Version: 13:02 May 28, 2014

LOG-CONCAVITY OF GENUS DISTRIBUTIONS OF RING-LIKE FAMILIES OF GRAPHS

JONATHAN L. GROSS, TOUFIK MANSOUR, AND THOMAS W. TUCKER

Abstract. We calculate genus distribution formulas for several families of *ring-like* graphs and prove that they are log-concave. The graphs in each of our ring-like families are obtained by applying the *self-bar-amalgamation operation* to the graphs in a *linear family* (linear in the sense of Stahl). That is, we join the two root-vertices of each graph in the linear family. Although log-concavity has been proved for many linear families of graphs, the only other ring-like sequence of graphs of rising maximum genus known to have log-concave genus distributions is the recently reinvestigated sequence of Ringel ladders. These new log-concavity results are further experimental evidence in support of the long-standing conjecture that the genus distribution of every graph is log-concave. Further evidence in support of the general conjecture is the proof herein that each partial genus distribution, relative to face-boundary walk incidence on root vertices, of an iterative bar-amalgamations of copies of various given graphs is log-concave, which is an unprecedented result for partitioned genus distributions. Our results are achieved via introduction of the concept of a *vectorized production matrix*, which seems likely to prove a highly useful operator in the theory of genus distributions and via a new general result on log-concavity.

1. Introduction

1.1. **Genus polynomials.** Our graphs are implicitly taken to be connected, and our ***graph embeddings*** are cellular and orientable. For general background in topological graph theory, see [GrTu87, BWGT09, PiPo14, Ch14]. Prior acquaintance with the concepts of *partitioned genus distribution* (abbreviated here as ***pgd***) and *production* (e.g., [GKP10, Gr11]) are necessary preparation for reading this paper. The exposition here is otherwise intended to be accessible both to graph theorists and to combinatorialists.

The number of combinatorially distinct embeddings of a graph G in the orientable surface of genus i is denoted by $g_i(G)$. The sequence $g_0(G)$, $g_1(G)$, $g_2(G)$, $\ldots$, is called the ***genus distribution*** of G. A genus distribution contains only finitely many positive numbers, and there are no zeros between the first and last positive numbers. The ***genus polynomial*** is the polynomial

$$\Gamma_G(z) \,=\, g_0(G) + g_1(G)z + g_2(G)z^2 + \ldots .$$

1.2. **Log-concave sequences.** A sequence $A = (a_k)_{k=0}^n$ is said to be ***nonnegative***, if $a_k \geq 0$ for all k. An element a_k is said to be an ***internal zero*** of A if $a_k = 0$ and if there exist indices i and j with $i < k < j$, such that $a_i a_j \neq 0$. If $a_{k-1}a_{k+1} \leq a_k^2$ for all k, then A is said to be ***log-concave***. If there exists an index h with $0 \leq h \leq n$ such that

$$a_0 \,\leq\, a_1 \,\leq\, \cdots \,\leq\, a_{h-1} \,\leq\, a_h \,\geq\, a_{h+1} \,\geq\, \cdots \,\geq\, a_n,$$

2000 *Mathematics Subject Classification.* 05A15, 05A20, 05C10.

then A is said to be ***unimodal***. It is well-known that any nonnegative log-concave sequence without internal zeros is unimodal, and that any nonnegative unimodal sequence has no internal zeros. A prior paper [GMTW13a] by the present authors provides additional contextual information regarding log-concavity and genus distributions.

For convenience, we sometimes abbreviate the phrase "log-concave genus distribution" as ***LCGD***. Proofs that closed-end ladders and doubled paths have LCGDs [FGS89] were based on explicit formulas for their genus distributions. Proof that bouquets have LCGDs [GRT89] was based on a recursion. A conjecture that all graphs have LCGDs was published by [GRT89].

We have elsewhere formulated various surgical operations on graph embeddings as simultaneous recurrences [FGS89] or, more recently, as transpositions of *production systems*. Stahl [Stah91, Stah97] has demonstrated the utility of representing such surgical operations by a matrix of polynomials, which are equivalent to our production systems. In a new kind of matrix that we develop herein, called a *vectorized production matrix*, each element of the matrix is a *pgd-vector*, in which each entry is a polynomial in z. Whereas a matrix has one vector as its operand, corresponding to a topological operation on the embeddings of a single graph, a vectorized matrix has two operands, corresponding to topological operations on the embeddings of a pair of graphs.

1.3. **Bar-amalgamations .** A general approach to understanding the structure of the set of embeddings of a given graph is to see what happens when an edge is added. Given two rooted connected graphs, (G, u) and (H, v), the ***bar-amalgamation*** $(G, u)\overline{*}H(, v)$ is the connected graph obtained by adding an edge uv to join the two roots. It is not hard to see [GrFu87] that if $g(x)$ and $h(x)$ are the genus polynomials of G and H, then the genus polynomial of the bar-amalgamation is $d_u d_v g(x)h(x)$, where the valences of roots u and v are d_u and d_v, respectively. Indeed, in any embedding of $(G, u)\overline{*}(H, v)$ of genus i, the edge uv must appear on some face twice, so the removal of edge uv gives disjoint embeddings of G and H, whose genera sum to i. This fact is very useful for constructing graphs with various specified properties. For example, if $g(x)$ and $h(x)$ are known to be real-rooted or log-concave, then so is the genus polynomial of the bar-amalgamation $(G, u)\overline{*}(H, v)$. Also, as long as the product $d_u d_v$ of the root-vertex valences remains the same, the bar amalgamations on different root pairs u, v have the same genus polynomial, even though the actual graphs may be non-isomorphic. Bar-amalgamations can also be used to construct graphs with specified genus ranges or with specified maximum, minimum, or average genus.

When we bar-amalgamate two doubly rooted graphs (G, u_1, u_2) and (H, v_1, v_2), the implicit meaning of the notation

$$(G, u_1, u_2)\overline{*}(H, v_1, v_2)$$

is that the new edge joins the second root u_2 of the first graph to the first root v_1 of the second graph. We might specify one of the other three possible bar-amalgamations by the notation

$$(G, u_1, u_2)\overline{*}_{u_i v_j}(H, v_1, v_2)$$

A more complicated kind of edge-addition is between two vertices u, v in the same connected graph. This is called a ***self-bar*** and is denoted $(G, u, v)\overline{*}$. Now there may be faces containing both u and v, so that the edge uv may be added either inside a face, which does not change the genus of the embedding, or between two different faces, which increases the genus by one. To determine the genus distribution of the resulting graph, we need information about the genus distribution of the original graph G partitioned according to the type of face-boundary walks that contain u and v. Very little is known about the nature of these partial genus distributions. For example, suppose that u and v have valence 2 in G and that we are interested in the genus distribution of all embeddings of G, where u and v are never in the same face-boundary walk and each lies on two distinct faces. It is unknown even whether such a distribution can have gaps. By way of contrast, it has long been known that the total genus distribution has no such gaps, and the proof is short and simple (e.g., see [GrTu87]).

Within this paper we shall see how partial genus distributions behave under bar-path amalgamation. We give explicit formulas for the partial genus distributions of bar-path using a repeated base graph (G, u, v) and the genus distribution when we complete the bar-path with a self-bar to form a bar-ring. We apply these formulas to six different base graphs (G, u, v) with log-concave genus distributions and log-concave partial genus distributions, and we show that all the partial genus distributions for the bar-path are log-concave and that the genus distribution for the bar-ring is log-concave. We also note in passing that the associated genus polynomials are easily seen to have at most three real roots. Since the degrees can be arbitrarily high for these examples, this lays to rest Stahl's hope [Stah97] (see also [LW07, Ch08, ChLi10]) that real-rootedness might play a critical role in understanding the log-concavity of genus polynomials.

1.4. **Bar-paths and bar-rings of copies of a graph.** The ***bar-path*** of a sequence of doubly vertex-rooted (disjoint) graphs

$$(G_1, u_1, v_1),\ (G_2, u_2, v_2),\ \ldots,\ (G_k, u_k, v_k)$$

is the graph obtained by joining the pairs $(v_1, u_2),\ (v_2, u_3),\ \ldots, (v_{n-1}, u_n)$ and taking u_1 and v_n as its root-vertices. The ***bar-ring*** of that same sequence is obtained from the bar-path by joining the vertices u_1 and v_n, by self-bar-amalgamation. We are presently focused on sequences of isomorphic copies of the same graph.

Our first example of bar-paths and bar-rings is based on the doubly rooted bouquet $(\ddot{B}_2, u, v)$, which is obtained from the bouquet B_2 by subdividing the two self-loops and taking the two new vertices u and v as roots. Figure 1.1 illustrates a bar-path of three copies of $(\ddot{B}_2, u, v)$.

