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On the treatment of nonlinear unilateral contact problems

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Summary: This paper is concerned with finite deformations of elastic bodies in the presence of unilateral constraints. The penalty formulation is applied to introduce the contact constraints. We develop special isoparametric contact elements. Starting from their Gaussian points the distance between the body and the obstacle is determined, where the obstacle is given as a C^2 continuous function. Variation and subsequent consistent linearization yield the tangent matrix of the contact elements in its general form, which can be incorporated into standard finite element schemes.

Zur Behandlung nichtlinearer unilateraler Kontaktprobleme

Übersicht: Es wird das Kontaktverhalten eines deformierbaren Körpers beschrieben, der endliche Deformationen erfährt, wenn er auf ein starres Hindernis gedrückt wird. Dabei findet die Penalty-Formulierung Anwendung. Zur Kontakterkennung werden isoparametrische Kontaktelemente verwendet. Ausgehend von deren Gausspunkten wird der Abstand des Körpers zum Hindernis bestimmt, das als C^2-stetige Funktion beschrieben wird. Variation und anschließende konsistente Linearisierung liefern die Tangentenmatrix für die Kontaktelemente in allgemeiner Form, die dann in ein standardmäßiges Finit-Element-Programm eingebaut werden kann.

1 Introduction

Many technical problems involve contact between rigid tools and deformable bodies, see e.g. forming simulation. Since in most of these problem classes the bodies undergo large deformations one has to develop a contact formulation which can handle these situations. This will be done here for the case of frictionless contact.

Due to its technical importance many different contact formulations have been developed and are discussed in the literature. In the last years more effort has been devoted to nonlinear problems. Within finite element methods normally the approach is followed which assumes surfaces that are parametrized by linear or bi-linear shape functions. One body arbitrarily is denoted as master body which defines the surface normals during the contact computations. Then the contact detection is performed via a check at the element nodes, see e.g. Hallquist [1]. Based on these geometrical representations consistent tangent moduli for contact elements have been developed, see Wriggers et al. [2] or Parisch [3]. A rigid tool is in this approach also approximated in the same way as a rigid body. However it seems more natural to approximate the tool by its geometrical representation – e.g. CAD-model – which also defines the contact normal in a consistent manner. This approach has been used in e.g. Hansson et al. [4] for three dimensional problems. In this paper we will use a spline interpolation of the tool.

On the other hand most formulations establish the contact conditions on nodal basis, see Hallquist [1], Wriggers et al. [2] or Hansson et al. [4]. In our work we will check for the contact in the Gaussian points which is consistent with the element formulation of the continuum body. This approach has also been proposed by Laursen et al. [5] for frictional contact.

The algorithmic treatment in this paper is based on the penalty method which is well established in finite element methods for contact problems, see Oden [6], Hallquist [1] or Papadopoulos et al. [7].

2 The problem

Let $K \subset R^2$ be a rigid obstacle which comes into contact with an elastic body Ω (Fig. 1). The contact is supposed to be frictionless throughout this paper. Furthermore we will not consider time dependent processes. The body is not loaded in its initial state and therefore the initial stresses are zero.

We will denote points in the reference configuration by capital letters. Small letters are associated with points in the current configuration. The mapping deforming Ω is called Φ. Thus we have:

$$\mathbf{x} = \Phi(\mathbf{X}) = \mathbf{X} + \mathbf{u}\,. \tag{1}$$

Now we are looking for the deformations of Ω which the body undergoes when it is loaded by surface loads $\hat{\mathbf{t}}$ or body forces $\varrho\mathbf{b}$. The equilibrium follows from the minimization of the strain energy stored in Ω:

$$\Pi(\mathbf{u}) = \int_\Omega W(\mathbf{u})\,\mathrm{d}\Omega - \int_\Omega \varrho\mathbf{b}\cdot\mathbf{u}\,\mathrm{d}\Omega - \int_{\partial\Omega} \hat{\mathbf{t}}\cdot\mathbf{u}\,\mathrm{d}(\partial\Omega) \to \min\,. \tag{2}$$

Here $W(\mathbf{u})$ denotes a hyperelastic strain energy function e.g.

$$W(\mathbf{u}) = \frac{\mu}{2}(I_C - 3) + \lambda \ln J + \frac{\lambda}{2}(J-1)^2 \tag{3}$$

for a compressible Neo Hookeian material with the first invariant $I_C = \mathrm{tr}\,\mathbf{C}$ of the right Cauchy Green tensor $\mathbf{C}$.

In addition a constraint condition has to be fulfilled: Ω is not allowed to penetrate the obstacle. For describing this condition we introduce a distance function for every point $x(\mathbf{u}) \in \Phi(\Omega)$ as can be found in the literature, ([5] or [4]):

$$g(\mathbf{u}) = \mathrm{sign}\,(g)\cdot g_+(\mathbf{u})\,, \tag{4}$$

where

$$g_+(\mathbf{u}) = \min_{y\in\partial K} \|\mathbf{x}(\mathbf{u}) - \mathbf{y}\| \stackrel{\text{def}}{=} \|\mathbf{x}(\mathbf{u}) - \bar{\mathbf{y}}\| \quad \text{and} \quad \mathrm{sign}\,(g) = \begin{cases} 1, & \mathbf{n}\cdot(\mathbf{x}(\mathbf{u}) - \bar{\mathbf{y}}) \geqq 0 \\ -1, & \text{otherwise}. \end{cases}$$

The geometric situation is shown in Fig. 2. $\mathbf{n}$ stands for the outward normal of the obstacle, $\gamma_c \subset \partial\Phi(\Omega)$ defines the domain of the body's surface that contacts the obstacle. It is:

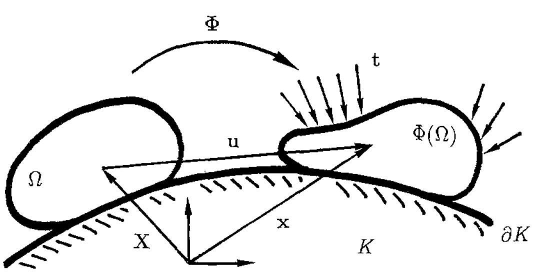
$$\gamma_c = \{\mathbf{x}(\mathbf{u}) : g(\mathbf{u}) = 0\}\,. \tag{5}$$

