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CALCULATION OF IMPACT–CONTACT PROBLEMS OF THIN ELASTIC SHELLS TAKING INTO ACCOUNT GEOMETRICAL NONLINEARITIES WITHIN THE CONTACT REGION

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During impact of elastic bodies, contact stresses are transmitted in time-depending contact surfaces. In many impact contact problems, large displacements and rotations appear only in the contact surface and in a certain neighbourhood. Therefore, it is efficient to consider geometrical nonlinearities only in this region, and to describe the remainder of the body within the geometrical linear theory. This leads to substructure techniques where only properties of the nonlinear elements need be modified during the impact contact process.

The principle of virtual work for nonlinear thin shells is expressed using the total Lagrangian formulation, and the geometrical nonlinearity of thin shells is described in the frame of moderate rotation theory.

The contact conditions lead to inequalities for the normal stresses and displacements in the contact interfaces. Therefore, the numerical algorithm involves two superposed iterations: for the computation of contact forces and contact areas and for the geometrical nonlinearity. The iteration procedure has to be carried out in each time step.

The spatial discretization using finite element techniques leads to a system of ordinary differential equations which is integrated over the time using the Newmark algorithm.

Numerical results were obtained for the impact contact problem of spherical shells. For these examples, the impact forces and the contact pressure distribution are presented for several parameter combinations. Results are controlled by conservation laws in integral form, and compared with results from geometrical linear theory.

1. Introduction

The treatment of impact problems is of growing interest in modern technology. To guarantee the safety of structures—e.g. containments in reactor engineering—against striking objects, one needs large scale experiments. From such measurements, assumptions for the impact forces over the time are deduced. In view of high costs for experiments, questionable extrapolations for the impact forces of different systems are sometimes introduced for the response analysis. Therefore, numerical algorithms are necessary which allow to calculate all time dependent quantities during the impact–contact process. Because of changing contact surfaces during impact and complicated structures, FE-discretizations are advantageous. During the impact process there exist time-dependent contact regions within which the contact stresses are transmitted. The contact surfaces are a priori unknown. The contact conditions are given in the form of inequalities for displacements and contact stresses. This leads to an iterative algorithm for the determination of contact regions and stresses in each time step, so that nonlinear algebraic systems arise even in the case of a geometrically linear description of the systems in the frame of continuum mechanics. Such linear impact–contact problems of

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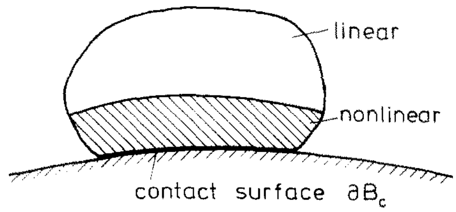


Fig. 1. Geometrically nonlinear description of contact surface and a sufficient neighbourhood.

elastic bodies were treated by Wriggers [1]. Especially two spherical shells impacting a rigid wall, i.e., the case of two disconnected contact regions was investigated using FE-displacement models, introducing the contact conditions directly. A substructure technique was used so that contact algorithms only concerned the contact domains whereas the remaining domains could be pre-eliminated in advance.

Hughes, Taylor et al. [2, 3] treated impact–contact processes of axisymmetric elastic bodies within a geometrically nonlinear theory. Here, contact conditions for the contact stresses are introduced via Lagrangian multipliers in the variational functional, and special contact elements are developed.

In this paper impact–contact problems of thin elastic shells are investigated within a geometrically nonlinear bending theory, or more exactly, within a moderate rotation theory. Compared with the magnitude of the bodies concerned, the contact surfaces are small in many cases. Often large displacements and rotations arise only within and near the contact regions. This is, for example, valid for impact–contact problems of thin spherical shells. Therefore, it is efficient to consider geometrical nonlinearities only in contact regions. The remainder of the body can be described within a geometrically linear theory, see Fig. 1. A considerable reduction of computational effort can be achieved by taking advantage of the nonlinear localization, using a modified substructure technique. Clough and Wilson [4] and Noor [5] discussed substructure concepts applied to dynamic response analysis of structures with local nonlinearities. The substructure technique will be used in this paper within the direct dynamic analysis of subsequent contact configurations. The direct analysis involves the step-by-step integration of the equations of motion. In this case a substructure technique analogous to static condensation can be used. Nonlinear properties of the elements only arise within and near the contact domains whereas the linear remainders can be pre-eliminated in the same way as in the overall geometrically linear theory for shells.