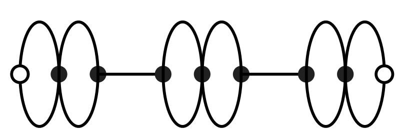

FIGURE 1.1. A bar-path of three copies of the bouquet $\ddot{B}_2$.

Figure 1.2 illustrates a bar-ring of three copies of $(\ddot{B}_2, u, v)$.

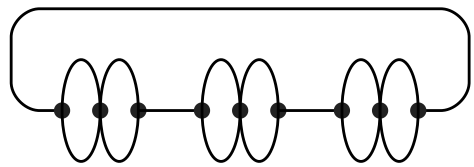

FIGURE 1.2. A bar-ring of three copies of the bouquet $\ddot{B}_2$.

1.5. **Context for this paper.** The algebraization of topological graph theory began with the Ringel-Youngs solution [RY68] to the Heawood map-coloring problem. There are two enumerative branches of topological graph theory. The map-theoretic branch, which dates back to the work of Tutte [Tut63] on planar maps, is concerned with all of the maps on a fixed surface. Foundations for enumerative work on higher genus surfaces were established by Jones and Singerman [JS78]. A survey of map theory is given by Nedela and Škoviera [NeSk14]. The present paper is in the complementary enumerative branch, initiated by Gross and Furst [GrFu87], whose concern is inventories of all the imbeddings of a fixed graph.

1.6. **Outline of this paper.** This paper is organized as follows. Section 2 describes a representation of partitioning of the genus distribution into ten parts as a *pgd-vector*. Section 3 describes how *productions* are used to describe the effect of a graph operation on the pgd-vector and introduces the concept of a *vectorized production matrix*. Theorem 3.11 is a new general result on constructing log-concave sequences. Subsection 4.1 uses Theorem 3.11 to prove that bar-rings of copies of the bouquet $\ddot{B}_2$ are log-concave. Subsequent subsections do the same for bar-rings of copies of various other graphs. We include some proofs that the partial genus polynomials (with respect to incidence of face-boundary walks on the roots) are log-concave. Section 5 concludes the paper with some research problems.

2. Partitioned Genus Distributions

A fundamental strategy in the calculation of genus distributions, from the outset [FGS89], has been to partition a genus distribution according to the incidence of face-boundary walks on one or more roots. We abbreviate "face-boundary walk" as ***fb-walk***. For a graph (G, u, s) with two 2-valent root-vertices, we can partition the number $g_i(G)$ into the following four parts:

- $dd_i(G)$**:** the number of embeddings of (G, u, v) in the surface S_i such that two distinct fb-walks are incident on root u and two on root v;
- $ds_i(G)$**:** the number of embeddings in S_i such that two distinct fb-walks are incident on root u and only one on root v;
- $sd_i(G)$**:** the number of embeddings in S_i such that one fb-walk is twice incident on root u and two distinct fb-walks are incident on root v;
- $ss_i(G)$**:** the number of embeddings in S_i such that one fb-walk is twice incident on root u and one is twice incident on root v.

Clearly, $g_i(G) = dd_i(G) + ds_i(G) + sd_i(G) + ss_i(G)$. Each of the four parts is sub-partitioned into ***embedding subtypes***, which we simply call ***embedding types***, when the context at hand requires them. For instance,

$$dd(G) \;=\; dd^0(G) \,+\, dd'(G) \,+\, dd''(G)$$

We define the ***pgd-vector*** of the graph(G, u, v) to be the following vector of ten embedding types, given here in what we call the ***canonical ordering***:

$$\big(dd^0(G)\ \ dd'(G)\ \ dd''(G)\ \ ds^0(G)\ \ ds'(G)\ \ sd^0(G)\ \ sd'(G)\ \ ss^0(G)\ \ ss^1(G)\ \ ss^2(G)\big) \tag{2.1}$$

- **$dd_i^0(G)$:** the number of type-dd embeddings of (G, u, v) in S_i such that neither fb-walk incident at root u is incident at root v;
- **$dd_i'(G)$:** the number of type-dd embeddings in S_i such that one fb-walk incident at root u is incident at root v;
- **$dd_i''(G)$:** the number of type-dd embeddings in S_i such that both fb-walks incident at root u are incident at root v;
- **$ds_i^0(G)$:** the number of type-ds embeddings in S_i such that neither fb-walk incident at root u is incident at root v;
- **$ds_i'(G)$:** the number of type-ds embeddings in S_i such that one fb-walk incident at root u is incident at root v;
- **$sd_i^0(G)$:** the number of type-sd embeddings in S_i such that the fb-walk incident at root u is not incident on root v;
- **$sd_i'(G)$:** the number of type-sd embeddings in S_i such that the fb-walk at root u is also incident at root v;
- **$ss_i^0(G)$:** the number of type-ss embeddings in S_i such that the fb-walk incident at root u is not incident on root v;
- **$ss_i^1(G)$:** the number of type-ss embeddings in S_i such that the fb-walk incident at root u is incident at root v, and the incident pattern is $uuvv$;
- **$ss_i^2(G)$:** the number of type-ss embeddings in S_i such that the fb-walk incident at root u is incident at root v, and the incident pattern is $uvuv$.

Each coordinate of the pgd-vector (2.1) is a polynomial in z, analogous to $\Gamma_G(z)$. For instance,

$$ds'(G) = ds_0'(G) + ds_1'(G)z + ds_2'(G)z^2 + \cdots .$$

3. Bar-Amalgamations and Vectorized Production Matrices

As mentioned in the introduction, we are presently using a *vectorized production matrix*, in which the entries are of the same algebraic type as pgd-vectors. We discuss the multiplication rules for a vectorized matrix immediately after the derivation of Corollary 3.8, which indicates how it is used, and further within the proof of Theorem 3.9. Concrete applications are given in Section 4.

Since the graphs under consideration have two 2-valent roots, their genus distributions are partitioned into 10 parts, as described in Section 2. Accordingly, the production matrix is 10×10. We regard the production matrix as having row labels and column labels according to the canonical ordering. The following six propositions relieve us of the task of justifying 100 productions one at a time. All six can be proved by face-tracing.

Proposition 3.1. *Let (G, a, b) and (H, u, v) be graphs, both with two 2-valent root-vertices, with embeddings of types wx_i and yz_j, respectively. Then each of the four extensions of those embeddings to the bar-amalgamation $(G, a, b)\overline{*}(H, u, v)$ has embedding type wz_{i+j}.*

Proposition 3.2. *Let (G, a, b) and (H, u, v) be graphs, both with two 2-valent root-vertices, with embeddings such that at least one has subtype 0 (i.e., the superscript is 0). Then each of the four extensions of those embeddings to the bar-amalgamation $(G, a, b)\overline{*}(H, u, v)$ has subtype 0.*

As a result of Propositions 3.1 and 3.2, we know the entries in every row and column of the vectorized production matrix for each subtype with superscript 0. That is, we now know 64 of the 100 entries in the matrix.

Relative to the an embedding of the bar-amalgamation $(G, a, b)\overline{*}(H, u, v)$, we call the types of the induced embeddings of (G, a, b) at root b and of (H, u, v) at root u the ***interior types***.

Proposition 3.3. *Let (G, a, b) and (H, u, v) be embedded graphs, both with two 2-valent root-vertices, such that*

(a) neither embedding has subtype 0;
(b) both embeddings have interior type s.

Then each of the four extensions of those embeddings to the bar-amalgamation $(G, a, b)\overline{*}(H, u, v)$ has subtype $'$ or 1.

Proposition 3.4. *Let (G, a, b) and (H, u, v) be embedded graphs, both with two 2-valent root-vertices, such that*

(a) neither embedding has subtype 0;
(b) at least one of the embeddings has subtype dd'';
(c) unless both embeddings have subtype dd'', the other embedding has interior type s.

Then each of the four extensions of those embeddings to the bar-amalgamation $(G, a, b)\overline{*}(H, u, v)$ has subtype $'$ or 1.

Together, Propositions 3.3 and Proposition 3.4 specify another 16 entries in the vectorized production matrix. Proposition 3.5 specifies another four entries, and Proposition 3.6 specifies the remaining entries.

Proposition 3.5. *Let (G, a, b) and (H, u, v) be embedded graphs, both with two 2-valent root-vertices, such that*

(a) both embeddings have subtype $'$;
(b) both embeddings have interior type d.

Then three of the four extensions of those embeddings to the bar-amalgamation $(G, a, b)\overline{*}(H, u, v)$ have subtype 0, and the other has subtype $'$ or 1.