In Fig. 2 body and obstacle are drawn separately only for the reason of clearness.
Finally we obtain the following formulation of our problem:

$$\Pi(\mathbf{u}) \to \min\,. \tag{6}$$

with the constraint condition

$$g(\mathbf{u}) \geqq 0 \quad \forall \mathbf{u}\,. \tag{7}$$



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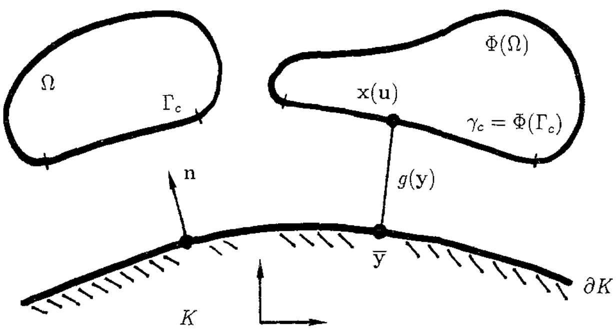

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Fig. 1 and 2. **1** The problem; **2** Distance function

3 Penalty method for solution

There exist various methods to transform optimization problems with constraint conditions to those without constraint conditions. All of them refer more or less to the method of Lagrange multipliers which has the drawback that the multipliers appear as additional unknowns. On the other hand this method enforces the contact constraints exactly. Another often applied technique is the penalty method which does not lead to an exact fulfillment of the contact conditions. However this approach does not introduce additional variables and thus is computationally advantageous.

Let us again state the optimization problem associated with contact of elastic bodies in the presence of large deformations

$$\Pi(\mathbf{u}) \to \min. \tag{8}$$

with the inequality constraint condition

$$\mathbf{u} \in S = \{\mathbf{u} : g(\mathbf{u}) \geqq 0\}, \tag{9}$$

which means we do not search for the minimum of Π for all possible $\mathbf{u}$. Hence the displacement field must be in the set of the $\mathbf{u}$ that fulfill the constraint condition. This problem is replaced by the optimization problem without any constraint condition

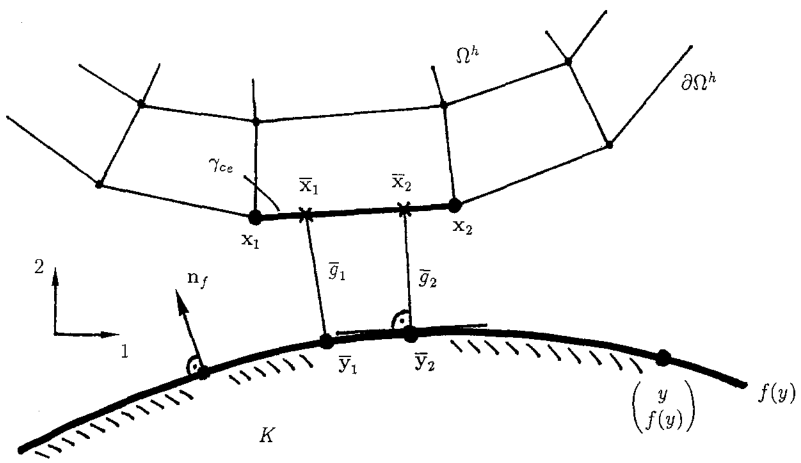
$$\Pi(\mathbf{u}) + \varepsilon P(\mathbf{u}) \to \min. \tag{10}$$

Here ε and $P(\mathbf{u})$ have to satisfy (see Luenberger [8]):

$$\begin{aligned} &\bullet\ \ \varepsilon > 0 \text{ arbitrary but fixed} \\ &\bullet\ \ P(\mathbf{u}): \quad \text{(i) continuous} \\ &\qquad\qquad\quad \text{(ii) } P(\mathbf{u}) \geqq 0 \quad \forall \mathbf{u} \\ &\qquad\qquad\quad \text{(iii) } P(\mathbf{u}) = 0 \Leftrightarrow \mathbf{u} \in S. \end{aligned} \tag{11}$$

It can be seen immediately that

$$P(\mathbf{u}) \overset{\text{def}}{=} \frac{1}{2} \int_{\gamma_c} g_+{}^2(\mathbf{u}) \, \mathrm{d}\Gamma \tag{12}$$



$\mathbf{x}_I = \mathbf{X}_I + \mathbf{u}_I$	:	element nodes
$\bar{\mathbf{x}}_I$	:	Gaussian points of the element
$\bar{\mathbf{y}}_I$	:	obstacle points to compute that define distance
$\bar{g}_I$	:	discrete distance

Fig. 3. Interpretation of penalty term

fulfills these conditions. If $\mathbf{u}$ is exact the penalty term $\varepsilon P(\mathbf{u})$ vanishes and we obtain the solution of the primary problem. In case $\mathbf{u}$ is not exact the penalty parameter has to be a large number. Then $P(\mathbf{u})$ will become small, i.e. the constraint condition has to be fulfilled as good as possible so that the minimum is reached. However the condition number of the problem increases with increasing ε which may lead to ill conditioning. A somewhat optimal choice of the penalty parameter is given by the estimation

$$\varepsilon = k/\sqrt{n\alpha}, \tag{13}$$

where k is the bulk stiffness of the finite elements that are used to discretize the body Ω. n is the number of the unknowns of the problem and α stands for the accuracy of the computer used for the solution of (10) (Wriggers et al. [9]).

In case of contact, i.e. $g(\mathbf{u}) < 0$, we can interpret ε as the stiffness of linear springs which support the body in the contact area γ_e (Fig. 3). The penalty term then describes the total energy of all these springs. If the solution is exact this energy vanishes because then no penetration is allowed.