2. The contact problem

The step-by-step integration of the equations of motion for the discretized system in the impact–contact process implies the solution of a contact problem at each time step. In the contact regions, certain inequalities for displacements and stresses must be fulfilled. In this paper, friction is excluded. This problem is treated by Stein and Wriggers in [6].

2.1. Contact conditions

The following inequalities must be satisfied during the impact process, see Fig. 2.

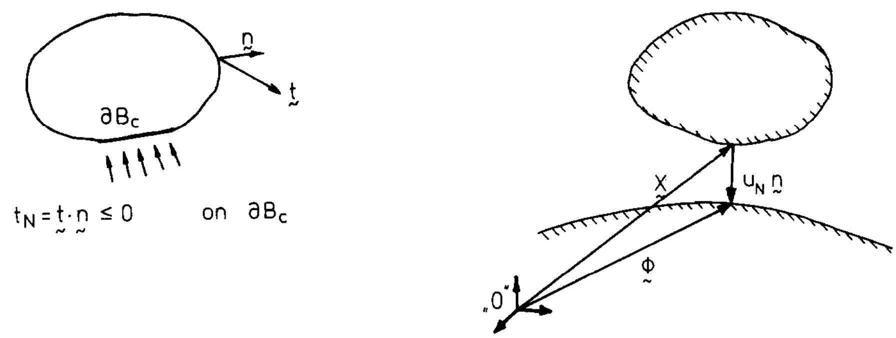


Fig. 2. Contact conditions. (a) Condition for normal stresses; (b) Kinematical condition.

(i) Adhesion and friction are not permitted; therefore, only compression stresses are transmitted,

$$t_n = \boldsymbol{t} \cdot \boldsymbol{n} \leq 0 \quad \text{on } \partial B_c . \tag{2.1}$$

(ii) Penetrations of material points are not permitted,

$$u_n - \Delta X_n \leq 0 \quad \text{on } \partial B_c \tag{2.2}$$

with

$$u_n = \boldsymbol{u} \cdot \boldsymbol{n} \tag{2.3a}$$

and

$$\Delta X_n = (\boldsymbol{\phi} - \boldsymbol{X}) \cdot \boldsymbol{n} . \tag{2.3b}$$

2.2. Contact algorithm

The direct implementation of the above contact conditions into the FE-algorithm leads to the following iteration scheme at a time step. Substructure technique is treated in Section 5.

(1) Assume contact surface ∂B_c.
(2) Solve the associated boundary value problem by FEM.
(3) Calculate normal stresses t_N on ∂B_c.
(4) Check condition (2.1) (no adhesion!)
if violated: release points where $t_N > 0$: go to (2)
if satisfied: go to (5).
(5) Check condition (2.2) (no penetration!)
if violated: increase ∂B_c: go to (2)
if satisfied: iteration finished.

3. Geometrically nonlinear theory of thin shells

Nonlinear shell theories were developed first for stability analysis, [7, 8]. In this theory the only nonlinear terms in the membrane strains $\gamma_{\alpha\beta}$ are $w_{,\alpha} \cdot w_{,\beta}$ whereas the changes of curvatures $\kappa_{\alpha\beta}$ remain linear. It is difficult to derive nonlinear shell theories of consistent accuracy for different orders of error magnitudes with respect to characteristic measures of the shell.

In this paper we assume thin shells, linear and isotropic elastic material, and small strains, so that the specific elastic potential energy is given by a quadratic functional of the strains. Based on works of Koiter [9] and John [10], Pietraszkiewicz [11, 12] gave four stages of consistent nonlinear theories where the rotations in the frame of Kirchhoff–Love hypothesis are used as a measure for classification.

