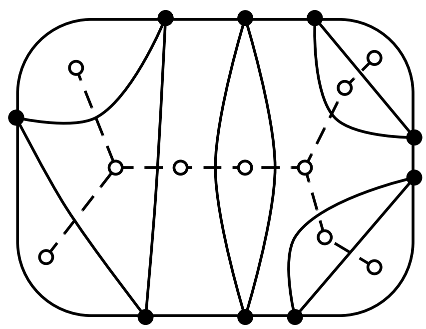

FIGURE 3.1. A 2-connected outerplane graph, with solid vertices and solid lines; its characteristic tree, with hollow vertices and broken lines

3.1. **Restricted tree colorings.** A ***leaf*** of a tree $T$ is a vertex of degree one. A proper coloring of a tree $T$ is called a ***restricted coloring*** if all leaves of $T$ have the same color.

Since every tree is bipartite, the chromatic number $\chi(T)$ of a non-trivial tree is 2. However, the restricted chromatic number $\chi_R(T)$ of a tree may be greater than 2. We observe that $\chi_R(T) \geq \chi(T) = 2$. For example, the restricted coloring of a path $P_{2n}, (n \geq 2)$ equals 3. Figure 3.2 shows two different trees with restricted chromatic numbers 3 and 2.

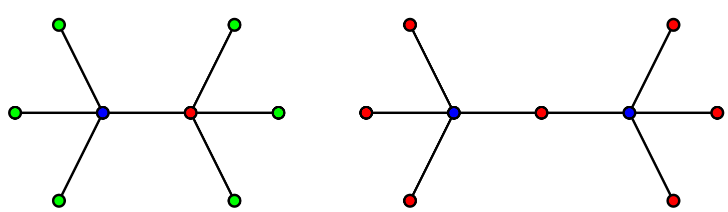

FIGURE 3.2. Optimal restricted colorings of two trees.

**Proposition 3.2.** *Let $T$ be an $n$-vertex tree ($n \geq 3$). Then we have either $\chi_R(T) = 2$ or $\chi_R(T) = 3$.*

*Proof.* Let $T^-$ be the subtree obtained from $T$ by deleting all the leaves. Since $T^-$ is a tree, we can give it a proper 2-coloring with the colors red and blue. Then we color all the leaves of $T$ yellow, to obtain a restricted coloring of $T$. Thus, $\chi_R(T) \leq 3$. □

By an $n$-***star***, with $n \geq 2$, we mean any graph isomorphic to the complete bipartite graph $K_{1,n}$. It is easy to see that the $\chi_R(K_{1,n}) = 2$. An $n$-star is called a ***red-blue star*** if all the leaves are red and the other vertex is blue. We now characterize a tree $T$ with $\chi_R(T) = 2$.

**Theorem 3.3.** *Let $T$ be a red-leafed plane tree with $\chi_R(T) = 2$. Then $T$ can be obtained by a sequence of vertex-amalgamations of plane red-blue stars.*

*Proof.* We proceed by induction on the number of vertices $k$ of $T$. For $k = 3, 4$, the tree $T$ is the star graph $K_{1,2}$ or $S_{1,3}$, respectively, i.e., already a red-blue star. Now we suppose that the theorem is true whenever $T$ has at most $n$ vertices, and we let $T$ have $n+1$ vertices. If $T$ is an $n$-star, the conclusion is clear. Accordingly, we suppose that $T$ is not a $n$-star. Let the vertex $v$ be the neighbor of any leaf. Since the leaf is red, the color of $v$ is blue. Moreover, the vertex $v$ and its neighbors form a red-blue star $T_0$ isomorphic to $K_{1,deg(v)}$.

Each of the components $T_1, T_2, \ldots, T_{deg(v)}$ of the graph obtained by deleting the blue vertex $v$ is either a tree with all leaves red, or a red isolated vertex. By induction, each of the trees $T_i$, except for the isolated vertices, can be obtained by a series of vertex-amalgamations of red-blue stars, as illustrated in Figure 3.3. The tree $T$ can be reconstructed by iterative vertex-amalgamation of subtrees $T_0, T_1, T_2, \ldots, T_{d(v)}$. The result follows. □

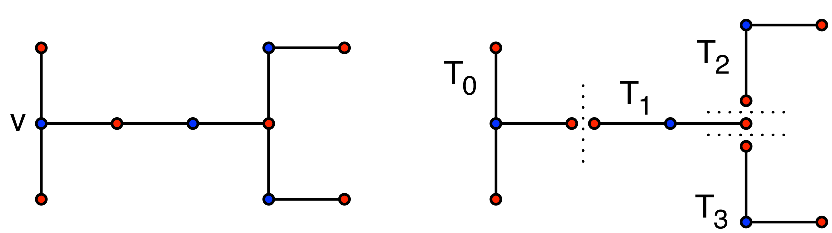


FIGURE 3.3. A plane tree with restricted chromatic number 2, and its red-blue star decomposition

3.2. **Characterizing 4-regular outerplane graphs.** In this subsection, we establish that a tree $T$ is a characteristic tree of a 4-regular outerplane graph if and only if $\chi_R(T) = 2$.