Proposition 3.6. *Let (G, a, b) and (H, u, v) be embedded graphs, both with two 2-valent root-vertices, such that the embeddings do not meet the premises of any of the Propositions 3.2, 3.3, 3.4, or 3.5. Then two of the four extensions of those embeddings to the bar-amalgamation $(G, a, b)\overline{*}(H, u, v)$ have subtype 0, and the other two have subtype $'$ or 1.*

Theorem 3.7 summarizes Propositions 3.1 to 3.6.

Theorem 3.7. *Let (G, a, b) and (H, u, v) be graphs, both with two 2-valent root-vertices. The vectorized production matrix for the operation $(G, a, b)\overline{*}(H, u, v)$ of bar-amalgamation is as follows:*

$$
\mathbf{M} = \begin{array}{c|cccccccccc}
 & dd^0 & dd' & dd'' & ds^0 & ds' & sd^0 & sd' & ss^0 & ss^1 & ss^2 \\
\hline
dd^0 & 4dd^0 & 4dd^0 & 4dd^0 & 4ds^0 & 4ds^0 & 4dd^0 & 4dd^0 & 4ds^0 & 4ds^0 & 4ds^0 \\
dd' & 4dd^0 & 3dd^0{+}dd' & 2dd^0{+}2dd' & 4ds^0 & 3ds^0{+}ds' & 4dd^0 & 2dd^0{+}2dd' & 4ds^0 & 2ds^0{+}2ds' & 2ds^0{+}2ds' \\
dd'' & 4dd^0 & 2dd^0{+}2dd' & 4dd' & 4ds^0 & 2ds^0{+}2ds' & 4dd^0 & 4dd' & 4ds^0 & 4ds' & 4ds' \\
ds^0 & 4dd^0 & 4dd^0 & 4dd^0 & 4ds^0 & 4ds^0 & 4dd^0 & 4dd^0 & 4ds^0 & 4ds^0 & 4ds^0 \\
ds' & 4dd^0 & 2dd^0{+}2dd' & 4dd' & 4ds^0 & 2ds^0{+}2ds' & 4dd^0 & 4dd' & 4ds^0 & 4ds' & 4ds' \\
sd^0 & 4sd^0 & 4sd^0 & 4sd^0 & 4ss^0 & 4ss^0 & 4sd^0 & 4sd^0 & 4ss^0 & 4ss^0 & 4ss^0 \\
sd' & 4sd^0 & 3sd^0{+}sd' & 2sd^0{+}2sd' & 4ss^0 & 3ss^0{+}ss^1 & 4sd^0 & 2sd^0{+}2sd' & 4ss^0 & 2ss^0{+}2ss^1 & 2ss^0{+}2ss^1 \\
ss^0 & 4sd^0 & 4sd^0 & 4sd^0 & 4ss^0 & 4ss^0 & 4sd^0 & 4sd^0 & 4ss^0 & 4ss^0 & 4ss^0 \\
ss^1 & 4sd^0 & 2sd^0{+}2sd' & 4sd' & 4ss^0 & 2ss^0{+}2ss^1 & 4sd^0 & 4sd' & 4ss^0 & 4ss^1 & 4ss^1 \\
ss^2 & 4sd^0 & 2sd^0{+}2sd' & 4sd' & 4ss^0 & 2ss^0{+}2ss^1 & 4sd^0 & 4sd' & 4ss^0 & 4ss^1 & 4ss^1
\end{array} \tag{3.1}
$$

3.1. **Vectorized production matrices.** In Expression (3.1) for the vectorized production matrix M, each entry is a pgd-vector for the category of rooted graph under consideration. In this case, with two 2-valent root-vertices, it is a 10-coordinate vector, in which each coordinate corresponds to the partial genus distributions for that category. We first define the elementary pgd-vectors:

(3.2)

abbreviation	elementary pgd-vector
dd^0	$\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$
dd'	$\begin{pmatrix}0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$
dd''	$\begin{pmatrix}0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$
$\vdots$	$\vdots$
ss^2	$\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$

Thus, here are the meanings of some entries for matrix M in Expression (3.1).

abbreviation	pgd-vector
$4dd^0$	$\begin{pmatrix}4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$
$3dd^0 + dd'$	$\begin{pmatrix}3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$
$2ss^0 + 2ss^1$	$\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0\end{pmatrix}$

We have seen that each of the coordinates of a pgd-vector for a graph is a univariate polynomial, in which the power of the indeterminate z represents the genus of the embedding surface. In a ***vectorized production matrix*** (abbr. ***vp-matrix***), in which each entry of the matrix is a pgd-vector, the powers of the indeterminate z in a coordinate represent increments of genus, when positive, or decrements, when negative. Since a bar-amalgamation does not involve addition or deletion of handles, the polynomials of the corresponding vp-matrix (3.1) appear as constant terms.

An $n \times n$ ***vectorized production matrix*** M has two operands, which are vectors with A and B, each with n coordinates, each of which is a polynomial in a single indeterminate. The result of the operation AMB is a vector C with n coordinates. Instead of doing matrix multiplication AMB in two steps as usual, it may be simpler to apply the following one-step rule, where we process the entries of M row-by-row and column-by-column:

$$C = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j m_{ij} \tag{3.3}$$

We observe that each summand in Equation (3.3) is the product of two univariate polynomials a_i and b_j with a pdg-vector m_{ij}, which means that each coordinate of the pgd-vector m_{ij} is multiplied by the product of the two polynomials. Accordingly, the sum C has the type of a pgd-vector.

Corollary 3.8. *Let X_G be the pgd-vector of a doubly rooted graph (G, u, v) such that root-vertices u and v are both 2-valent. Then the pgd-vector of a bar-path of n copies of (G, u, v)*

$$\overbrace{(G,u,v)\overline{*}(G,u,v)\overline{*}\cdots\overline{*}(G,u,v)}^{n \text{ copies of } (G,u,v)} \tag{3.4}$$

is

$$\overbrace{X_G M X_G M X_G \cdots M X_G}^{n \text{ copies of } X_G \text{ and } n-1 \text{ of } M} \tag{3.5}$$

where the vp-matrix M is given by (3.1).

Since the vp-matrix M is an operator with two arguments, the formal meaning of Expression (3.5) is

$$(((X_G M X_G^t) M X_G^t) \cdots M X_G^t). \tag{3.6}$$

It is sometimes convenient to omit notational distinction between a row vector and a column vector, because the evaluation need not proceed in the order given by Expression (3.6). When a pgd-vector is to the left of a vectorized production matrix, it is processed like a row vector; when to the right, it is processed like a column vector. Further discussion of the evaluation of Expression (3.5) is given with the aid of an example, within the proof of Theorem 3.9, which gives the general formula for the pgd-vector of a bar-path of copies of a generic graph (G, u, v) with two 2-valent roots.

Theorem 3.9. *Let $X_G = (x_1, x_2, \ldots, x_{10})$ be the pgd-vector of a doubly rooted graph (G, u, v) such that the root-vertices u and v are both 2-valent. Denote the bar-path*

$$\overbrace{(G,u,v)\overline{*}(G,u,v)\overline{*}\cdots\overline{*}(G,u,v)}^{n \text{ copies of } (G,u,v)}$$

by H_n. Then the pgd-vector of the graph H_n is given by $h = \big(h_1, h_2, 0, h_4, h_5, h_6, h_7, h_8, h_9, 0\big)$, for $n \geq 2$, where

$h_1 = 4^{n-1}(x_1 + x_2 + x_3 + x_6 + x_7)(x_1 + x_2 + \cdots + x_5)\xi^{n-2} - (x_2 + 2x_3 + 2x_7)(x_2 + 2x_3 + 2x_5)\psi^{n-2}$,
$h_2 = (x_2 + 2x_3 + 2x_7)(x_2 + 2x_3 + 2x_5)\psi^{n-2}$,
$h_4 = 4^{n-1}(x_4 + x_5 + x_8 + x_9 + x_{10})(x_1 + x_2 + \cdots + x_5)\xi^{n-2} - (x_5 + 2x_9 + 2x_{10})(x_2 + 2x_3 + 2x_5)\psi^{n-2}$,
$h_5 = (x_5 + 2x_9 + 2x_{10})(x_2 + 2x_3 + 2x_5)\psi^{n-2}$,
$h_6 = 4^{n-1}(x_1 + x_2 + x_3 + x_6 + x_7)(x_6 + x_7 + x_8 + x_9 + x_{10})\xi^{n-2} - (x_7 + 2x_9 + 2x_{10})(2x_7 + x_2 + 2x_3)\psi^{n-2}$,
$h_7 = (x_7 + 2x_9 + 2x_{10})(x_2 + 2x_3 + 2x_7)\psi^{n-2}$,
$h_8 = 4^{n-1}(x_4 + x_9 + x_{10} + x_8 + x_5)(x_6 + x_7 + x_8 + x_9 + x_{10})\xi^{n-2} - (x_7 + 2x_9 + 2x_{10})(x_5 + 2x_9 + 2x_{10})\psi^{n-2}$,
$h_9 = (x_7 + 2x_9 + 2x_{10})(x_5 + 2x_9 + 2x_{10})\psi^{n-2}$,

with genus polynomial $\xi = x_1 + x_2 + \cdots + x_{10}$ for G and with $\psi = x_2 + 2x_3 + 2x_5 + 2x_7 + 4x_9 + 4x_{10}$.