4 Finite element discretization

In the previous section the continuous problem is formulated finally:

$$\Pi_P(\mathbf{u}) = \Pi(\mathbf{u}) + \frac{\varepsilon}{2} \int_{\gamma_c} g_+{}^2(\mathbf{u})\, \mathrm{d}\Gamma \to \min. \tag{14}$$

We have to minimize the sum of strain and penalty energy. $\Pi(\mathbf{u})$ is discretized in Ω by standard finite elements which leads to the residual vectors and tangent matrices for every element. In the same way we introduce contact elements and attach them to the surface of Ω so that they are able to decide whether Ω contacts the obstacle and thus determine the contact region. This formulation leads also to residuals and tangent matrices for every element.

In the following our objective is the discretization of the penalty term. For that purpose we develop contact elements with two or three nodes which can be used for two dimensional situations for the discretization of Ω by means of four or nine node elements. The surface of the obstacle is realized by a C^2 continuous function $f(y)$, $y \in R$ which will be specified within the examples. This function either is given as analytical function or is obtained by interpolation of several obstacle points (y_i, f_i), $i = 0, \ldots, n$. The interpolation is based on natural cubic splines:

$$S_\Delta(f; y) = a_i \frac{(y_{i+1} - y)^3}{6h_{i+1}} + a_{i+1} \frac{(y - y_i)^3}{6h_{i+1}} + b_i(y - y_i) + c_i \quad \text{if} \quad y \in [y_i, y_{i+1}], \quad i = 0, \ldots, n-1, \tag{15}$$

where the a_i follow from the linear system of equations

$$\frac{h_i}{6} a_{i-1} + \frac{h_i + h_{i+1}}{3} a_i + \frac{h_{i+1}}{6} a_{i+1} = \frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}, \qquad i = 1, \ldots, n-1.$$

In addition we define

$$a_0 = 0 \quad \text{and} \quad a_n = 0$$

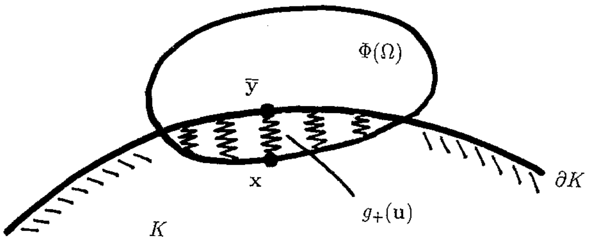
which gives reasons for the term "natural". Furthermore it is

$$b_i = \frac{f_{i+1} - f_i}{h_{i+1}} - \frac{h_{i+1}}{6}(a_{i+1} - a_i) \quad \text{and} \quad c_i = f_i - a_i \frac{h_{i+1}^2}{6}, \qquad i = 0, \ldots, n-1.$$

Thus we have polynomials of third order on every interval (Törnig et al. [10]). Beyond that the notations of Fig. 4 hold.

Discretization of (12) now yields:

$$\frac{\varepsilon}{2} \int_{\gamma_c} g_+{}^2\, \mathrm{d}\Gamma \approx \frac{\varepsilon}{2} \sum_{e=1}^{\text{numel}} \int_{\gamma_{c_e}} \bar{g}_+{}^2\, \mathrm{d}\Gamma, \tag{16}$$



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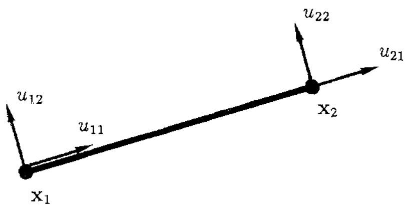

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Fig. 4 and 5. **4** Notations; **5** Degrees of freedom of the contact element

where $\bar{g}_+$ is the distance between the element surface γ_{c_e} and the obstacle. Points of γ_{c_e} are discretized by isoparametric functions:

$$\gamma_{c_e}: \bar{\mathbf{x}} = \sum_{I=1}^{N} N_I(\xi)\,\mathbf{x}_I, \qquad \xi \in [-1, 1]. \tag{17}$$

With the introduction of the displacement vector $\mathbf{u}$ between the deformed and undeformed state, see Fig. 4,

$$\mathbf{x}_I = \mathbf{X}_I + \mathbf{u}_I \tag{18}$$

we can express (17) in terms of $\mathbf{u}_I$. For two dimensional problems we have to introduce $2N$ degrees of freedom u_{Ii} for every contact element, $I = 1, \ldots, N$, where N is the number of nodal points per element and $i = 1, 2$ the spatial dimension of the problem, see Fig. 5.

Noting that $d\Gamma_e = \|\gamma_{c_e,\xi}\|\, d\xi$ we carry out a coordinate transformation to the isoparametric reference element:

$$\int_{\gamma_{c_e}} \bar{g}_+^{\,2}\, d\Gamma = \int_{-1}^{1} \bar{g}_+^{\,2}\, \|\gamma_{c_e,\xi}\|\, d\xi = \int_{-1}^{1} \bar{g}_+^{\,2} \left\| \sum_{I=1}^{N} N_{I,\xi}(\xi)\, \mathbf{x}_I \right\| d\xi. \tag{19}$$

Here and in what follows $(.)_{,\xi}$ denotes the derivate of (.) with respect to ξ.

Numerical integration (Gauss quadrature) gives

$$\int_{\gamma_{c_e}} \bar{g}_+^{\,2}\, d\Gamma \approx \sum_{k=1}^{n_p} \tilde{g}_k^{\,2} \left\| \sum_{I=1}^{N} N_{I,\xi}(\xi_k)\, \mathbf{x}_I \right\| \omega_k \tag{20}$$

$$\tilde{g}_k = \begin{cases} \bar{g}_k, & \text{if contact} \\ 0, & \text{otherwise,} \end{cases}$$

where the ξ_k stand for the coordinates of the Gaussian integration points and ω_k are the weights of the integration formula. n_p is the number of the Gaussian points per element. $\tilde{g}_k$ represents the distance function at ξ_k which is only in the case of contact different from zero. Let p describe the number of the Gaussian points for which we have contact. Then (20) yields:

$$\int_{\gamma_{c_e}} \bar{g}_+^{\,2}\, d\Gamma \approx \sum_{k=1}^{p} \bar{g}_k^{\,2} \left\| \sum_{I=1}^{N} N_{I,\xi}(\xi_k)\, \mathbf{x}_I \right\| \omega_k. \tag{21}$$