A ***bracelet*** $BR_n$ is a 4-regular graph obtained from the $n$-cycle $C_n$ by doubling every edge. It is easy to see that $BR_n$ has an outerplane embedding, whose characteristic tree is the star graph $K_{1,n}$. Figure 3.4 shows the outerplane embedding of $BR_4$ and its characteristic tree $K_{1,4}$.

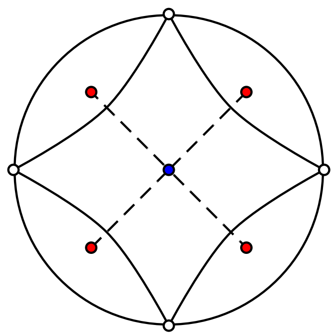

FIGURE 3.4. The bracelet $BR_4$ and its characteristic tree $K_{1,4}$.

We observe that there are exactly two ways to assign directions to the edges of a bracelet graph $BR_n$, so as to obtain an Eulerian digraph:

(1) In a ***bi-directional bracelet*** $BB_r$, the two parallel edges in each pair are assigned opposite directions: the head of one arc is the tail of the other. For instance, in Figure 3.4, the direction of the outer cycle could be counterclockwise, and the direction of the inner cycle could be clockwise. This would be a directed embedding of $BB_4$.
(2) In a ***uni-directional bracelet*** $UB_r$, the two parallel edges in each pair are assigned the same direction: the two arcs have the same head and the same tail. In Figure 3.4, the outer and inner cycles might both be clockwise or both counterclockwise. Neither would be a directed embedding of $UB_4$, since the boundary walks of the four digons would not be directed walks.

**Proposition 3.4.** *For $r \geq 3$, the uni-directional bracelet $UB_r$ is non-planar.*

*Proof.* In an alternating rotation system for $UB_r$, the edges joining any vertex $v$ with its two neighbors would have to alternate. It follows that the restriction of the corresponding embedding to the vertex $v$, its two neighbors, and the four edges incident at $v$ is already non-planar. □

Let $G_1$ and $G_2$ be disjoint connected graphs. Let $e = uv \in E(G_1)$ and let $f = xy \in E(G_2)$. The ***cross-connection*** $(G_1, uv) \rightleftarrows (G_2, xy)$ is the graph is obtained from $G_1 \cup G_2$ by adding two edges $ux$ and $vy$ and deleting the edges $e$ and $f$, as shown in Figure 3.5.

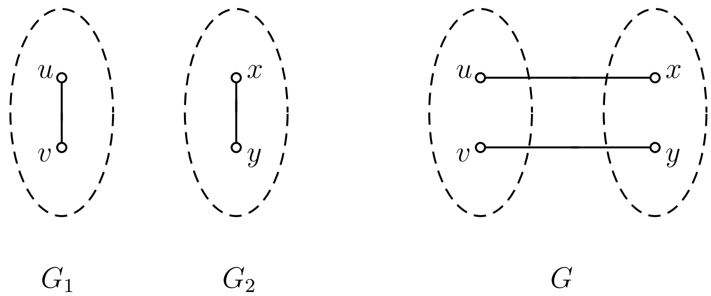


FIGURE 3.5. A cross-connection of graphs $G_1$ and $G_2$.

We partition the interior faces of a 4-regular outerplane graph into two types:

**Type I:** Every edge of the face-boundary is a chord of the outer spanning cycle.
**Type II:** At least one edge of the face-boundary lies on the outer cycle.

**Theorem 3.5.** *A tree $T$ is the characteristic tree of a 4-regular outerplane graph $G$, if and only if $\chi_R(T) = 2$.*

*Proof.* First, let $T$ be a characteristic tree of a 4-regular outerplanar graph $G$. We color all the Type I faces blue and all the Type II faces red. Since $G$ is a 4-regular outerplanar graph, the Type I faces are adjacent only to Type II faces. By transferring the face colors to the dual tree $T$, we obtain a restricted 2-coloring of $T$. Thus, $\chi_R(T) = 2$.

Conversely, let $\chi_R(T) = 2$. Then, by Theorem 3.3, the tree $T$ can be constructed by a sequence of vertex-amalgamations of red-blue stars. Each red-blue star $K_{1,n}$ is the weak dual of a bracelet $BR_n$. The vertex-amalgamation of two stars dualizes to a cross-connection of two bracelets, in which the four vertices involved all lie on the exterior region, so that the two new edges also lie on the exterior region. Thus, such a cross-connection preserves outerplanarity. By reiterating this process, we construct a 4-regular outerplane graph whose characteristic tree is $T$. □

The ***preorder traversal*** of an $n$-vertex rooted plane tree $(T, v)$ is defined recursively, as a sequence of vertices of $T$:

***Basis:*** If $n = 1$, then the traversal is the root $v$.

***Recursive Step:*** For $n > 1$, we consider the principal subtrees, $T_1, T_2, \ldots, T_k$ of $T$ at the root $v$. We start the traversal of $T$ at the root $v$, and then we concatenate the preorder traversals of the principal subtrees, choosing those subtrees in left-to-right order.