Proof. We observe that the quantity $h_1 + \cdots + h_{10} = g(z)$ is the genus polynomial of the the bar-path H_n. Since ξ is the genus polynomial of the base graph (G, u, v), we should have

$$g(z) \;=\; h_1 + \cdots + h_{10} \;=\; 4^{n-1}\xi^n,$$

because the genus polynomial of a bar-amalgamation is the product of the genus polynomials of the amalgamands, with additional factors for the degrees. Indeed, the formulas for $h_1, \ldots h_{10}$ come in four non-zero consecutive pairs, where the second member of each pair cancels a negatively signed term in the first member. Thus, when we add them, we get only the sum of the positive terms from h_1, h_4, h_6, h_8. When we add the positive terms from h_1 and h_4, we get $4^{n-1}\xi(x_1 + x_2 + x_3 + x_4 + x_5)\xi^{n-2}$, and when we add the positive terms from h_6 and h_8, we get $4^{n-1}\xi(x_6 + x_7 + x_8 + x_9 + x_{10})\xi^{n-2}$. Thus the overall sum is $g(z) = 4^{n-1}\xi\xi\xi^{n-2} = 4^{n-1}\xi^n$.

We can confirm, similarly, that $h_2 + 2h_3 + 2h_5 + 2h_7 + 4h_9 + 4h_{10} = \psi^n$.

We observe that any face containing the leftmost root u and the rightmost root v must use each of the bar-amalgamating edges twice, so there cannot be two such faces. Thus, $h_2 = 0$. Similarly, any face containing u and v twice cannot contain them in the cyclic order $uvuv$, since this would entail using each bar-amalgamating edge four times. Accordingly, $h_{10} = 0$.

We now derive the formulas. By the definitions, the vector h is given by $(X_G M)^{n-1} X_G$. Direct calculations show that the product matrix $X_G M$ is given by

$$X_G M = \begin{pmatrix}
x_{44444} & x_{43242} & x_{42040} & 0 & 0 & x_{44444} & x_{42040} & 0 & 0 & 0 \\
0 & x_{01202} & x_{02404} & 0 & 0 & 0 & x_{02404} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_{44444} & x_{43242} & 0 & 0 & x_{44444} & x_{42040} & x_{42040} \\
0 & 0 & 0 & 0 & x_{01202} & 0 & 0 & 0 & x_{02404} & x_{02404} \\
y_{44444} & y_{43422} & y_{42400} & 0 & 0 & y_{44444} & y_{42400} & 0 & 0 & 0 \\
0 & y_{01022} & y_{02044} & 0 & 0 & 0 & y_{02044} & 0 & 0 & 0 \\
0 & 0 & 0 & y_{44444} & y_{43422} & 0 & 0 & y_{44444} & y_{42400} & y_{42400} \\
0 & 0 & 0 & 0 & y_{01022} & 0 & 0 & 0 & y_{02044} & y_{02044} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{3.7}$$

Our representation (3.7) of the the product matrix $X_G M$ employs the following abbreviations:

$$x_{a_1 a_2 \cdots a_5} = \sum_{i=1}^{5} a_i x_i; \qquad y_{a_1 a_2 \cdots a_5} = \sum_{i=1}^{5} a_i x_{5+i}.$$

Each coordinate x_i in the pgd-vector $X_G = (x_1, x_2, \ldots, x_{10})$ is a polynomial in the ring $\mathbb{Z}[z]$. Each entry in a vp-matrix like M is a linear combination of elementary pgd-vectors (which are listed in (3.2)) with coefficients in $\mathbb{Z}[z]$, and, thus, it has the type of a pgd-vector. Each subscript a_i in the abbreviations $x_{a_1 a_2 \cdots a_5}$ and $y_{a_1 a_2 \cdots a_5}$ is a polynomial in $\mathbb{Z}[z]$. It follows that a sum $\sum_{i=1}^{5} a_i x_i$ or $\sum_{i=1}^{5} a_i x_{5+i}$ is a polynomial in $\mathbb{Z}[z]$.

We now define the generating function

$$F(t) \;=\; \sum_{n \geq 1} (X_G M)^{n-1} t^{n-1}.$$

Accordingly, we have

$$F(t) \;=\; (I - tX_G M)^{-1},$$

where I is the 10×10 unit matrix. After some rather tedious algebraic manipulations (with help from a mathematical programming system, for instance, Maple), we see that the coefficient of t^n, for $n \geq 2$, in the k^{th} coordinate of the vector $F(t)X_G$ is given by h_k. (Note that the explicit formulas for coefficients of the matrix $F(t)$ and the vector $F(t)X_G$ are too long to present here.) $\square$

Expression (3.7) illustrates how we may regard $X_G M$ either as a row vector whose entries are column vectors, or as a matrix whose entries are polynomials. Theorem 3.10 is highly useful in Section 4.

Theorem 3.10. *The genus polynomial for the bar-ring of n copies of (G, u, v) is*

$$z4^n\xi^n + (1-z)\psi^n,$$

where ξ and ψ are given in Theorem 3.9.

Proof. It follows immediately from Theorem 4.1 of [GMTW14b] that the (non-partitioned) genus polynomial of the graph $\overline{*}_{uv}(G, u, v)$, obtained by self-bar-amalgamation, is given by the dot product of the pgd-vector $h = \big(h_1, h_2, 0, h_4, h_5, h_6, h_7, h_8, h_9, 0\big)$, and

$$B \;=\; \begin{pmatrix} 4z & 1+3z & 2+2z & 4z & 2+2z & 4z & 2+2z & 4z & 4 & 4 \end{pmatrix}. \tag{3.8}$$

It is easily checked that one part of the dot product consists of $4z$ times the sum of the positive terms for h_1, h_4, h_6 and h_8; as we have already observed, this sum is $4^{n-1}\psi^n$, namely the genus polynomial of the bar path. The second part of the dot products is $(1-z)h_2 + (2-2z)h_5 + (2-2z)h_7 + (4-4z)h_9$; again, as we have observed, $h_2 + 2h_5 + 2h_7 + 4h_9 = \psi^n$ (since $h_3 = h_{10} = 0$). Thus the dot product is $z4^n\xi^n + (1-z)\psi^n$. $\square$

In Section 4, we study the log-concavity of the genus polynomials of several bar-paths and bar-rings. In all our applications of Theorem 3.10 to these graphs, the genus polynomial ξ is linear, which implies that ψ is also linear (and, in fact, the coefficients of 4ξ dominate those of ψ). Under these conditions, it is easily verified by elementary calculus that the polynomial $z(4\xi^n) = (z-1)\psi^n$ has at most three real roots. Hence, all our example applications for $n > 2$ have complex roots (see [Stah97, LW07]).

The following general result about log-concavity will be used on a variety of examples.

Theorem 3.11. *Let $0 \leq r \leq \frac{1}{2}$, $\frac{1}{4} \leq s \leq 1$, and $t \geq 2$ be real numbers such that $2st \geq 3$ and $n \geq \max\{m, \frac{1}{r}\}$, where m is the minimum number such that $1.5^n \geq \frac{n^2}{r}$, for all $n \geq m$. Then the polynomial*

$$p(x) \;=\; rx(sx+1)^n t^n \;+\; (1-rx)(x+1)^n$$

is log-concave.