Now we have to discretize the distance function $\bar{g}_k$:

$$\bar{g}_k = \|\bar{\mathbf{x}}_k - \bar{\mathbf{y}}_k\| = \|\bar{\mathbf{x}}(\xi_k) - \bar{\mathbf{y}}_k\| = \left\| \sum_{I=1}^{N} N_I(\xi_k)\, \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|. \tag{22}$$

Since the distance function is by definition positive we test a possible penetration using the outward obstacle's normal (Fig. 6). At a Gaussian point $\bar{\mathbf{x}}_k$ we have contact if the condition

$$\mathbf{n}_f^{\,k} \cdot (\bar{\mathbf{x}}_k - \bar{\mathbf{y}}_k) \leqq 0 \tag{23}$$

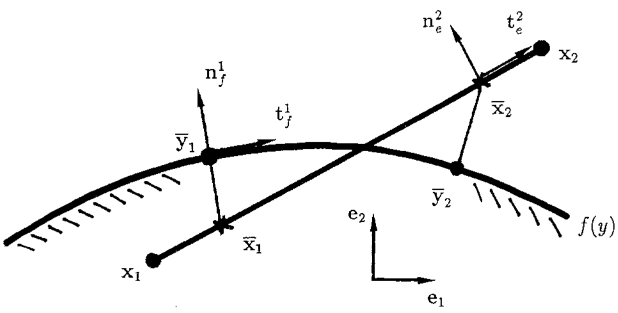


Fig. 6. Realization of contact

holds. Here we have introduced the outward normal of the obstacle

$$\mathbf{n}_f{}^k = \frac{1}{\sqrt{1+f'^2(\bar{y}_k)}} \begin{pmatrix} -f'(\bar{y}_k) \\ 1 \end{pmatrix} \tag{24}$$

which is defined in terms of the obstacle function $f(y)$. (In case of the deformable body lying below the obstacle we have to multiply $\mathbf{n}_f{}^k$ by (-1).) Associated with this definition we also obtain the tangent vector to the obstacle surface:

$$\mathbf{t}_f{}^k = \frac{1}{\sqrt{1+f'^2(\bar{y}_k)}} \begin{pmatrix} 1 \\ f'(\bar{y}_k) \end{pmatrix}. \tag{25}$$

Combining (21) and (22) the penalty term has the form

$$\frac{\varepsilon}{2} \int_{\gamma_c} g_+{}^2 \, \mathrm{d}\Gamma \approx \sum_{e=1}^{\mathrm{numel}} \left[\frac{\varepsilon}{2} \sum_{k=1}^{p} \omega_k \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\| \right], \tag{26}$$

where the arguments ξ_k have been replaced by the index k for clearness. The discretized version of problem (14) can now be stated:

$$\boxed{\Pi^h(\mathbf{u}) + \sum_{e=1}^{\mathrm{numel}} \left[\frac{\varepsilon}{2} \sum_{k=1}^{p} \omega_k \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\| \right] \to \min.} \tag{27}$$

This problem will be solved by using Newton's method: Let

$$F(x) \to \min. \tag{28}$$

be the problem. Then the standard argument for a minimum gives

$$\delta F(x) \stackrel{!}{=} 0.$$

Subsequent linearization (which comes out of the Taylor series) yields

$$\begin{aligned} L[\delta F(x)]_x &= \delta F(x) + \Delta\delta F(x) \\ &= \delta F(x) + D\delta F(x) \cdot \Delta x = 0 \\ \Rightarrow D\delta F(x) \cdot \Delta x &= -\delta F(x). \end{aligned} \tag{29}$$

In the next two sections we develop the variation and the linearization of the discretized contact penalty term (26) which is needed for the application of Newton's method (29).

4.1 Variation

The problem

$$\delta\Pi^h(\mathbf{u}) + \sum_{e=1}^{\text{numel}} \delta\left[\frac{\varepsilon}{2}\sum_{k=1}^{p} \omega_k \left\|\sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k\right\|^2 \left\|\sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I\right\|\right] \stackrel{!}{=} 0 \tag{30}$$

represents the weak form of the equilibrium which the displacement field **u** has to fulfill. The variation of the first term is a standard procedure (here we use an available element to discretize the body, Wriggers [12]). Thus we concentrate on the second term and compute the variation of the discretized penalty term:

Using the product rule we have

$$\delta\Pi^h(\mathbf{u}) + \sum_{e=1}^{\text{numel}} \frac{\varepsilon}{2}\sum_{k=1}^{p} \omega_k \left[\delta\left\|\sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k\right\|^2 \left\|\sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I\right\| + \left\|\sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k\right\|^2 \delta\left\|\sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I\right\|\right] = 0. \tag{31}$$

The particular variations are found to be (Fig. 6):

$$\delta\left\|\sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k\right\|^2 = 2\left(\sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k\right)\cdot\left(\sum_{I=1}^{N} N_I{}^k\,\delta\mathbf{x}_I - \delta\bar{\mathbf{y}}_k\right) = 2\bar{g}_k \mathbf{n}_f{}^k\cdot\left(\sum_{I=1}^{N} N_I{}^k\,\delta\mathbf{x}_I - \delta\bar{\mathbf{y}}_k\right) \tag{32}$$

$$\delta\left\|\sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I\right\| = \frac{\sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I}{\left\|\sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I\right\|}\cdot\left(\sum_{I=1}^{N} N_{I,\xi}^k\,\delta\mathbf{x}_I\right) = \mathbf{t}_e{}^k\cdot\left(\sum_{I=1}^{N} N_{I,\xi}^k\,\delta\mathbf{x}_I\right). \tag{33}$$

Note that with $\mathbf{x}_I = \mathbf{X}_I + \mathbf{u}_I$ we have $\delta\mathbf{x}_I = \delta\mathbf{u}_I$. Since $\bar{y}_k$ also depends on the displacement field we have to compute its variation, too. However to express $\delta\bar{\mathbf{y}}_k$ in terms of the variational displacements some more reflections are neccessary. Before we develop this expresssion we state the algorithm for the computation of $\bar{y}_k$. With

$$\bar{\mathbf{y}}_k = \begin{pmatrix} \bar{y}_k \\ f(\bar{y}_k) \end{pmatrix} \tag{34}$$

formally this leads to

$$\delta\bar{\mathbf{y}}_k = \begin{pmatrix} \delta\bar{y}_k \\ f'(\bar{y}_k)\,\delta\bar{y}_k \end{pmatrix}. \tag{35}$$