**Theorem 3.6.** *There is a bijection between the set of all 4-regular 2-connected outerplane graphs and the set of all plane trees $T$ with $\chi_R(T) = 2$.*

*Proof.* First, we see that given a 4-regular 2-connected outerplane graph $G$, its characteristic plane tree $T$ is uniquely determined, by the definition of a characteristic tree. By Theorem 3.5, we have $\chi_R(T) = 2$.

Now let $T$ be a plane tree with $\chi_R(T) = 2$. By Theorem 3.3, the tree $T$ can be obtained by a sequence of vertex-amalgamations of plane red-blue stars. We visit the blue vertices to $T$ according to a preorder traversal, and we obtain an outerplane graph by a sequence of cross-connections of bracelets, according to that traversal ordering, as shown in Figure 3.6. This reconstructs the unique 4-regular 2-connected outerplane graph $G$ whose characteristic tree is $T$.

An alternative reconstruction of the outerplane graph $G$ can be obtained from the plane tree $T$ as follows:

(1) Joint a new vertex $v_\infty$ to each red vertex $w$ of $T$ by a multi-edge whose multiplicity equals the valence of $w$, so that between each consecutive pair of edges from blue vertices incident on $w$ there is a single edge from $v$. This forms a 2-connected plane bipartite graph $H$, in which every face-boundary walk contains two edges of the tree $T$ and two edges incident on $v_\infty$.
(2) Then the 4-regular 2-connected outerplane graph $G$ is the dual of $H$.

□

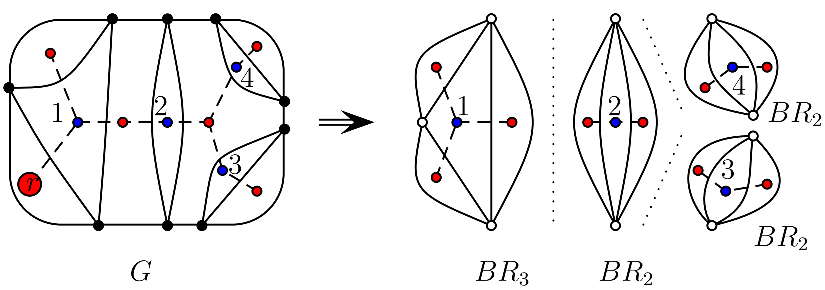


FIGURE 3.6. A 4-regular outerplane graph $G$ and the sum operation of its bracelet graphs. The large red vertex is the root, and the numbers within the bracelets indicate the preorder traversal.

3.3. **Directed genus distribution of a cross-connection.** Corresponding to the cross-connection operation for undirected graphs, we now introduce the directed counterpart.

Let $D_1$ and $D_2$ be disjoint connected digraphs. Let $a = \overrightarrow{uv} \in E(D_1)$ and $b = \overrightarrow{yx} \in E(D_2)$. The ***directed cross-connection*** $D_0 = (D_1, \overrightarrow{uv}) \rightleftarrows (D_2, \overrightarrow{yx})$

is the digraph obtained from $D_1 \cup D_2$ by adding arcs $\overrightarrow{ux}$ and $\overrightarrow{yv}$ and deleting the directed edges $a$ and $b$, as shown in Figure 3.7.

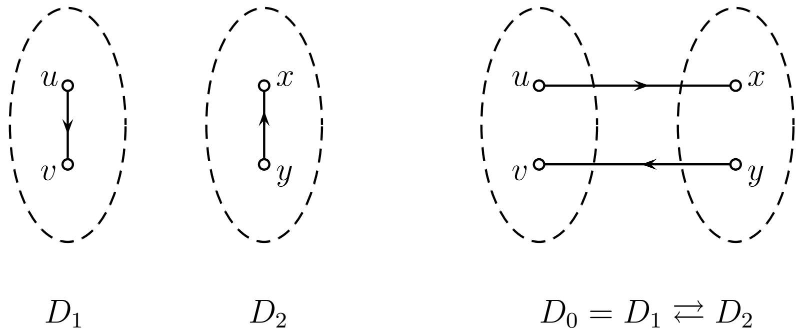


FIGURE 3.7. From the digraphs $D_1$ and $D_2$ to the digraph $D_0$

**Theorem 3.7.** *Let $D_1$ and $D_2$ be disjoint connected digraphs, with $\overrightarrow{uv} \in E(D_1)$ and $\overrightarrow{yx} \in E(D_2)$. Then the directed genus polynomial of the directed cross-connection $D_0 = (D_1, \overrightarrow{uv}) \rightleftarrows (D_2, \overrightarrow{yx})$ is given by*

$$\Gamma_{D_1 \rightleftarrows D_2}(x) \;=\; \Gamma_{D_1}(x)\, \Gamma_{D_2}(x). \tag{3.1}$$

*Proof.* Let $\rho_1$ be a rotation system for $D_1$ with rotations

$$u : (Av) \qquad v : (Bu)$$

where $A$ is a sequence of the neighbors of $u$, excluding $v$, and $B$ is sequence of the neighbors of $v$, excluding $u$. Similarly, let $\rho_2$ be a rotation system for $D_2$ with rotations

$$x : (Cy) \qquad y : (Dx)$$

where $C$ is a sequence of the neighbors of $x$, excluding $y$, and $D$ is is a sequence of the neighbors of $y$, excluding $x$. Then the rotation system $\rho_0$ for $D_0$ has rotations

$$u : (Ax) \qquad v : (By) \qquad x : (Cu) \qquad y : (Dv)$$

with all its other rotations as in $\rho_1$ and $\rho_2$.