Proof. Let a_j be the coefficient of x^j in the polynomial $p(x)$. Then

$$a_j \;=\; r\binom{n}{j-1}(t^n s^{j-1} - 1) \;+\; \binom{n}{j}.$$

We define

$$f_j \;=\; \frac{j!(j+1)!(n+2-j)!(n+1-j)!}{n!^2}(a_j^2 - a_{j-1}a_{j+1}).$$

In order to show that $p(x)$ is a log-concave polynomial, we prove that $f_j \geq 0$, for all $j = 0, 1, \ldots, n+1$. Since $f_0 = (n+2)(n+1)^2$, we have $f_0 \geq 0$. Now let us show that $f_1, f_2 \geq 0$:

$$f_1 \;=\; (n+1)((n-r)^2 + n + r^2) \;-\; 2(n+1)r(n(s-2) \;+\; 2r)t^n + 2r^2(n+1)t^{2n}.$$

By using the fact that $t^n \geq 2^n \geq 2n^2$, we obtain that

$$f_1 \;\geq\; (n+1)((n-r)^2 + n + r^2) + 2r(n+1)(n - 2r + n(2rn + 1 - s))t^n \geq 0,$$

which gives the case $j = 1$. When $j = 2$, the coefficient of t^{2n} in f_2 is $6r^2s^2(n+1) > 0$, which, by replacing t^{2n} by n^2t^n/r, implies that $f_2 \geq f_{2,0} + f_{2,1}t^n$, where

$$\begin{aligned}
f_{2,0} \;&=\; 6(1 - 2r + 3r^2) + (3 + 8(1 - 2r) + 6r^2)(n - 2) + 2(3 - 2r)(n - 2)^2 + (n - 2)^3, \\
f_{2,1} \;&=\; 6r(r + 10s^2 + 4s(1 - 2r) + rs^2) + 2r(18s(1 - 2r) + 39s^2 - 1 + 3r(s^2 + 8s + 1))(n - 2) \\
&\quad + 2r(18s^2 + 6s - 1)(n - 2)^2 + 6rs^2(n - 2)^3.
\end{aligned}$$

From the inequalities $0 \leq r \leq \frac{1}{2}$, $\frac{1}{4} \leq s \leq 1$ and $n \geq m \geq 2$, we infer that $f_2 \geq 0$. Therefore, we can assume that $j \geq 3$.

Note that the coefficient of t^{2n} in f_j is given by $jr^2s^{2j-2}(n+1)(j+1) > 0$. On the other hand,

$$t^n \;=\; t^{n+2-j}t^{j-2} \;\geq\; \frac{2^{n+2-j}1.5^{j-2}}{s^{j-2}} \;\geq\; \frac{n^2}{rs^{j-2}}.$$

Thus, by replacing t^{2n} by $\frac{n^2t^n}{rs^{j-2}}$ in the expression $f_j = f_{j,0} + f_{j,1}t^n + f_{j,2}t^{2n}$, we obtain

$$f_j \;\geq\; g_j \;=\; g_{j,0} + g_{j,1}(n + 1 - j) + g_{j,2}(n + 1 - j)^2 + g_{j,3}(n + 1 - j)^3,$$

where (in each step we perform some rather tedious algebraic manipulations)

$$g_{j,3} \;=\; 1 + rj(j + 1)s^jt^n \geq 0,$$

$$\begin{aligned}
g_{j,2} \;&=\; j + 1 + rj\left(s^{j-2}t^n - 2 + s^{j-2}(24s^2 + 8s - 3 + (8s^2 + 2s - 1)(j - 3) + 3s^2(j - 3)^2)t^n\right) \\
&\geq\; j + 1 + rj\left(1.5^n - 2 + s^{j-2}(24s^2 + 8s - 3 + (8s^2 + 2s - 1)(j - 3) + 3s^2(j - 3)^2)t^n\right) \geq 0,
\end{aligned}$$

$$\begin{aligned}
g_{j,1} \;&=\; j + rj((3(j - 3)^3 + 8(j - 3)^2 + 13(j - 3) + 46)s^jt^n + 2j(s^jt^n - 1) + (j - 1)s^{j-2}t^n) \\
&\quad + rj(j + 1)(2s + 2s^2 - r(1 + s^2))s^{j-2}t^n + r^2j^2(j + 1)(s - 1)2s^{j-2}t^n + r^2j(j + 1) \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
g_{j,0} \;&=\; s^jt^nj^2(j + 1)(j - 1)^2r + j^2(1 - 2t^ns^j - 1)(j + 1)r^2 \\
&\geq\; 2s^jt^nj^2(j + 1)(j - 1)2r^2 + j^2(1 - 2t^ns^j - 1)(j + 1)r^2 \\
&=\; j^2(j + 1)r^2((2(j - 3)^2s + 8(j - 3)s + 2(4s - 1))t^ns^{j-1} + 1) \geq 0.
\end{aligned}$$

Therefore, $g_j \geq 0$ for all $j = 3, 4, \ldots, n+1$. Hence, $f_j \geq 0$ for all $j = 0, 1, \ldots, n+1$, which completes the proof. $\square$

Theorem 3.12. *Let $1 \leq r$, $\frac{1}{4} \leq s \leq 1$ and $t \geq 2$ such that $st \geq 3/2$. For all $n \geq 8$, the polynomial*

$$q(x) \;=\; r(sx + 1)^nt^n - (x + 1)^n$$

is log-concave.

Proof. Let a_j be the coefficient of x^j in the polynomial $q(x)$. Let

$$b_j \;=\; (a_j^2 - a_{j+1}a_{j-1})\frac{j!(j+1)!(n+1-j)!(n-j)!}{n!^2}$$

Then

$$b_j \;=\; rt^n s^j(rnt^n s^j + rt^n s^j - 2jn + 2j^2 - 2n - 2 + js(n-j)) + 1 + n + j(n-j)rt^n s^{j-1}.$$

From the inequalities $1 \le r$, $\frac{1}{4} \le s \le 1$, $t \ge 2$, $st \ge 3/2$ and $0 \le j \le n$, we obtain

$$b_j \;\ge\; rt^n s^j((n+1)1.5^n - 2n^2 - 2n - 2) \;\ge\; rt^n s^j(2.25n(n+1) - 2n^2 - 2n - 2) \;\ge\; 0,$$

which proves that the polynomial $q(x)$ is a log-concave, as required. □

Using arguments similar to those in the proofs of the theorems above, we can prove the following additional result.

Theorem 3.13. *Let $r \ge 1$, $\frac{1}{2} \le s \le 1$ and $t \ge 4$. For all $n \ge 2$, the polynomials*

$$r(4+sz)(1+z/4)^n t^n - (1+sz)(1+z)^n$$

and

$$r(4+sz)^2(1+z/4)^n t^n - (1+sz)^2(1+z)^n$$

are log-concave.

4. Applications

In this section, we apply the results of Section 3 to six different examples, which are bar-paths and bar-rings of bouquets, dipoles, doubled paths, ladders, and the complete graph K_4.

4.1. **Bar-paths and bar-rings of bouquets.** It is known [GrKlRi93] that the smallest positive value of average genus for a graph is 1/3, and that the unique 2-edge-connected graph with average genus 1/3 is the bouquet B_2. The maximum genus of B_2 is 1. Since the average genus of a bar-amalgamation of two graphs is the sum of their average genera, and since the maximum genus is the sum of their genus maxima, it follows that the average genus and maximum genus of a bar-path of n copies of $\ddot{B}_2$ are $n/3$ and n, respectively. By way of contrast, the available anecdotal evidence supports the folk conjecture that the average genus of a graph tends to be much nearer to the maximum genus than to the minimum genus. In this regard, the bar-paths of copies of $\ddot{B}_2$ seem to be exceptions.

Theorem 4.1. *The pgd-vector of a bar-path of n copies of the bouquet $(\ddot{B}_2, u, v)$ is given by*

$$(X_{\ddot{B}_2} M)^{n-1} X_{\ddot{B}_2} \tag{4.1}$$

where

$$X_{\ddot{B}_2} \;=\; \begin{pmatrix} 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2z \end{pmatrix} \tag{4.2}$$

and where M is the vectorized production matrix (3.1)

Proof. Formula (4.1) is a simple application of Corollary 3.8. The pgd-vector $X_{\ddot{B}_2}$ is easily calculated by face-tracings. □

Theorem 4.2. *The pgd-vector of a bar-path of n copies of the bouquet $(\ddot{B}_2, u, v)$ is given, for $n \geq 2$, by*

$$\begin{aligned}(4.3)\quad \big(4^n\xi^{n-2} - 16\psi^{n-2} \quad 16\psi^{n-2} \quad 0 \quad 2z4^n\xi^{n-2} - 16z\psi^{n-2} \quad 16z\psi^{n-2} \\ -2z4^n\xi^{n-2} - 16z\psi^{n-2} \quad 16z\psi^{n-2} \quad z^24^n\xi^{n-2} - 16z^2\psi^{n-2} \quad 16z^2\psi^{n-2} \quad 0\big),\end{aligned}$$

where $\xi = 4 + 2z$ and $\psi = 4 + 8z$. Each coordinate is a log-concave polynomial.

Proof. The pgd-vector (4.3) is calculated directly from Theorem 3.9. In order to show that each coordinate of (4.3) is a log-concave polynomial, we have to show that the following two polynomials are log-concave:

$$f(z) = (1 + 2z)^{n-2} \text{ and } g(z) = c2^{n-2}(2 + z)^{n-2} - (1 + 2z)^{n-2}, \text{ with } c = 8, 4, 2.$$

Since the binomial coefficients are a log-concave sequence, we see that $f(z)$ is a log-concave polynomial. Applying Theorem 3.12 with $g(z/2) = c4^{n-2}(0.25z + 1)^{n-2} - (1 + z)^{n-2}$ completes the proof. □

Theorem 4.3. *The (non-partitioned) genus distribution of a bar-ring of n copies of the bouquet $\ddot{B}_2$, is given by the polynomial*

$$f(z) \;=\; z4^n(4 + 2z)^n + (1 - z)(4 + 8z)^n.$$

This polynomial is log-concave.