$\bar{y}_k$ is obtained from the definition of the minimum distance function

$$\bar{g}_k{}^2 = \min_{y\in D_f} d_k(y), \tag{36}$$

where (Fig. 7)

$$d_k(y) = (\bar{x}_{k1} - y)^2 + (\bar{x}_{k2} - f(y))^2. \tag{37}$$

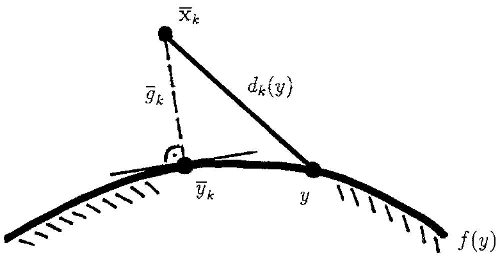
We determine $\bar{y}_k$ by means of Newton's method by finding the root of the equation

$$h_k(y) \stackrel{\text{def}}{=} \frac{1}{2}\,d_k{}'(y) = -(\bar{x}_{k1} - y) - f'(y)\,(\bar{x}_{k2} - f(y)) = 0:$$

$$y_{i+1} = y_i - \frac{h_k(y_i)}{h_k{}'(y_i)}, \qquad i = 0, 1, 2, \ldots \tag{38}$$

end of iteration for

$$\left|\frac{y_{n+1} - y_n}{y_{n+1}}\right| < \text{tol},$$



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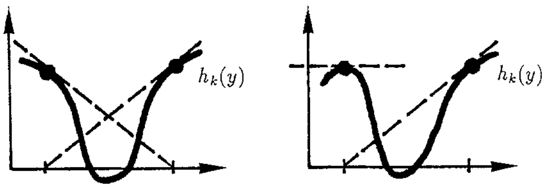

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Fig. 7 and 8. 7 Computation of $\bar{y}_k$; **8** Failure of Newton's method

where y_{n+1} is the result of the last iteration carried out. Several examples have shown that the iteration's starting point y_0 has to be chosen very careful by nesting of intervalls to avoid the situations shown in Fig. 8.

Finally it has to be tested whether the computed extremum point is a minimum.

Once $\bar{y}_k$ is determined we define

$$\bar{y}_k \overset{\text{def}}{=} y_{n+1} \qquad \text{and thereby} \qquad f(\bar{y}_k) = f(y_{n+1}). \tag{39}$$

Since $\bar{y}_k$ depends on the position of $\bar{\mathbf{x}}_k$

$$\bar{y}_k = \bar{y}_k(\bar{\mathbf{x}}_k) = \bar{y}_k(\bar{x}_{k1}, \bar{x}_{k2}) \tag{40}$$

we can write

$$\delta\bar{y}_k = \frac{\partial\bar{y}_k}{\partial\bar{x}_{k1}}\,\delta\bar{x}_{k1} + \frac{\partial\bar{y}_k}{\partial\bar{x}_{k2}}\,\delta\bar{x}_{k2} = \nabla\bar{y}_k\cdot\delta\bar{\mathbf{x}}_k = \nabla\bar{y}_k\cdot\left(\sum_{I=1}^{N} N_I{}^k\,\delta\mathbf{x}_I\right). \tag{41}$$

To compute the gradient $\nabla\bar{y}_k$ in (41) we use total differentiation of $h_k(y(\bar{\mathbf{x}}_k)) \overset{!}{=} 0$ (see Heuser [11]):

$$h_k(y) = h_k(y(\bar{\mathbf{x}}_k)) \overset{!}{=} 0: \tag{42}$$

$$0 = \frac{dh_k}{d\bar{x}_{k1}} = \frac{\partial h_k}{\partial\bar{x}_{k1}} + \frac{\partial h_k}{\partial y}\,\frac{\partial y}{\partial\bar{x}_{k1}}$$

$$\frac{\partial h_k}{\partial\bar{x}_{k1}} = -1 \qquad \frac{\partial h_k}{\partial y} = 1 - f''(y)\,(\bar{x}_{k2} - f(y)) + f'^2(y) \tag{43}$$

$$\Rightarrow \quad \frac{\partial y}{\partial\bar{x}_{k1}} = \frac{1}{1 - f''(y)\,(\bar{x}_{k2} - f(y)) + f'^2(y)}.$$

In the same way we obtain

$$\frac{\partial y}{\partial\bar{x}_{k2}} = \frac{f'(y)}{1 - f''(y)\,(\bar{x}_{k2} - f(y)) + f'^2(y)}. \tag{44}$$

Introducing the abbreviation

$$\bar{c}_k = \frac{1 + f'^2(y_k)}{1 - f''(\bar{y}_k)\,(\bar{x}_{k2} - f(\bar{y}_k)) + f'^2(\bar{y}_k)} \tag{45}$$

we derive

$$\nabla\bar{y}_k = \frac{\bar{c}_k}{1 + f'^2(\bar{y}_k)}\begin{pmatrix}1\\ f'(\bar{y}_k)\end{pmatrix} = \frac{\bar{c}_k}{\sqrt{1 + f'^2(\bar{y}_k)}}\,\mathbf{t}_f{}^k \tag{46}$$

which yields

$$\delta\bar{y}_k = \nabla\bar{y}_k \cdot \delta\bar{\mathbf{x}}_k = \frac{\bar{c}_k}{\sqrt{1+f'^2(\bar{y}_k)}}\, \mathbf{t}_f{}^k \cdot \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \tag{47}$$

and finally

$$\delta\bar{\mathbf{y}}_k = \delta\bar{y}_k \begin{pmatrix} 1 \\ f'(\bar{y}_k) \end{pmatrix} = \mathbf{c}_k \left(\mathbf{t}_f{}^k \cdot \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \right) \mathbf{t}_f{}^k . \tag{48}$$