For each $i \in \{0, 1, 2\}$, we let $p_i$, $q_i$, $r_i$, and $g_i$ denote the numbesr of vertices, arcs, and faces and the genus of the directed embedding $(D_i, \rho_i)$. Clearly,

$$p_0 \;=\; p_1 + p_2 \tag{3.2}$$

$$q_0 \;=\; q_1 + q_2 \tag{3.3}$$

We observe that each arc of a directed graph embedding is on the boundary of exactly two different regions.

In this context, the region whose boundary direction is consistent with the orientation of the surface is called a ***face***, and the other region is called an ***antiface***. Suppose that the arc $\overrightarrow{uv}$ of $D_1$ lies on the face $f_1^1 = (uvW)$ and the antiface

$f_1^2 = (vuQ)$. Suppose also that the arc $\overrightarrow{yx}$ of $D_2$ lies on the face $f_2^1 = (yxX)$ and the antiface $f_2^2 = (xyY)$. Then the new edges $\overrightarrow{ux}$ and $\overrightarrow{yv}$ both lie on the face $f_0^1 = (uxXyvW)$ and the antiface $f_0^2 = (vyYxuQ)$, as illustrated in Figure 3.8. The other faces of $(D_0, \rho_0)$ are the same as the faces of $(D_1, \rho_1)$ and $(D_2, \rho_2)$. It follows that

$$r_0 \;=\; r_1 + r_2 - 2 \tag{3.4}$$

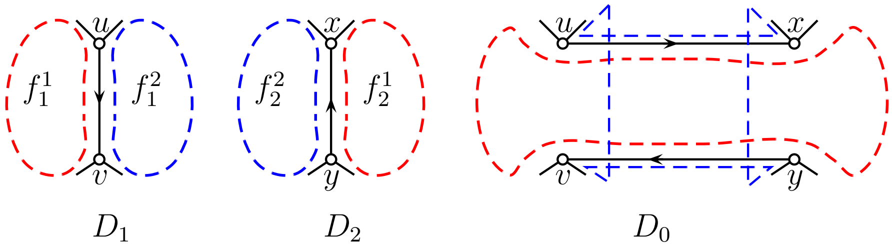


FIGURE 3.8. Cross-connecting two directed embeddings.

Using Euler's polyhedral equation $p - q + r = 2 - 2g$ and (3.2), (3.3), and (3.4), we have

$$\begin{aligned} 2 - 2g(D_0, \rho_0) &= p_0 - q_0 + r_0 \\ &= (p_1 + p_2) - (q_1 + q_2) + (r_1 + r_2 - 2) \\ &= (2 - 2g(D_1, \rho_1)) + (2 - 2g(D_2, \rho_2)) - 2 \end{aligned}$$

This implies that

$$g_0 \;=\; g_1 + g_2 \tag{3.5}$$

from which we infer

$$\Gamma_{D_1 \rightleftarrows D_2}(x) \;=\; \Gamma_{D_1}(x)\, \Gamma_{D_2}(x). \tag{3.6}$$

□

3.4. **Directed genus distribution of a vertex-amalgamation.** A formula for the genus distribution of the graph obtained by amalgamating two graphs at a 2-valent root in each, in terms of the partitioned genus distributions of the amalgamands, was derived by [9]. In this subsection, we derive a much simpler formula for the directed genus polynomial of an Eulerian digraph obtained by amalgamating two Eulerian digraphs at a 2-valent root in each. This simpler formula involves no partitioning of the directed genus distributions.

Let $(D_1, r_1)$ and $(D_2, r_2)$ be Eulerian digraphs, each with a 2-valent root. The ***directed vertex-amalgamation*** $D_1 *_{r_1 = r_2} D_2$, is the Eulerian digraph obtained by identifying the two roots.

**Theorem 3.8.** *Let $D_1$ and $D_2$ be disjoint connected Eulerian digraphs, with arcs $a = \overrightarrow{uv} \in E(D_1)$ and $b = \overrightarrow{yx} \in E(D_2)$. Let $D_1^{'}$ and $D_2^{'}$ be the digraphs obtained from $D_1$ and $D_2$ by inserting subdivision vertices $w_1$ and $w_2$ into arcs $a$ and $b$. Let $D_0 = D_1^{'} *_{w_1=w_2} D_2^{'}$. Then the directed genus polynomial of $D_0$ is given by*

$$\Gamma_{D_0}(x) \;=\; 2\Gamma_{D_1}(x)\,\Gamma_{D_2}(x). \tag{3.7}$$

*Proof.* Figure 3.9 illustrates the vertex amalgamation, in which we have taken $w$ to be the 4-valent vertex that results from merging the 2-valent vertices $w_1$ and $w_2$.