3.1. Moderate rotation shell theory

The shell strain energy function (3.1) for isotropic elastic material with the above assumption was given by Koiter [9] for a consistent linear first-approximation theory (small strains)

$$\Sigma = \frac{h}{2} H^{\alpha\beta\lambda\mu} \left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12} \kappa_{\alpha\beta}\kappa_{\lambda\mu} \right), \tag{3.1}$$

with the components of elasticity tensor

$$H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} \left(a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\lambda\mu} \right). \tag{3.2}$$

The same form of the strain energy was derived by Pietrasciewicz [10] using error estimates of the stresses and their derivatives which were introduced by John [9]. Using the strain energy function (3.1), the constitutive equations for the stress results are obtained

$$N^{\alpha\beta} = h H^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu} \tag{3.3}$$

and the stress couples

$$M^{\alpha\beta} = \frac{h^3}{12} H^{\alpha\beta\lambda\mu} \kappa_{\lambda\mu}. \tag{3.4}$$

In the above relations the assumption of small strains was used everywhere in the shell, but no restrictions for the magnitude of rotations of material elements were made. In many engineering problems of thin shells, rotations of the middle surface can be of considerable magnitude. Therefore, it is reasonable to use shell equations resulting from consistently restricted rotations of different order of magnitude. Pietraszkiewicz [11] give a classification of approximated shell equations in terms of the finite rotation vector $\boldsymbol{\Omega}$, which is defined by the angle of rotation ω with respect to the axis of rotation described by a unit vector $\boldsymbol{e}$,

$$\boldsymbol{\Omega} = \sin \omega \boldsymbol{e}. \tag{3.5}$$

Let the small parameter θ have the form [10]

$$\vartheta = \max\left(\frac{h}{L}, \frac{h}{d}, \sqrt{\frac{h}{R}}, \sqrt{\eta}\right), \tag{3.6}$$

where
h is shell thickness,
L the smallest wave length of deformation pattern,
d the distance of the considered point from the lateral boundary,
η the maximum eigenvalue of the Green strain tensor of the shell and
R the characteristic radius.
The magnitude of the rotation angle in terms of ϑ can be classified as

$$\begin{array}{lll} \text{(i)} & \omega \le \mathrm{O}(\vartheta^2) & \text{small rotations,} \\ \text{(ii)} & \omega = \mathrm{O}(\vartheta) & \text{moderate rotations,} \\ \text{(iii)} & \omega = \mathrm{O}(\sqrt{\vartheta}) & \text{large rotations,} \\ \text{(iv)} & \omega \ge \mathrm{O}(1) & \text{finite rotations.} \end{array} \tag{3.7}$$

In this paper a moderate rotations theory is used. Expanding (3.5) into Taylor series at $\omega = 0$ one obtains in the case of moderate rotations,

$$\sin\omega = \omega + \mathrm{O}(\vartheta^3), \qquad \cos\omega = 1 + \mathrm{O}(\vartheta^2). \tag{3.8}$$

With the further assumption that the rotation around the normal vector is small, while the rotations around tangents to the middle surface are moderate, one gets the nonlinear surface strain tensor

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \tfrac{1}{2}\varphi_\alpha\varphi_\beta \tag{3.9}$$

in the frame of moderate rotation theory [11]. Here,

$$\theta_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w \tag{3.10}$$

is the linear strain tensor with the surface curvature tensor $b_{\alpha\beta}$, and the displacement gradients of the middle surface $u_{\alpha|\beta}$. φ_α are the linearized rotations of the normal to the surface

$$\varphi_\alpha = w_{,\alpha} + b^\lambda_\alpha u_\lambda\,. \tag{3.11}$$

Within the moderate rotation theory, the curvature strain tensor of the middle surface

$$\kappa_{\alpha\beta} = -\tfrac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}) \tag{3.12}$$

remains linear.