One recognizes that the vector $\delta\bar{\mathbf{y}}_k$ is tangential to the obstacle – as could be expected.
We now can write the variation in the following form:

$$\delta\Pi^h(\mathbf{u}) + \sum_{e=1}^{\text{numel}} \frac{\varepsilon}{2} \sum_{k=1}^{p} \omega_k \left[2\bar{g}_k \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\| \left(\sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I - \bar{c}_k \left(\mathbf{t}_f{}^k \cdot \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \right) \mathbf{t}_f{}^k \right) \cdot \mathbf{n}_f{}^k \right.$$
$$\left. + \bar{g}_k{}^2 \sum_{I=1}^{N} N_{I,\xi}^k\, \delta\mathbf{x}_I \cdot \mathbf{t}_e{}^k \right] = 0. \tag{49}$$

Because of $\mathbf{t}_f{}^k \perp \mathbf{n}_f{}^k$ this expression can be simplified to its final form

$$\boxed{\delta\Pi^h(\mathbf{u}) + \sum_{e=1}^{\text{numel}} \sum_{k=1}^{p} \left\{ \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \cdot \left[\varepsilon\omega_k \bar{g}_k \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\| \mathbf{n}_f{}^k \right] + \sum_{I=1}^{N} N_{I,\xi}^k\, \delta\mathbf{x}_I \cdot \left[\frac{\varepsilon\omega_k}{2}\, \bar{g}_k{}^2 \mathbf{t}_e{}^k \right] \right\} = 0.} \tag{50}$$

Writing $\delta\mathbf{u}_I$ instead of $\delta\mathbf{x}_I$ and introducing the abbreviations

$$(\delta\mathbf{u}_e)^T \overset{\text{def}}{=} (\delta u_{11}, \delta u_{12}, \delta u_{21}, \delta u_{22}, \ldots, \delta u_{N1}, \delta u_{N2}) \tag{51}$$

and

$$\mathbf{N}_e{}^k \overset{\text{def}}{=} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ \vdots & \vdots \\ N_N & 0 \\ 0 & N_N \end{bmatrix} \quad \text{and} \quad \mathbf{N}_{\xi_e}^k \overset{\text{def}}{=} \begin{bmatrix} N_{1\xi} & 0 \\ 0 & N_{1\xi} \\ N_{2\xi} & 0 \\ 0 & N_{2\xi} \\ \vdots & \vdots \\ N_{N\xi} & 0 \\ 0 & N_{N\xi} \end{bmatrix}, \tag{52}$$

respectively, as well as the constants

$$A_k \overset{\text{def}}{=} \varepsilon\omega_k \bar{g}_k \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\| \tag{53}$$

and

$$B_k \overset{\text{def}}{=} \frac{\varepsilon\omega_k}{2}\, \bar{g}_k{}^2, \tag{54}$$

yields

$$\sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{u}_I = (\delta\mathbf{u}_e)^T \cdot \mathbf{N}_e{}^k . \tag{55}$$

Therewith the variation takes the form

$$\delta\Pi^h(\mathbf{u}) + \sum_{e=1}^{\text{numel}} (\delta\mathbf{u}_e)^T \cdot \sum_{k=1}^{p} \{ A_k \mathbf{N}_e{}^k \cdot \mathbf{n}_f{}^k + B_k \mathbf{N}_{\xi_e}^k \cdot \mathbf{t}_e{}^k \} = 0. \tag{56}$$

4.2 Linearization

To compute the tangent matrix $D\,\delta F(x)$ for the NEWTON scheme we have to linearize equation (50) of section 4.1. For one element we have

$$\Delta\delta \sum_{k=1}^{p} \frac{\varepsilon\omega_k}{2} \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\| = \sum_{k=1}^{p} \frac{\varepsilon\omega_k}{2} \Bigg[\Delta\delta \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\|$$
$$+ \delta \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 \Delta \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\|$$
$$+ \Delta \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 \delta \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\|$$
$$+ \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 \Delta\delta \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\| \Bigg]. \tag{57}$$

Since we already have computed

$$\delta \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 = 2\bar{g}_k \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \cdot \mathbf{n}_f{}^k \tag{58}$$

in (49) or (50), respectively, we find in an analogous way

$$\Delta \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 = 2\bar{g}_k \mathbf{n}_f{}^k \cdot \sum_{I=1}^{N} N_I{}^k\, \Delta\mathbf{x}_I. \tag{59}$$

Similarly using (36) we obtain

$$\Delta \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\| = t_e{}^k \cdot \sum_{I=1}^{N} N_{I,\xi}^k\, \Delta\mathbf{x}_I\,. \tag{60}$$

Thus the only terms to be determined in (57) are

$$\Delta\delta \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 \quad \text{and} \quad \Delta\delta \left\| \sum_{I=1}^{N} N_{I,\xi}^k \mathbf{x}_I \right\|.$$

Straight forward computation yields

$$\Delta\delta \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 = \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \cdot \Delta(2\bar{g}_k \mathbf{n}_f{}^k)$$
$$= \sum_{I=1}^{N} \mathrm{N}_I{}^k\, \delta\mathbf{x}_I \cdot 2\Delta \left(\sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right)$$
$$= \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \cdot 2 \left[\sum_{I=1}^{N} N_I{}^k\, \Delta\mathbf{x}_I - \bar{c}_k \left(\mathbf{t}_f{}^k \cdot \sum_{I=1}^{N} N_I{}^k\, \Delta\mathbf{x}_I \right) \mathbf{t}_f{}^k \right]$$
$$= \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \cdot 2 \left[\sum_{I=1}^{N} N_I{}^k\, \Delta\mathbf{x}_I - \bar{c}_k (\mathbf{t}_f{}^k \otimes \mathbf{t}_f{}^k) \cdot \sum_{I=1}^{N} N_I{}^k\, \Delta\mathbf{x}_I \right]$$
$$= \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \cdot 2[\mathbf{1} - \bar{c}_k (\mathbf{t}_f{}^k \otimes \mathbf{t}_f{}^k)] \cdot \sum_{I=1}^{N} N_I{}^k\, \Delta\mathbf{x}_I. \tag{61}$$

In the same way we find

$$\Delta\delta \left\| \sum_{I=1}^{N} N_{I,\xi}^{k} \mathbf{x}_I \right\| = \sum_{I=1}^{N} N_{I,\xi}^{k}\, \delta\mathbf{x}_I \cdot \frac{1}{\left\| \sum_{I=1}^{N} N_{I,\xi}^{k} \mathbf{x}_I \right\|} (\mathbf{n}_e{}^k \otimes \mathbf{n}_e{}^k) \cdot \sum_{I=1}^{N} N_{I,\xi}^{k}\, \Delta\mathbf{x}_I . \tag{62}$$

For this algebra we have used the relations for dyadic tensor products $(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \otimes \mathbf{a}) \cdot \mathbf{b}$ and $(\mathbf{a} \cdot \mathbf{b})\,(\mathbf{c} \cdot \mathbf{d}) = \mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{c}) \cdot \mathbf{d}$.