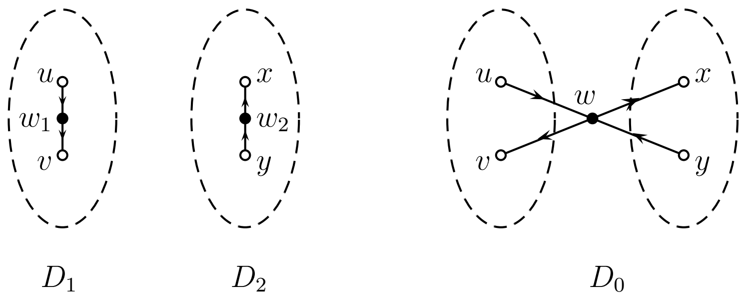


FIGURE 3.9. Vertex-amalgamation of the digraphs $D_1$ and $D_2$.

Suppose that the rotation system $\rho_1$ for $D_1$ has rotations

$$u : (Av) \qquad v : (Bu)$$

where $A$ is a sequence of the neighbors of $u$, excluding $v$, and $B$ is a sequence of the neighbors of $v$, excluding $u$. Suppose also that the rotation system $\rho_2$ for$D_2$ has rotations

$$x : (Cy) \qquad y : (Dx)$$

where $C$ is a sequence of the neighbors of $x$, excluding $y$, and $D$ is a sequence of the of neighbors of $y$, excluding $x$. Since the vertex $w$ has two different alternating rotations, each pair of rotation systems with one for $D_1$ and the other for $D_2$ will induce two rotation systems for $D_0$. The proof has two cases.

**Case 1** Let $\rho_0$ be the rotation system for $D_0$ with rotations

$$u : (Ax) \qquad v : (By) \qquad x : (Cu) \qquad y : (Dv) \qquad w : (uxyv)$$

and all other vertex rotations as in $\rho_1$ and $\rho_2$.

For each $i \in \{0, 1, 2\}$, we let $p_i$, $q_i$, $r_i$, and $g_i$ denote the numbers of vertices, arcs, faces, and genus of $(D_i, \rho_i)$. Clearly,

$$p_0 \;=\; p_1 + p_2 + 1 \tag{3.8}$$

$$q_0 \;=\; q_1 + q_2 + 2 \tag{3.9}$$

Suppose that the arc $\overrightarrow{uv}$ of $(D_1, \rho_1)$ lies on the face $f_1^1 = (uvW)$ and the antiface $f_1^2 = (vuQ)$. Suppose also that the arc $\overrightarrow{yx}$ of $(D_2, \rho_2)$ lies on the face $f_2^1 =$

$(yxX)$ and the antiface $f_2^2 = (xyY)$. Then the new edges $\overrightarrow{uw}$, $\overrightarrow{yw}$, $\overrightarrow{wx}$, and $\overrightarrow{wv}$ collectively lie on the three different faces: $f_0^1 = (Wuwv)$, $f_0^2 = (ywxX)$, and $f_0^3 = (uwxY^-ywvQ^-)$. The other faces of $\rho_0$ are the same as the faces of $\rho_1$ and $\rho_2$. It follows that

$$r_0 = r_1 + r_2 - 1 \tag{3.10}$$

Figure 3.10 illustrates the situation.

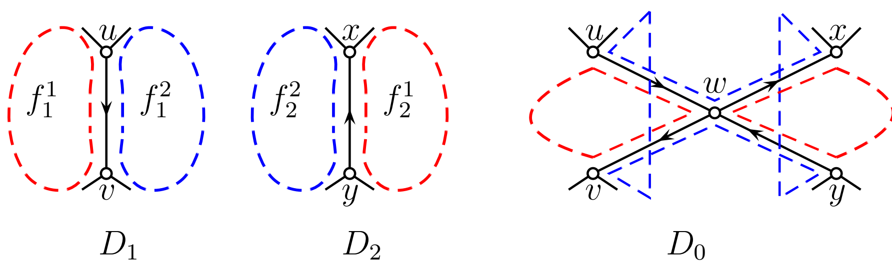


FIGURE 3.10. Vertex-amalgamation of two directed embeddings.

By Euler's polyhedral equation and (3.8), (3.9), and (3.10), we have

$$\begin{aligned} 2 - 2g(D_0, \rho_0) &= p_0 - q_0 + r_0 \\ &= (p_1 + p_2 + 1) - (q_1 + q_2 + 2) + (r_1 + r_2 - 1) \\ &= (2 - 2g(D_1, \rho_1)) + (2 - 2g(D_2, \rho_2)) + 2 \end{aligned}$$

which implies that

$$g_0 = g_1 + g_2$$

**Case 2:** Now let $\rho_0$ be the rotation system of $D_0$ with rotations

$$u : (Ax) \qquad v : (By) \qquad x : (Cu) \qquad y : (Dv) \qquad w : (yxuv)$$

and all other vertex rotations as in $\rho_1$ and $\rho_2$. We omit the details, which are similar to Case 1, with the same result:

$$g_0 = g_1 + g_2$$

Accordingly, we conclude that

$$\Gamma_{D_0}(x) = 2\Gamma_{D_1}(x)\,\Gamma_{D_2}(x).$$

□

3.5. **Cobblestone paths and bracelets.** The ***cobblestone digraph*** $J_n$ is a directed path $P_n$ with a reversal of each arc added. Figure 3.11 illustrates the cobblestone digraph $J_5$.