Proof. The formula follows directly from Theorem 3.10 with $\xi = 4 + 2z$ and $\psi = 4 + 4(2z) = 4 + 8z$. Note that

$$4^{-n}f(z/2) = \frac{1}{2}z\left(\frac{1}{2}z + 2\right)^n 2^n + (1 - \frac{1}{2}z)(z + 1)^n.$$

Thus, by Theorem 3.11, the polynomial is log-concave. □

4.2. **Bar-paths and bar-rings of the dipole $\ddot{D}_3$.** The graph obtained from the dipole D_3 by subdividing two edges and taking the two new vertices as roots is denoted $\ddot{D}_3$. It is illustrated in Figure 4.1.

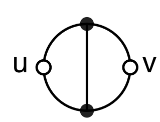

FIGURE 4.1. The dipole $(\ddot{D}_3, u, v)$.

It is also known [GrKlRi93] that the second smallest positive value of average genus for a graph is 1/2, and the unique 2-edge-connected graph with that average genus is the dipole D_3. The maximum genus of D_3 is 1. Thus, the average genus and maximum genus of a bar-path of n copies of $\ddot{D}_3$ are $n/2$ and n, respectively. Thus, a bar-path of copies of $\ddot{D}_3$ is another exception to the widespread experience that the average genus of a graph tends to be much nearer to the maximum genus than to the minimum genus.

Theorem 4.4. *The pgd-vector of a bar-path of n copies of the dipole $(\ddot{D}_3, u, v)$ is given by*

$$(X_{\ddot{D}_3} M)^{n-1} X_{\ddot{D}_3} \tag{4.4}$$

where

$$X_{\ddot{D}_3} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2z \end{pmatrix} \tag{4.5}$$

and M is the vectorized production matrix (3.1)

Proof. Formula 4.4 is a simple application of Corollary 3.8. The pgd-vector $X_{\ddot{D}_3}$ is easily calculated by face-tracings. □

Theorem 4.5. *The pgd-vector of a bar-path of n copies of the dipole $(\ddot{D}_3, u, v)$ is given, for $n \geq 2$, by*

$$\begin{aligned}\big(4^n\xi^{n-2} - 4\psi^{n-2} \quad 4\psi^{n-2} \quad 0 \quad z4^n\xi^{n-2} - 8z\psi^{n-2} \quad 8z\psi^{n-2} \\ z4^n\xi^{n-2} - 8z\psi^{n-2} \quad 8z\psi^{n-2} \quad z^2 4^n\xi^{n-2} - 16z^2\psi^{n-2} \quad 16z^2\psi^{n-2} \quad 0\big),\end{aligned}$$

where $\xi = 2 + 2z$ and $\psi = 2 + 8z$. Each coordinate is a log-concave polynomial.

Proof. The first assertion follows from Theorem 4.4 and Theorem 3.9 with $X_G = X_{\ddot{D}_3}$. In order to show that each coordinate of (4.3) is a log-concave polynomial, we have to show that the following two polynomials are log-concave:

$$f(z) = (1 + 4z)^{n-2} \text{ and } g(z) = 4^n\xi^{n-2} - c\psi^{n-2}, \text{ with } c = 4, 8, 16.$$

Since the binomial coefficients are a log-concave sequence, we infer that $f(z)$ is a log-concave polynomial. Applying Theorem 3.12 with $2^{-n+2}g(z/4) = 4^n(1 + z/4)^{n-2} - c(1 + z)^{n-2}$, we complete the proof. □

Theorem 4.6. *The (non-partitioned) genus distribution of a bar-ring of n copies of the dipole $\ddot{D}_3$, is given by the polynomial*

$$f(z) = z8^n(1 + z)^n + 2^n(1 - z)(1 + 4z)^n.$$

This polynomial is log-concave.

Proof. The formula follows directly from Theorem 3.10, with $\xi = 2 + 2z$ and $\psi = 2 + 8z$. By Theorem 3.11, with

$$2^{-n}f(z/4) = \frac{1}{4}z\left(\frac{1}{4}z + 1\right)^n 4^n + (1 - \frac{1}{4}z)(z + 1)^n,$$

we complete the proof. □

4.3. **The doubled path DP_3.** The ***doubled path*** DP_n is formed by doubling the edges in a path P_{n+1} with n edges, and then taking the two 2-valent vertices at opposite ends as the roots. It is illustrated in Figure 4.2. A singly rooted version of this graph appears in [FGS89], where it is called a *cobblestone path*.

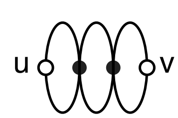

FIGURE 4.2. The doubled path (DP_3, u, v).

Theorem 4.7. *The pgd-vector of a bar-path of n copies of the doubled path (DP_3, u, v), for $n \geq 2$, is given by*

$$\begin{aligned} R(z) = \Big(&4(4+3z)^2 4^n \xi^{n-2} - 64(1+3z)^2\psi^{n-2} \quad 64(1+3z)^2\psi^{n-2} \quad 0 \quad 8z(4+3z)4^n\xi^{n-2} - 64z(1+3z)\psi^{n-2} \\ &64z(1+3z)\psi^{n-2} \quad 8z(4+3z)4^n\xi^{n-2} - 64z(1+3z)\psi^{n-2} \quad 64z(1+3z)\psi^{n-2} \\ &16z^2 4^n\xi^{n-2} - 64z^2\psi^{n-2} \quad 64z^2\psi^{n-2} \quad 0\Big), \end{aligned} \tag{4.6}$$

where $\xi = 16 + 20z$ and $\psi = 8 + 40z$. Each coordinate is a log-concave polynomial.

Proof. The doubled path DP_3 has the following pgd-vector:

$$X_{DP_3} = \begin{pmatrix} 8 & 8 & 4z & 0 & 8z & 0 & 8z & 0 & 0 & 0 \end{pmatrix} \tag{4.7}$$

Thus the first assertion follows from Theorem 3.9 with $X_G = X_{DP_3}$. Since the polynomials ψ^{n-2} and $z^i(1+3z)^j$ are both log-concave with no internal zeros, we infer immediately that the polynomials $R_2(z)$, $R_5(z)$, $R_7(z)$ and $R_9(z)$ are log-concave. Thus, it remains to show that the polynomial

$$f_m(z) \;=\; 2^{4-m}(4+3z)^m 4^n \xi^{n-2} - 64(1+3z)^m \psi^{n-2}$$

is log-concave for all $m = 0, 1, 2$. The case $m = 0$ follows immediately from Theorem 3.12, thus $R_8(z)$ is a log-concave polynomial. The case $m = 1$ follows immediately from Theorem 3.13 with

$$8^{-n} f_1(z/5) \;=\; 2(4+3z/5)(1+z/4)^{n-2}8^{n-2} - (1+3z/5)(1+z)^{n-2},$$

which shows that both $R_4(z)$ and $R_6(z)$ are log-concave polynomials. The case $m = 2$ follows immediately from Theorem 3.13 with

$$8^{-n} f_2(z/5) \;=\; (4+3z/5)^2(1+z/4)^{n-2}8^{n-2} - (1+3z/5)^2(1+z)^{n-2},$$

which shows that $R_1(z)$ is a log-concave polynomial. □

Theorem 4.8. *The (non-partitioned) genus distribution of a bar-ring of n copies of the doubled path DP_3, is given by the polynomial*

$$f(z) = 8^n(z(4+5z)^n 2^n + (1-z)(1+5z)^n).$$

This polynomial is log-concave.

Proof. The formula follows directly from Theorem 3.10, with $\xi = 16 + 20z$ and $\psi = 8 + 40z$. By Theorem 3.11 with $8^{-n}f(z/5) \;=\; \frac{1}{5}z(1+z)^n 2^n + (1 - \frac{1}{5}z)(1+z)^n$, we complete the proof. □

4.4. **The symmetric (closed-end) ladder $\ddot{L}_3$.** The ***symmetric (closed-end) ladder*** $\ddot{L}_3$ is obtained by subdividing an end-rung at each end of the closed-end ladder L_3, as illustrated in Figure 4.3.

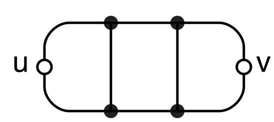

FIGURE 4.3. The symmetric closed-end ladder $\ddot{L}_3$.