The final result for the linearization of (50) is now obtained by inserting (58) to (62) into (57):

$$\Delta\delta \sum_{k=1}^{p} \frac{\varepsilon\omega_k}{2} \left\| \sum_{I=1}^{N} N_I{}^k \mathbf{x}_I - \bar{\mathbf{y}}_k \right\|^2 \left\| \sum_{I=1}^{N} N_{I,\xi}^{k} \mathbf{x}_I \right\|$$

$$= \sum_{k=1}^{p} \left\{ \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \cdot \varepsilon\omega_k \left\| \sum_{I=1}^{N} N_{I,\xi}^{k} \mathbf{x}_I \right\| [\mathbf{1} - \bar{c}_k(\mathbf{t}_f{}^k \otimes \mathbf{t}_f{}^k)] \cdot \sum_{I=1}^{N} N_I{}^k\, \Delta\mathbf{x}_I \right.$$

$$+ \sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{x}_I \cdot \varepsilon\omega_k \bar{g}_k (\mathbf{n}_f{}^k \otimes \mathbf{t}_e{}^k) \cdot \sum_{I=1}^{N} N_{I,\xi}^{k}\, \Delta\mathbf{x}_I + \sum_{I=1}^{N} N_{I,\xi}^{k}\, \delta\mathbf{x}_I \cdot \varepsilon\omega_k \bar{g}_k (\mathbf{t}_e{}^k \otimes \mathbf{n}_f{}^k) \cdot \sum_{I=1}^{N} N_I{}^k\, \Delta\mathbf{x}_I$$

$$\left. + \sum_{I=1}^{N} N_{I,\xi}^{k}\, \delta\mathbf{x}_I \cdot \frac{\varepsilon\omega_k}{2} \frac{\bar{g}_k{}^2}{\left\| \sum_{I=1}^{N} N_{I,\xi}^{k} \mathbf{x}_I \right\|} (\mathbf{n}_e{}^k \otimes \mathbf{n}_e{}^k) \cdot \sum_{I=1}^{N} N_{I,\xi}^{k}\, \Delta\mathbf{x}_I \right\} .$$

If we now replace $\delta\mathbf{x}_I$ and $\Delta\mathbf{x}_I$ by $\delta\mathbf{u}_I$ and $\Delta\mathbf{u}_I$ and introduce the matrices $\mathbf{N}_e{}^k$ and $\mathbf{N}_{\xi_e}^k$ defined in section 4.1 we can write

$$\sum_{I=1}^{N} N_I{}^k\, \delta\mathbf{u}_I = (\delta\mathbf{u}_e)^T \cdot \mathbf{N}_e{}^k \quad \text{and} \quad \sum_{I=1}^{N} N_{I,\xi}^{k}\, \delta\mathbf{u}_I = (\delta\mathbf{u}_e)^T \cdot \mathbf{N}_{\xi_e}^k \tag{64}$$

as well as

$$\sum_{I=1}^{N} N_I{}^k\, \Delta\mathbf{u}_I = (\mathbf{N}_e{}^k)^T \cdot \Delta\mathbf{u}_e \quad \text{and} \quad \sum_{I=1}^{N} N_{I,\xi}^{k}\, \Delta\mathbf{u}_I = (\mathbf{N}_{\xi_e}^k)^T \cdot \delta\mathbf{u}_e . \tag{65}$$

Together with the abbreviation

$$(\Delta\mathbf{u}_e)^T = (\Delta u_{11}, \Delta u_{12}, \Delta u_{21}, \Delta u_{22}, \ldots, \Delta u_{N1}, \Delta u_{N2}) \tag{66}$$

we can describe the linearized penalty term in its final form:

$$\Delta\delta \frac{\varepsilon}{2} \int_{\gamma_c} g_+{}^2 \, \mathrm{d}\Gamma \approx (\Delta\mathbf{u}_e)^T \cdot \mathbf{K}_e \cdot \Delta\mathbf{u}_e , \tag{67}$$

where the tangent matrix $\mathbf{K}_e$ for one element has the representation

$$\mathbf{K}_e = \sum_{k=1}^{p} \left\{ \mathbf{N}_e{}^k \cdot \varepsilon\omega_k \left\| \sum_{I=1}^{N} N_{I,\xi}^{k} \mathbf{x}_I \right\| [\mathbf{1} - \bar{c}_k(\mathbf{t}_f{}^k \otimes \mathbf{t}_f{}^k)] \cdot (\mathbf{N}_e{}^k)^T + \mathbf{N}_e{}^k \cdot \varepsilon\omega_k \bar{g}_k (\mathbf{n}_f{}^k \otimes \mathbf{t}_e{}^k) \cdot (\mathbf{N}_{\xi_e}^k)^T \right.$$

$$\left. + \mathbf{N}_{\xi_e}^k \cdot \varepsilon\omega_k \bar{g}_k (\mathbf{t}_e{}^k \otimes \mathbf{n}_f{}^k) \cdot (\mathbf{N}_e{}^k)^T + \mathbf{N}_{\xi_e}^k \cdot \frac{\varepsilon\omega_k}{2} \frac{\bar{g}_k{}^2}{\left\| \sum_{I=1}^{N} N_{I,\xi}^{k} \mathbf{x}_I \right\|} (\mathbf{n}_e{}^k \otimes \mathbf{n}_e{}^k) \cdot (\mathbf{N}_{\xi_e}^k)^T \right\} . \tag{68}$$

The corresponding right side $\mathbf{P}_e$ for one contact element is the result of the variation of the penalty term, except a multiplication by (-1) (compare the remarks concerning Newton's method):

$$\mathbf{P}_e = -\sum_{k=1}^{p} \{A_k \mathbf{N}_e^k \cdot \mathbf{n}_f^k + B_k \mathbf{N}_{\xi_e}^k \cdot \mathbf{t}_e^k\}. \tag{69}$$

5 Numerical examples

We give two examples: one classical for the linear case and one for the nonlinear one where large deformations are involved. The computations have been carried out within the environment of the Finite Element Analysis Program (FEAP), see Zienkiewicz, Taylor [13].