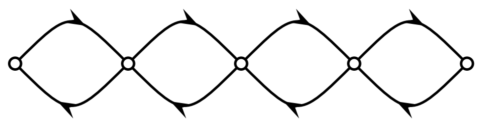

FIGURE 3.11. The cobblestone digraph $J_5$

**Theorem 3.9.** *The directed genus distribution of the cobblestone digraph $J_n$ consists of $2^{n-2}$ embeddings of genus 0.*

*Proof.* By Proposition 2.1, the number of directed embeddings of $J_n$ is $2^{n-2}$. By Theorem 3.8, we know that the cobblestone digraph $J_n$ has maximum genus 0. □

Bonnington, Conder, Morton and McKenna [2] have proved that the directed genus distribution of the bi-directional bracelet digraph $BB_{2k}$ consists of two embeddings of genus 0 and $2^{2k} - 2$ embeddings of genus 1. We now extend this directed genus distribution formula to $BB_n$ for all integers $n \geq 2$.

**Theorem 3.10.** *The directed genus distribution of the bi-directional bracelet $BB_n$, for $n \geq 2$, consists of two embeddings of genus 0 and $2^n - 2$ embeddings of genus 1.*

*Proof.* The bi-directional bracelet $BB_n$ can be obtained by identifying the two end-vertices $u$ and $v$ of a cobblestone digraph $J_{n+1}$. Consider a cellular embedding of $J_{n+1}$ in the surface $S_h$ with $k$ faces. (Of course, we have $k = n + 1 - 2h$.) Since each arc of $J_n$ lies in two different faces, each vertex must lie in at least two different faces. In particular, suppose that the vertex $u$ lies in faces $f_1$ and $f_2$ and that vertex $v$ lies in faces $f_3$ and $f_4$, as shown in Figure 3.12.

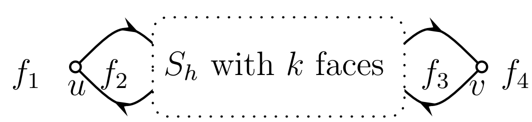


FIGURE 3.12. A directed embedding of the cobblestone digraph $J_{n+1}$.

The two kinds of directed embeddings of $BB_n$ that correspond to this embedding of $J_{n+1}$ occur as determined by the following two cases.

(1) The faces $f_1$ and $f_2$ that border vertex $u$ are both different from the faces $f_3$ and $f_4$ that border vertex $v$. In this case, both resulting directed embeddings of $BB_n$ are on the surface $S_{h+1}$ with $k - 1$ faces. See Figure 3.13 for details.

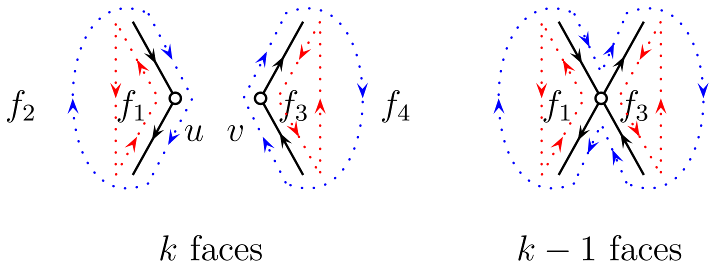


FIGURE 3.13. Case 1: Vertices $u$ and $v$ do not lie on a common face.

(2) One of the faces bordering vertex $u$ also borders vertex $v$. In this case, there is one imbedding of $BB_n$ on $S_h$, with $k+1$ faces, and one embedding on $S_{h+1}$, with $k-1$ faces. See Figure 3.14 for details.

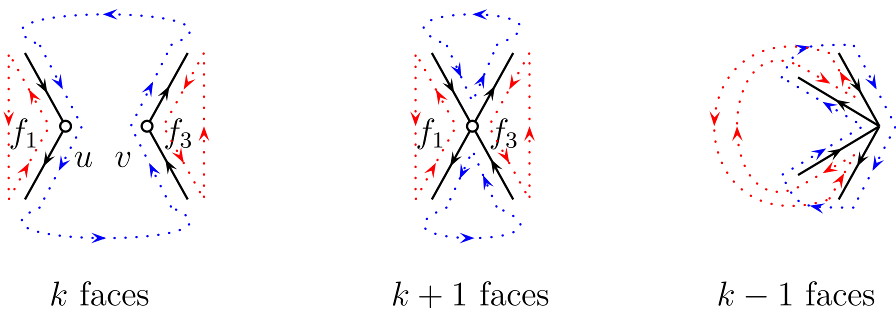


FIGURE 3.14. Case 2: Vertices $u$ and $v$ lie on a common face.