Theorem 4.9. *The pgd-vector of a bar-path of n copies of the symmetric (closed-end) ladder $(\ddot{L}_3, u, v)$ is given, for $n \geq 2$, by*

$$\begin{aligned}R(z) = \big(&4(1+2z)^2 4^n\xi^{n-2} - 4(1+8z)^2\psi^{n-2} \quad 4(1+8z)^2\psi^{n-2} \quad 0 \quad 4z(1+2z)4^n\xi^{n-2} - 8z(1+8z)\psi^{n-2} \\ &8z(1+8z)\psi^{n-2} \quad 4z(1+2z)4^n\xi^{n-2} - 8z(1+8z)\psi^{n-2} \quad 8z(1+8z)\psi^{n-2} \\ &4z^2 4^n\xi^{n-2} - 16z^2\psi^{n-2} \quad 16z^2\psi^{n-2} \quad 0\big),\end{aligned} \tag{4.8}$$

where $\xi = 4 + 12z$ and $\psi = 2 + 24z$. Each coordinate is a log-concave polynomial.

Proof. The symmetric ladder $(\ddot{L}_3, u, v)$ has the following pgd-vector:

$$X_{L_3} = \begin{pmatrix} 2 & 2 & 4z & 0 & 4z & 0 & 4z & 0 & 0 & 0 \end{pmatrix}. \tag{4.9}$$

Thus the first assertion follows from Theorem 3.9, with $X_G = X_{L_3}$. Since the polynomials ψ^{n-2} and $z^i(1+8z)^j$ are both log-concave with no internal zeros, we infer immediately that the polynomials $R_2(z)$, $R_5(z)$, $R_7(z)$ and $R_9(z)$ are log-concave. Thus, it remains to show that the polynomial

$$f_m(z) = (1+2z)^m 4^{n+1}\xi^{n-2} - 2^{4-m}(1+8z)^m\psi^{n-2}$$

is log-concave for all $m = 0, 1, 2$. The case $m = 0$ follows immediately from Theorem 3.12, thus $R_8(z)$ is a log-concave polynomial. The case $m = 1$ follows from Theorem 3.13 with

$$2^{-n-1}f_1(z/12) = 2(4+2z/3)(1+z/4)^{n-2}8^{n-2} - (1+2z/3)(1+z)^{n-2},$$

which implies that the polynomials $R_4(z)$ and $R_6(z)$ are log-concave. Also, the case $m = 2$ follows from Theorem 3.13 with

$$2^{-n}f_2(z/12) = (4+2z/3)^2(1+z/4)^{n-2}8^{n-2} - (1+2z/3)^2(1+z)^{n-2},$$

which implies that $R_1(z)$ is a log-concave polynomial. □

Theorem 4.10. *The (non-partitioned) genus distribution of a bar-ring of n copies of the symmetric (closed-end) ladder L_3, is given by the polynomial*

$$f(z) = 2^n(z(1+3z)8^n + (1-z)(1+12z)^n).$$

This polynomial is log-concave.

Proof. The formula follows directly from Theorem 3.10, with $\xi = 4 + 12z$ and $\psi = 2 + 24z$. By Theorem 3.11 with $2^{-n}f(z/12) = \frac{1}{12}z(1+\frac{3}{12}z)^n 8^n + (1-\frac{1}{12}z)(1+z)^n$, we complete the proof. □

4.5. **The doubled-root ladder (L''_n, u, v).** The ***doubled-root ladder*** (L''_n, u, v) is obtained from the closed-end ladder L_n by twice subdividing one end-rung, as illustrated in Figure 4.4 for L_3.

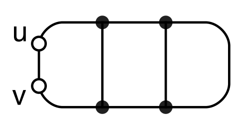

FIGURE 4.4. The ladder L''_3.

Theorem 4.11. *Let $n \geq 2$. The pgd-vector of a bar-path of n copies of the double-root ladder (L''_3, u, v) is given by*

$$\begin{aligned} R(z) = \big((1+2z)^2(4^{n+1}\xi^{n-2} - 64\psi^{n-2}) \quad 64(1+2z)^2\psi^{n-2} \quad 0 \quad z(1+2z)(\xi^(n-2)4^{n+1} - 64\psi^{n-2}) \\ 64z(1+2z)\psi^{n-2} \quad z(1+2z)(\xi^(n-2)4^{n+1} - 64\psi^{n-2}) \quad 64z(1+2z)\psi^{n-2} \\ z^2(\xi^(n-2)4^{n+1} - 64\psi^{n-2}) \quad 64z^2\psi^{n-2} \quad 0\big), \end{aligned} \tag{4.10}$$

where $\xi = 4 + 12z$ and $\psi = 8 + 32z$. Each coordinate is a log-concave polynomial.

Proof. The double-rooted ladder (L''_3, u, v) has the following pgd-vector:

$$X_{L''_3} = \begin{pmatrix} 0 & 0 & 4+8z & 0 & 0 & 0 & 0 & 0 & 4z & 0 \end{pmatrix}. \tag{4.11}$$

Thus the first assertion follows from Theorem 3.9 with $X_G = X_{L''_3}$. Since the polynomials ψ^{n-2} and $z^i(1+2z)^j$ are both log-concave with no internal zeros, we infer immediately that the polynomials $R_2(z)$, $R_5(z)$, $R_7(z)$ and $R_9(z)$ are log-concave. Thus, it remains to show that the polynomial $f(z) = 4^{n+1}\xi^{n-2} - 64\psi^{n-2}$ has no internal zero and is log-concave. It is not hard to see that $f(z)$ has no internal zeros. By applying Theorem 3.12, with $2^{-3n}f(z/4) = 2^{n-2}(1+3z/4)^{n-2} - (1+z)^{n-2}$, we complete the proof. □

Theorem 4.12. *The (non-partitioned) genus distribution of a bar-ring of n copies of the double-root ladder L''_3, is given by the polynomial*

$$f(z) = 8^n(z(1+3z)^n 2^n + (1-z)(1+4z)^n).$$

This polynomial is log-concave.

Proof. The formula follows directly for Theorem 3.10 with $\xi = 4 + 12z$ and $\psi = 8 + 32z$. By theorem 3.11 with $8^{-n}f(z/4) = \frac{1}{4}z(1+\frac{3}{4}z)^n 2^n + (1-\frac{1}{4}z)(1+z)^n$, we complete the proof. □

4.6. **The complete graph K_4.** The rooted graph $(\ddot{K}_4, u, v)$ is obtained by sub diving two non-adjacent edges and taking the two new vertices as roots. This is illustrated in Figure 4.5.

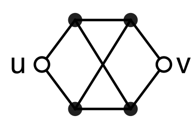

FIGURE 4.5. The complete graph K_4.

Theorem 4.13. *Let $n \geq 2$. The pgd-vector of a bar-path of n copies of the graph $(\ddot{K}_4, u, v)$ is given by*

$$\begin{aligned} R(z) = \big((1+4z)^2 4^n\xi^{n-2} - 256z^2\psi^{n-2} \quad 256z^2\psi^{n-2} \quad 0 \quad 3z(1+4z)4^n\xi^{n-2} - 128z^2\psi^{n-2} \\ 128z^2\psi^{n-2} \quad 3z(1+4z)4^n\xi^{n-2} - 128z^2\psi^{n-2} \quad 128z^2\psi^{n-2} \\ 9z^2 4^n\xi^{n-2} - 64z^2\psi^{n-2} \quad 64z^2\psi^{n-2} \quad 0\big), \end{aligned} \tag{4.12}$$

where $\xi = 2 + 14z$ and $\psi = 32z$. Each coordinate is a log-concave polynomial.

Proof. The complete graph $\ddot{K}_4$ has the following pgd-vector:

$$X_{\ddot{K}_4} \;=\; \begin{pmatrix} 2 & 0 & 4z & 0 & 4z & 0 & 4z & 0 & 0 & 2z \end{pmatrix}. \tag{4.13}$$

Thus the first assertion follows from Theorem 3.9 with $X_G = X_{\ddot{K}_4}$. Since the polynomial ψ^{n-2} is a both log-concave, we infer immediately that the polynomials $R_2(z)$, $R_5(z)$, $R_7(z)$ and $R_9(z)$ are log-concave. In order to show that the polynomials $R_1(z)$, $R_4(z)$, $R_6(z)$, and $R_8(z)$ are log-concave, we have to show that the polynomials

- $(1+4z)^2(1+7z)^{n-2} - 4^n z^n$,
- $3(1+4z)(1+7z)^{n-2} - 24^{n-1}z^{n-1}$, and
- $9z(1+7z)^{n-2} - 4^{n-1}z^{n-1}$

are log-concave. Since the polynomials $(1+4z)^2(1+7z)^{n-2}$, $3(1+4z)(1+7z)^{n-2}$ and $9z(1+7z)^{n-2}$ are of degrees n, $n-1$ and $n-1$, respectively, the log-concavity follows immediately. Hence, we have shown that each coordinate in the pgd-vector of the graph

$$\overbrace{(L_3'', u, v)\overline{*}(L_3'', u, v)\overline{*}\cdots\overline{*}(L_3'', u, v)}^{n \text{ copies of } (L_3'',u,v)},$$

namely, the vector $R(z)$, is a log-concave polynomial. □

Theorem 4.14. *The (non-partitioned) genus distribution of a bar-ring of n copies of the dipole L_3'', is given by the polynomial*

$$f(z) \;=\; 8^n(z(1+7z)^n + (1-z)(4z)^n).$$

This polynomial is log-concave.