3.2. Incremental principle of virtual work for the shell in elasto-kinetics

In the following, an incremental principle of virtual work is derived in a 'total Lagrangian representation (T.L)' which has been discussed by Bathe, Ramm and Wilson [13], Larsen and Popov [14], Klee, Paulun and Stein [15] and others. Here, the total Lagrangian formulation will be used in the frame of moderate rotation theory for shells.

From Fig. 3 one gets the relations

$$ {}^{1}\boldsymbol{x} = {}^{0}\boldsymbol{x} + {}^{1}\boldsymbol{u}\,, \tag{3.13a}$$

$$ {}^{2}\boldsymbol{x} = {}^{1}\boldsymbol{x} + \Delta\boldsymbol{u}\,, \tag{3.13b}$$

$$ {}^{2}\boldsymbol{u} = {}^{1}\boldsymbol{u} + \Delta\boldsymbol{u}\,. \tag{3.13c}$$

The principle of virtual work for the neighbouring configuration can be written as

$$\int_{\mathcal{M}} ({}^{2}N^{\alpha\beta}\,{}^{2}\delta\gamma_{\alpha\beta} + {}^{2}M^{\alpha\beta}\,{}^{2}\delta\kappa_{\alpha\beta})\,\mathrm{d}a = \int_{\mathcal{M}} {}^{2}\boldsymbol{f}\cdot\delta{}^{2}\boldsymbol{u}\,\mathrm{d}a + \int_{\mathcal{M}} \rho h\,{}^{2}\ddot{\boldsymbol{u}}\cdot\delta{}^{2}\boldsymbol{u}\,\mathrm{d}a\,. \tag{3.14}$$

With the definition of increments

$$\begin{aligned} &\Delta N^{\alpha\beta} := {}^{2}N^{\alpha\beta} - {}^{1}N^{\alpha\beta}\,, \qquad \Delta M^{\alpha\beta} := {}^{2}M^{\alpha\beta} - {}^{1}M^{\alpha\beta}\,,\\ &\Delta\gamma_{\alpha\beta} := {}^{2}\gamma_{\alpha\beta} - {}^{1}\gamma_{\alpha\beta}\,, \qquad \Delta\kappa_{\alpha\beta} := {}^{2}\kappa_{\alpha\beta} - {}^{1}\kappa_{\alpha\beta}\,, \end{aligned} \tag{3.15}$$

we obtain from (3.14)

$$\begin{aligned} &\int_{\mathcal{M}} [({}^{1}N^{\alpha\beta} + \Delta N^{\alpha\beta})^{2}\delta\gamma_{\alpha\beta} + ({}^{1}M^{\alpha\beta} + \Delta M^{\alpha\beta})^{2}\delta\kappa_{\alpha\beta}]\,\mathrm{d}a\\ &\quad = \int_{\mathcal{M}} {}^{2}\boldsymbol{f}\cdot\delta{}^{2}\boldsymbol{u}\,\mathrm{d}a + \int_{\mathcal{M}} \rho h\,{}^{2}\ddot{\boldsymbol{u}}\cdot\delta{}^{2}\boldsymbol{u}\,\mathrm{d}a\,. \end{aligned} \tag{3.16}$$

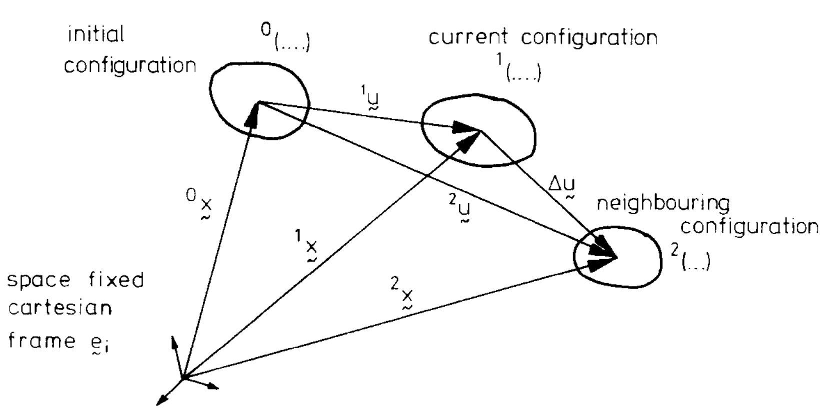


Fig. 3. Configurations of a deformable solid, e.g. a shell.