By Theorem 3.9, all $2^{n-1}$ directed embeddings of the cobblestone digraph $J_{n+1}$ have genus 0. Suppose that we consider the alternating rotation system for a cobblestone digraph with corresponding $S_0$-embedding as shown in Figure 3.11 as an initial position. We observe that the end-vertices lie on the same face, so we are in Case (2). If the rotations at all the interior vertices are changed, then we are also in Case (2). These two are the only alternating rotation systems that correspond to Case (2). From each of these two, we have one embedding in $S_0$ and one in $S_1$. The other $2^{n-1}-2$ embeddings are in Case (1). From each of them we have two embeddings in $S_1$. Therefore,

$$\begin{aligned}\Gamma_{BR_n}(x) &= 2(1+x) + \left(2^{n-1}-2\right)x \\ &= 2 + (2^n - 2)\,x\end{aligned}$$

□

3.6. **4-regular outerplanar digraphs.** We recall that the characteristic tree $T$ of a 4-regular outerplane digraph $D$ is the weak dual of the underlying graph of $D$. In Theorem 3.11, the 2-connectedness restriction is to defer the treatment of self-arcs, which are allowed under the premises of Theorem 3.12.

**Theorem 3.11.** *Let $D$ be a 2-connected 4-regular outerplanar digraph, and let $T$ be the characteristic tree for any planar embedding of $D$. Let $K_{1,i_1}, K_{1,i_2}, \ldots, K_{1,i_k}$ be the stars in a red-blue star decomposition of $T$. Then the directed genus polynomial of the digraph $D$ is given by*

$$\Gamma_D(x) \;=\; \prod_{j=1}^{k} \left(2 + (2^{i_j} - 2)x\right). \tag{3.11}$$

*Proof.* It follows from Theorem 3.3 and Theorem 3.5 that the characteristic tree $T$ of a 4-regular outerplane digraph has a red-blue star decomposition. As we have discussed in §3.2, the duals of these red-blue stars are bracelet graphs. Accordingly, the digraph $D$ is constructible by iterated amalgamation of directed bracelets, that is, of copies of $BB_r$ and $UB_s$, for $r, s \in \{i_1, i_2, \ldots, i_k\}$. However, Proposition 3.4 and Theorem 3.7 imply that, if any of these directed bracelets actually was a uni-directional bracelet $UB_s$, then the digraph $D$ would be non-planar. It follows from Theorem 3.10 that

$$\Gamma_D(x) \;=\; \prod_{j=1}^{k} \Gamma_{BB_{i_j}}(x) \;=\; \prod_{j=1}^{k} \left(2 + (2^{i_j} - 2)x\right). \qquad \square$$

**Example 3.1.** Let $D$ be the 2-edge connected 4-regular outerplanar digraph of Figure 3.15, and let us find the directed genus polynomial $\Gamma_D(x)$. According to Equation (3.11) of Theorem 3.11, we obtain

$$\Gamma_D(x) \;=\; \Gamma_{BB_3}(x)\Gamma_{BB_2}(x)^3 \;=\; (2+6x)(2+2x)^3$$

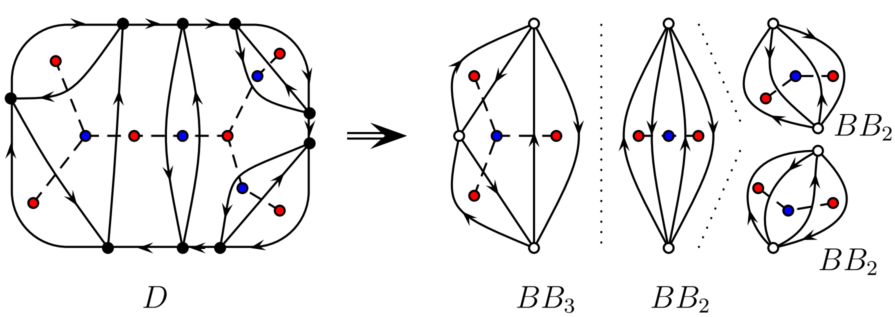


FIGURE 3.15. Bracelet decomposition of a 2-edge-connected 4-regular outerplanar digraph.

When a 4-regular outerplanar digraph $D$ has self-arcs, we use a normalized outerplane embedding to calculate its directed genus distribution. Whenever $D$ is not 2-connected, the weak dual of a normalized outerplane embedding of $D$ is a forest, rather than a tree, with a tree for each 2-connected component or self-arc, as illustrated by Figure 3.16. In the case of a self-arc, the tree is trivial. Here, in a stretch of our definition of red-blue star, we regard $K_1$ as the star $K_{1,0}$. The following theorem is implied by Theorem 3.8 and Theorem 3.11, and the details of the proof are omitted.

