CS1001

Lecture 22

Overview

- Mechanizing Reasoning
- Gödel's Incompleteness Theorem

Natural Deduction

- Start with Axioms (fundamental rules) and Facts
- Apply Rules of logic
- Deduce additional facts

Can Deduction be Performed by Computer?

- Assuming all facts about the natural world were to be described as facts in a logical system, can all other facts be derived using the laws of math/logic?
- Punch line: No! *Any* formal system breaks down; there are truths that can not be derived

Why?

- Paradox
- Self Reference
- As shown in the past, paradox and self reference are fundamental parts of a “real world” or generic system. We must allow these.
- If we don’t, we have no way of reasoning about the infinite case and therefore can’t develop generic algorithms

Mechanical Reasoning

- Aristotle (~350BC): *Organon*
  - We can explain logical deduction with rules of inference (syllogisms)

    Every B is A

    C is B

    ➔ C is A

    Every human is mortal.

    Godel is human.

    Godel is mortal.

More Mechanical Reasoning

- Euclid (~300BC): *Elements*
  - We can reduce geometry to a few axioms and derive the rest by following rules
- Newton (1687): *Philosophiæ Naturalis Principia Mathematica*
  - We can reduce the motion of objects (including planets) to following axioms (laws) mechanically

Mechanical Reasoning

- Late 1800s – many mathematicians working on codifying "laws of reasoning"
  - George Boole, *Laws of Thought*
  - Augustus De Morgan
  - Whitehead and Russell

All true
statements about number theory

Perfect Axiomatic System

Derives **all** true statements, and **no** false statements starting from a finite number of axioms and following mechanical inference rules.

Incomplete Axiomatic System

incomplete

Derives
some, but not all true statements, and **no** false statements starting from a finite number of axioms and following mechanical inference rules.

Inconsistent Axiomatic System

Derives
all true
statements, and **some** false
statements starting from a
finite number of axioms
and following mechanical
inference rules.

some false
statements

Principia Mathematica

- Whitehead and Russell (1910– 1913)
  - – Three Volumes, 2000 pages
- Attempted to axiomatize mathematical reasoning
  - – Define mathematical entities (like numbers) using logic
  - – Derive mathematical “truths” by following mechanical rules of inference
  - – Claimed to be *complete* and *consistent*
    - All true theorems could be derived
    - No falsehoods could be derived

Russell's Paradox

- Some sets are not members of themselves
  - In a certain town in Spain, there lives an excellent barber who shaves all the men who do not shave themselves.

    Who shaves the barber?
- Some sets are members of themselves
- Call the set of all sets that are not members of themselves *S*
- Is *S* a member of itself?

Russell's Paradox

- *S*: set of all sets that are not members of themselves
- Is *S* a member of itself?
  - If *S* is an element of *S*, then *S* is a member of itself and should not be in *S*.
  - If *S* is not an element of *S*, then *S* is not a member of itself, and should be in *S*.

Ban Self-Reference?

- *Principia Mathematica* attempted to resolve this paragraph by banning self-reference
- Every set has a type
  - The lowest type of set can contain only "objects", not "sets"
  - The next type of set can contain objects and sets of objects, but not sets of sets

Russell's Resolution?

Set ::= Set$_n$

Set$_0$ ::= { *x* | *x* is an *Object* }

Set$_n$ ::= { *x* | *x* is an *Object* or a *Set*$_{\underline{n\text{ - }1}}$ }

S: Set$_n$

Is *S* a member of itself?

No, it is a Set$_n$ so, it can't be a member of a Set$_n$

Epimenides Paradox

Epidenides (a Cretan):
“All Cretans are liars.”

Equivalently:
“This statement is false.”

Russell’s types can help with the set paradox, but not with this one.

Gödel's Solution

All consistent axiomatic formulations of number theory include *undecidable* propositions.

(GEB, p. 17)

undecidable – cannot be proven either true or false inside the system.

Kurt Gödel

- Born 1906 in Brno (now Czech Republic, then Austria-Hungary)
- 1931: publishes *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*

  (*On Formally Undecidable Propositions of Principia Mathematica and Related Systems*)

- 1939: flees Vienna
- Institute for Advanced Study, Princeton
- Died in 1978 – convinced everything was poisoned and refused to eat

Gödel’s Theorem

In the Principia Mathematica system, there are statements that cannot be proven either true or false.

Gödel’s Theorem

In any interesting rigid system, there are statements that cannot be proven either true or false.

Gödel's Theorem

All logical systems of any complexity are incomplete: there are statements that are *true* that cannot be proven within the system.

Proof – General Idea

- Theorem: In the Principia Mathematica system, there are statements that cannot be proven either true or false.
- Proof: Find such a statement

Gödel's Statement

G: This statement of number theory does not have any proof in the system of *Principia Mathematica*.

G is unprovable, but true!

Gödel's Proof

G: This statement of number theory does not have any proof in the system of *PM*.

If *G* were provable, then PM would be inconsistent.

If *G* is unprovable, then PM would be incomplete.

PM cannot be complete and consistent!

Finishing The Proof

- Turn *G* into a statement in the *Principia Mathematica* system
- Is *PM* powerful enough to express "This statement of number theory does not have any proof in the system of *PM*."?

How to express "does not have any proof in the system of *PM*"

- What does it mean to have a proof of *S* in PM?
  - There is a sequence of steps that follow the inference rules that starts with the initial axioms and ends with *S*
- What does it mean to **not** have **any** proof of *S* in PM?
  - There is **no** sequence of steps that follow the inference rules that starts with the initial axioms and ends with *S*

Can PM express unprovability?

- There is **no** sequence of steps that follow the inference rules that starts with the initial axioms and ends with *S*

Can we express “This statement of number theory”

- We can write turn every statement into a number, so we can turn “This statement of number theory does not have any proof in the system of *PM*” into a number

Gödel’s Proof

G: This statement of number theory does not have any proof in the system of *PM*.

If *G* were provable, then PM would be inconsistent.

If *G* is unprovable, then PM would be incomplete.

PM cannot be complete and consistent!

Generalization

All logical systems of any complexity are incomplete: there are statements that are *true* that cannot be proven within the system.

Practical Implications

- Mathematicians will ***never*** be completely replaced by computers
  - There are mathematical truths that cannot be determined mechanically
  - We can build a computer that will prove only true theorems about number theory, but if it cannot prove something we do not know that that is not a true theorem.

Russell’s Doctrine

“I wish to propose for the reader's favourable consideration a doctrine which may, I fear, appear wildly paradoxical and subversive. The doctrine in question is this: that it is undesirable to believe a proposition when there is no ground whatever for supposing it true.”
(Russell, *Introduction: On the Value of Scepticism*, 1928)