Proof. The formula follows directly from Theorem 3.10, with $\xi = 2 + 14z$ and $\psi = 32z$. Let $d_{n,j}$ be the $8^n z^j$ coefficient of the polynomial $f(z)$. Then, for $j = 0, 1, \ldots, n-1$, we have

$$\begin{aligned} d_{n,n+1} &= 7^n - 4^n, \\ d_{n,n} &= 4^n + n7^{n-1}, \text{ and} \\ d_{n,j} &= \binom{n}{j-1} 7^{j-1}. \end{aligned}$$

Clearly, the polynomial $f(z)$ is log-concave if and only if $d_{n,j}^2 - d_{n,j+1}d_{n,j-1} \geq 0$, for $j = n-1, n$. By the definitions,

$$d_{n,n}^2 - d_{n,n+1}d_{n,n-1} \;=\; (4^n + n7^{n-1})^2 - (7^n - 4^n)\binom{n}{2}7^{n-2}$$

and

$$d_{n,n-1}^2 - d_{n,n}d_{n,n-2} \;=\; \binom{n}{2}^2 49^{n-2} - \binom{n}{3}7^{n-3}(4^n + n7^{n-1}).$$

It is routine to check the nonnegativity of these two expressions, which completes the proof. □

5. Conclusions and Research Problems

This is one of a series of papers concerned with the 25-year-old LCGD Conjecture that the genus distribution of a graph is always log-concave. After affirmations of the LCGD conjecture for various linear families of graphs, we focus here on a variety of ring-like classes, of which a principal case is classes that are obtained by applying the self-bar-amalgamation or the self-amalgamation operation to linear classes.

We have further algebraized the approach to calculation of genus distributions and proof of their log-concavity by introducing *vectorized production matrices.* Along with Theorem 3.7, Corollary 3.8, and Theorem 3.9, the use of vectorized production matrices has enabled us not only to derive closed forms for genus distributions of bar-paths of graphs, but also to prove that their partial genus distributions are log-concave. Although having log-concave partial genus distributions has seemed reasonable to conjecture, it is not yet known even whether various kinds of partial genus distributions can have internal zeros. Furthermore, by applying Theorem 4.1 of [GMTW13a] and then Theorem 3.11 to these closed forms, we have obtained genus distributions of the corresponding ring-like graphs and proved that they are log-concave.

We conclude by posing three new research problems on graph genus distributions.

Research Problem 5.1. Determine whether a partial genus distribution of a graph with two 2-valent root vertices can have any gaps (i.e., internal zeros). As mentioned in Subsection 1.3, there is an easy proof using rotation systems that there are no gaps in the total genus distribution of any graph. Observe that we specify the kind of partitioning to be used. A different paradigm for partitioning a genus distribution introduced in [Gr14] is a principal concern of [GMTW14a].

Research Problem 5.2. Given a finite sequence of graphs $G_1, G_2, \ldots, G_n$ with log-concave partial genus distributions and log-concave total genus distribution, such that the graphs $G_1, G_2, \ldots, G_n$ are not mutually isomorphic, derive a set of conditions under which the total genus distribution and the partial genus distributions of a bar-path of these graphs is log-concave. By way of contrast, the results of Section 3 are applied in Section 4 only to a bar-path of copies of a fixed graph.

Research Problem 5.3. Given a finite sequence of graphs $G_1, G_2, \ldots, G_n$ with log-concave partial genus distributions and a log-concave total genus distribution, such that the graphs $G_1, G_2, \ldots, G_n$ are not mutually isomorphic, derive conditions under which the genus distribution of the corresponding bar-ring of graphs is log-concave. Our results in Section 4 are only for a bar-ring of copies of a fixed graph.

References

[BWGT09] L.W. Beineke, R.J. Wilson, J.L. Gross, and T.W. Tucker, editors, *Topics in Topological Graph Theory*, Cambridge Univ. Press, 2009.

[Ch14] J. Chen, Minimum genus and maximum genus, Section 7.2 of the *Handbook of Graph Theory, 2nd Edition* (J.L. Gross, J. Yellen, and P. Zhang, eds.) CRC Press, 2014.

[Ch08] Y. Chen, A note on a conjecture of Stahl, *Canad. J. Math* **60** (2008), 958–959.

[ChLi10] Y. Chen and Y. Liu, On a conjecture of Stahl, *Canad. J. Math* **62** (2010), 1058–1059.

[FGS89] M. Furst, J.L. Gross and R. Statman, Genus distributions for two class of graphs, *J. Combin. Theory Ser. B* **46** (1989), 523–534.

[Gr11] J.L. Gross, Genus distribution of graph amalgamations: Self-pasting at root-vertices, *Australasian J. Combin.* **49** (2011), 19–38.

[Gr14] J.L. Gross, Embeddings of graphs of fixed treewidth and bounded degree, *Ars Math. Contemp.* **7** (2014), 379–403. Presented at AMS Annual Meeting at Boston, January 2012.

[GrFu87] J.L. Gross and M.L. Furst, Hierarchy for imbedding-distribution invariants of a graph, *J. Graph Theory* **11** (1987), 205–220.

[GKP10] J.L. Gross, I.F. Khan, and M.I. Poshni, Genus distribution of graph amalgamations: Pasting at root-vertices, *Ars Combin.* **94** (2010), 33–53.

[GrKlRi93] J.L. Gross, E.W. Klein, and R.G. Rieper, On the average genus of graphs, *Graphs and Combinatorics* **9** (1993), 153–162.

[GMTW13a] J.L. Gross, T. Mansour, T.W. Tucker, and D.G.L. Wang, Log-concavity of combinations of sequences and applications to genus distributions, draft manuscript, 2013.

[GMTW14a] J.L. Gross, T. Mansour, T.W. Tucker, and D.G.L. Wang, Calculating genus polynomials via string operations and matrices, draft manuscript, 2014.

[GMTW14b] J.L. Gross, T. Mansour, T.W. Tucker, and D.G.L. Wang, Log-concavity of the genus polynomials of Ringel ladders, draft manuscript, 2014.

[GRT89] J.L. Gross, D.P. Robbins, and T.W. Tucker, Genus distributions for bouquets of circles, *J. Combin. Theory Ser. B* **47** (1989), 292–306.

[GrTu87] J.L. Gross and T.W. Tucker, *Topological Graph Theory*, Dover, 2001 (original ed. Wiley, 1987).

[JS78] G.A. Jones and D. Singerman, Theory of maps on orientable surfaces, *Proc. London Math. Soc.* **37** (1978), 273–301.

[LW07] L.L. Liu and Y. Wang, A unified approach to polynomial sequences with only real zeros, *Adv. in Appl. Math.* **38** (2007), 542–560.

[NeSk14] R. Nedela and M. Škoviera, Maps, Section 7.6 of the *Handbook of Graph Theory, 2nd Edition* (J.L. Gross, J. Yellen, and P. Zhang, eds.) CRC Press, 2014.

[PiPo14] T. Pisanski and P. Potočnik, Graphs on surfaces, Section 7.1 of the *Handbook of Graph Theory, 2nd Edition* (J.L. Gross, J. Yellen, and P. Zhang, eds.) CRC Press, 2014.

[RY68] G. Ringel and J.W.T. Youngs, Solution of the Heawood map-coloring problem, *Proc. Nat. Acad. Sci. USA* **60** (1968), 438–445.

[Stah91] S. Stahl, Permutation-partition pairs. III. Embedding distributions of linear families of graphs, *J. Combin. Theory, Ser. B* **52** (1991), 191–218.

[Stah97] S. Stahl, On the zeros of some genus polynomials, *Canad. J. Math.* **49** (1997), 617–640.

[Tut63] W.T. Tutte, A census of planar maps, *Canad. J. Math.* **15** (1963), 249–271.

Department of Computer Science, Columbia University, New York, NY 10027, USA;
email: gross@cs.columbia.edu

Department of Mathematics, University of Haifa, 3498838 Haifa, Israel;
email: tmansour@univ.haifa.ac.il

Department of Mathematics, Colgate University, Hamilton, NY 13346, USA;
email: ttucker@colgate.edu