**Theorem 3.12.** *Let $D$ be a normalized 4-regular outerplane digraph, and let $T_1, T_1, \ldots, T_k$ be the components of its weak dual forest $F$. Let*

$$\{K_{1,i_1}, K_{1,i_2}, \ldots, K_{1,i_l}\} \quad \textit{with } (l \geq 1)$$

*be the union of the stars in the red-blue star decompositions of $T_1, T_1, \ldots, T_k$. Then the directed genus polynomial of $D$ is given by*

$$\Gamma_D(x) \;=\; 2^{k-1}\prod_{i=1}^{k}\prod_{j=1}^{l}\Gamma_{BB_{i_j}}(x) \tag{3.12}$$

*Where*

$$\Gamma_{BB_{i_j}}(x) \;=\; \begin{cases} 2+(2^{i_j}-2)x, & \textit{if } i_j > 1 \\ 1, & \textit{otherwise.} \end{cases}$$ □

**Example 3.2.** Let $D$ be the 4-regular outerplanar digraph of Figure 3.16, and let us find the directed genus polynomial $\Gamma_D(x)$. According to formula (3.12) of Theorem 3.12, we obtain

$$\Gamma_D(x) \;=\; 2^3\Gamma_{BB_1}(x)\Gamma_{BB_1}(x)\Gamma_{BB_3}(x)\Gamma_{BB_2}(x) \;=\; 8(2+6x)(2+2x)$$

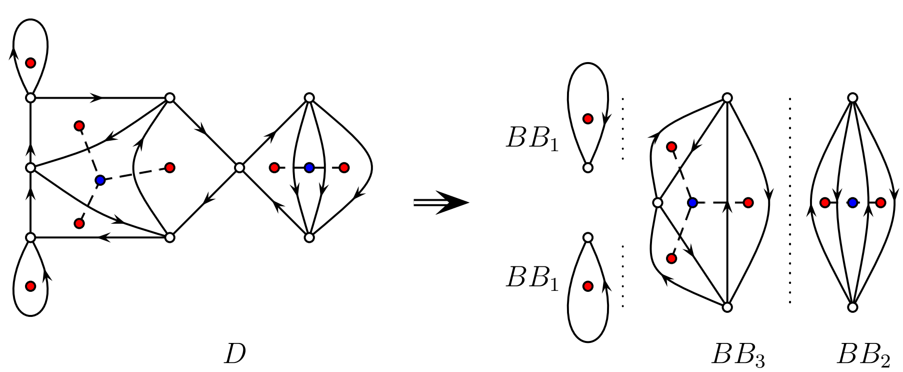


FIGURE 3.16. The directed genus polynomial of a normalized 4-regular outerplanar digraph $D$.

3.7. **Unimodality.** We show in this subsection that the directed genus distribution of a 4-regular outerplanar digraph is strongly unimodal. We recall that a real sequence $a_0\, a_1, \ldots, a_n$ is called ***unimodal*** if for some number $m$ such that $0 \leq m \leq n$, we have

$$a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots \geq a_n,$$

in which case, $m$ is called the ***mode*** of the sequence. Moreover, if for every $j$ such that $1 \leq j \leq n-1$, we have ${a_j}^2 \geq a_{j-1}a_{j+1}$, then the sequence is called ***log-concave*** or ***strongly unimodal***. Obviously, a strongly unimodal sequence is unimodal.

**Theorem 3.13.** *The directed genus distribution of a 4-regular outerplanar digraph is strongly unimodal.*

*Proof.* The directed genus distribution of the directed bracelet $BB_k$ is strongly unimodal, by Theorem 3.10. Accordingly, by Theorem 3.12, recalling that the convolution of two strongly unimodal sequences is strongly unimodal, we conclude that the directed genus distribution of a 4-regular outerplanar digraph is strongly unimodal. Indeed, from Theorem 3.12, we see that the genus polynomial is a product of binomials, which implies that it is real-rooted, a stronger condition than strong unimodality. □

## 4. Conclusions

Whereas the calculations of genus distributions, inaugurated by [8], are now quite numerous, the calculation of directed genus distributions, starting with [2], is a new venture. The genus distribution of 4-regular outerplanar graphs was calculated by Poshni et al.[15]. Here we have calculated the directed genus distribution for this same class of graphs, and proved that this distribution is strongly unimodal. In the course of so doing, we have developed methods that can be used to calculate directed genus distributions of various other Eulerian digraphs. For example, by Theorem 3.7 and Theorem 3.8, we can calculate (1) the directed genus distribution of the directed cross-connection $G \rightleftarrows H$ of any two digraphs $G$ and $H$ whose directed genus distributions are known, and (2) the directed genus distribution of the vertex amalgamation $(G, u) * (H, v)$ of any two digraphs $G$ and $H$ with two 2-valent roots whose directed genus distributions are known, with arbitrarily large degrees at vertices of $G$ and $H$ other than at the roots. We have also proved a splitting theorem for directed embeddings, which implies that the enumeration of digraph embeddings with any vertex degrees larger than four can be converted into a problem of enumerating embeddings of some 4-regular digraphs. We note that the *medial graph* of a graph is 4-regular, an additional reason for interest among topological graph theorists in 4-regular graphs.

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