According to moderate rotation theory, one gets the strains of the middle surface for the neighbouring configuration as

$$ {}^2\gamma_{\alpha\beta} = {}^2\theta_{\alpha\beta} + \tfrac{12}{2}\varphi_\alpha\, {}^2\varphi_\beta = {}^1\theta_{\alpha\beta} + \Delta\theta_{\alpha\beta} + \tfrac{1}{2}({}^1\varphi_\alpha + \Delta\varphi_\alpha)({}^1\varphi_\beta + \Delta\varphi_\beta) \tag{3.17} $$

and the increments

$$ \Delta\gamma_{\alpha\beta} = \Delta\theta_{\alpha\beta} + \tfrac{1}{2}(\Delta\varphi_\alpha\, {}^1\varphi_\beta + {}^1\varphi_\alpha \Delta\varphi_\beta + \Delta\varphi_\alpha\Delta\varphi_\beta)\,. \tag{3.18} $$

The variation of these quantities yields

$$ \delta^2\boldsymbol{u} = \delta({}^1\boldsymbol{u} + \Delta\boldsymbol{u}) = \delta\boldsymbol{u}\,, \tag{3.19} $$

$$ \delta\Delta\gamma_{\alpha\beta} = \delta\Delta\theta_{\alpha\beta} + \tfrac{1}{2}(\delta\Delta\varphi_\alpha\, {}^1\varphi_\beta + {}^1\varphi_\alpha\delta\Delta\varphi_\beta + \delta\Delta\varphi_\alpha\Delta\varphi_\beta + \Delta\varphi_\alpha\delta\Delta\varphi_\beta)\,, \tag{3.20a} $$

$$ \delta\Delta\kappa_{\alpha\beta} = \delta\Delta\kappa_{\alpha\beta} \tag{3.20b} $$

and

$$ {}^2\delta\gamma_{\alpha\beta} = \delta\Delta\gamma_{\alpha\beta}\,. \tag{3.21} $$

$\delta\Delta\gamma_{\alpha\beta}$ can be splitted into the following linear and nonlinear terms

$$ \delta\Delta\gamma^{\mathrm{L}}_{\alpha\beta} = \delta\Delta\theta_{\alpha\beta} + \tfrac{1}{2}(\delta\Delta\varphi_\alpha\, {}^1\varphi_\beta + {}^1\varphi_\alpha\delta\Delta\varphi_\beta) \tag{3.22a} $$

$$ \delta\Delta\gamma^{\mathrm{NL}}_{\alpha\beta} = \tfrac{1}{2}(\delta\Delta\varphi_\alpha\Delta\varphi_\beta + \Delta\varphi_\beta\delta\Delta\varphi_\beta)\,. \tag{3.22b} $$

Then the work principle can be written as

$$ \begin{aligned} &\int_{\mathcal{M}} [({}^1N^{\alpha\beta} + \Delta N^{\alpha\beta})(\delta\Delta\gamma^{\mathrm{L}}_{\alpha\beta} + \delta\Delta\gamma^{\mathrm{NL}}_{\alpha\beta}) + ({}^1M^{\alpha\beta} + \Delta M^{\alpha\beta})\delta\Delta\kappa_{\alpha\beta}]\,\mathrm{d}a \\ &\quad = \int_{\mathcal{M}} {}^2\boldsymbol{f}\cdot\delta\Delta\boldsymbol{u}\,\mathrm{d}a + \int_{\mathcal{M}} \rho h\, {}^2\ddot{\boldsymbol{u}}\cdot\delta\Delta\boldsymbol{u}\,\mathrm{d}a \end{aligned} \tag{3.23} $$