5.1 *Hertz contact: cylinder on a rigid foundation*

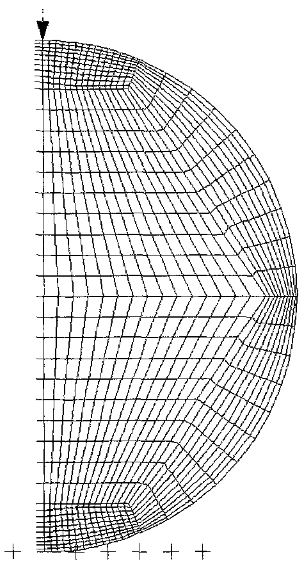
The first example is a Hertz contact problem which has an analytical solution. We compute the Hertz solution for a cylinder under a single load on a rigid foundation. We use the mesh shown in Fig. 9, the material constants $E_{\text{cyl}} = 500$, $\nu_{\text{cyl}} = 0.3$ and a penalty parameter of $\varepsilon = 10^6$. The radius of the cylinder is $R_{\text{cyl}} = 8$.

Assuming plane strain conditions we obtain the results that are plotted in Fig. 10.

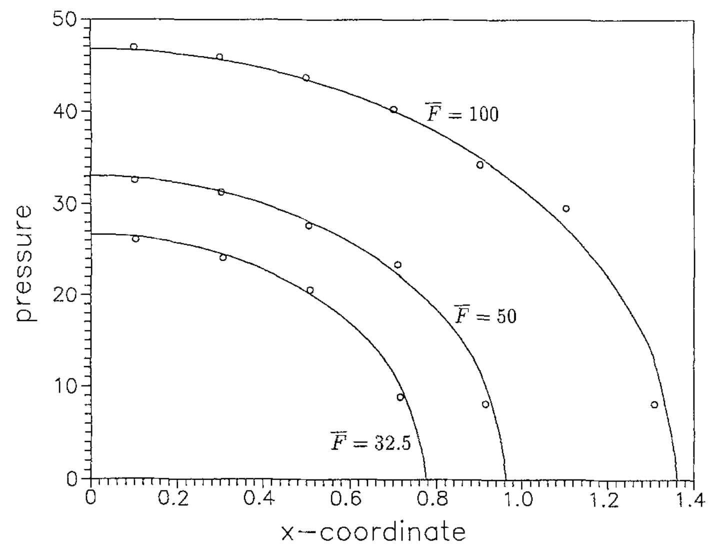
The solid lines show the analytic solution for Hertz contact of a cylinder with a plane rigid foundation which can be found in Johnson [14] or Goldsmith [15]:

$$p = \frac{E}{2(1-\nu^2)} \frac{1}{R} \sqrt{4\bar{F}\,\frac{1-\nu^2}{E\pi}\,R - x^2}\,.$$

Here E is the elastic modulus of the cylinder, ν is its Poisson ratio and R its radius. $\bar{F}$ denotes the load per unit length. The numerical solution is in good agreement with the analytical one.



9



10

Fig. 9 and 10. **9** Mesh for Hertz contact; **10** Finite element solution in comparison to analytic solution

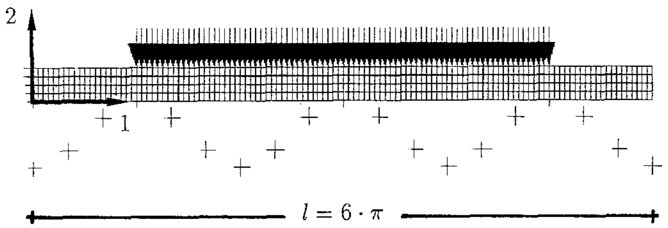


Fig. 11. Elastic bar on a rigid foundation

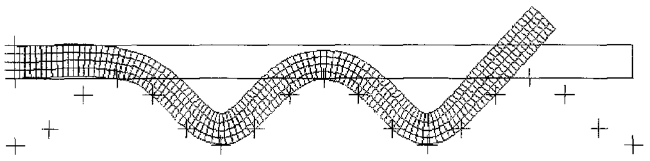


Fig. 12. Deformed configuration of the elastic bar

5.2 Large deformations: bar on a curved rigid foundation

We consider an elastic bar with a hyperelastic material law as mentioned in (3). The bar lies on a rigid foundation which is described by the function

$$f(x) = -\cos x - 1,$$

see Fig. 11.

At the bar's left end all displacements are restricted. In the middle part of the bar we apply a constant load $p = 3000/\pi$. The material parameters are $\lambda = 12000$, $\mu = 6000$ and the penalty parameter is chosen as $\varepsilon = 10^5$. Fig. 12 shows the deformation: The bar exactly takes the form of the foundation. The crosses represent the points of the foundation that are used to interpolate the obstacle function $f(x)$.

6 Conclusion

The present paper is concerned with unilateral contact problems assuming large elastic deformations.

For the deduction of the algorithm great importance is attributed to obtain a description of the problem that is as realistic as possible:

- the check for contact at the Gaussian points of the contact elements is consistent to the discretization of the deformable body.
- the interpolation of the obstacle by cubic splines gives a consistent contact normal.
- special attention is directed to a consistent linearization.

Two examples show the usability and performance of the algorithm.

In future work it is planed to extend the research to contact problems of two deformable bodies. Furthermore heat conduction and the influence of friction shall be included.

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