or in the alternate form

$$ \begin{aligned} &\int_{\mathcal{M}} (\Delta N^{\alpha\beta}\delta\Delta\gamma_{\alpha\beta} + \Delta M^{\alpha\beta}\delta\Delta\kappa_{\alpha\beta})\,\mathrm{d}a + \int_{\mathcal{M}} {}^1N^{\alpha\beta}\delta\Delta\gamma^{\mathrm{NL}}_{\alpha\beta}\,\mathrm{d}a \\ &\quad = \int_{\mathcal{M}} {}^2\boldsymbol{f}\cdot\delta\Delta\boldsymbol{u}\,\mathrm{d}a + \int_{\mathcal{M}} \rho h^2\ddot{\boldsymbol{u}}\cdot\delta\Delta\boldsymbol{u}\,\mathrm{d}a - \int_{\mathcal{M}} ({}^1N^{\alpha\beta}\delta\Delta\gamma^{\mathrm{L}}_{\alpha\beta} + {}^1M^{\alpha\beta}\delta\Delta\kappa_{\alpha\beta})\,\mathrm{d}a\,. \end{aligned} \tag{3.24} $$

Introducing the constitutive relations (3.3) and (3.4)

$$\Delta N^{\alpha\beta} = hH^{\alpha\beta\lambda\delta}\Delta\gamma_{\lambda\delta}\,, \tag{3.25a}$$

$$\Delta M^{\alpha\beta} = \frac{h^3}{12}H^{\alpha\beta\lambda\delta}\Delta\kappa_{\lambda\delta}\,, \tag{3.25b}$$

leads to the material dependent work principle, i.e., the first variation of the kinetic potential

$$\begin{aligned}&\int_{\mathcal{M}} hH^{\alpha\beta\lambda\delta}\Big(\Delta\gamma_{\alpha\beta}\delta\Delta\gamma_{\lambda\delta} + \frac{h^2}{12}\Delta\kappa_{\alpha\beta}\delta\Delta\kappa_{\lambda\delta}\Big)\,\mathrm{d}a + \int_{\mathcal{M}} hH^{\alpha\beta\lambda\delta}({}^1\gamma_{\alpha\beta}\delta\Delta\gamma^{\mathrm{NL}}_{\lambda\delta})\,\mathrm{d}a\\ &\quad = \int_{\mathcal{M}} {}^2\boldsymbol{f}\cdot\delta\Delta\boldsymbol{u}\,\mathrm{d}a + \int_{\mathcal{M}} \rho h\,{}^2\ddot{\boldsymbol{u}}\cdot\delta\Delta\boldsymbol{u}\,\mathrm{d}a - \int_{\mathcal{M}} hH^{\alpha\beta\delta\mu}\Big({}^1\gamma_{\alpha\beta}\delta\Delta\gamma^{\mathrm{L}}_{\delta\mu} + \frac{h^2}{12}{}^1\kappa_{\alpha\beta}\delta\Delta\kappa_{\delta\mu}\Big)\,\mathrm{d}a\,.\end{aligned} \tag{3.26}$$

Following Bathe, Ramm and Wilson [13], the first term on the left-hand side is linearized as

$$\int_{\mathcal{M}} hH^{\alpha\beta\lambda\delta}(\Delta\gamma^{\mathrm{L}}_{\alpha\beta}\delta\Delta\gamma^{\mathrm{L}}_{\lambda\delta} + \cdots)\,\mathrm{d}a + \cdots = \cdots\,. \tag{3.27}$$

In the linear case, we have to put $\Delta\theta_{\alpha\beta}$ here instead of $\Delta\gamma^{\mathrm{L}}_{\alpha\beta}$. The linearization necessitates a post-iteration process in order to fulfil the equilibrium conditions.

3.3. Formulation for axisymmetric spherical shells

In the axisymmetric case, we have the conditions $u_2 = 0$, $(\cdots)_{,2} = 0$ which simplify the formalism. Then the metric coefficients of a spherical shell are (see [1], e.g., and Fig. 4)

$$a^{11} = 1\,, \qquad a^{22} = \frac{1}{r^2\sin^2 s/r}\,, \qquad a^{12} = a^{21} = 0\,, \tag{3.28}$$

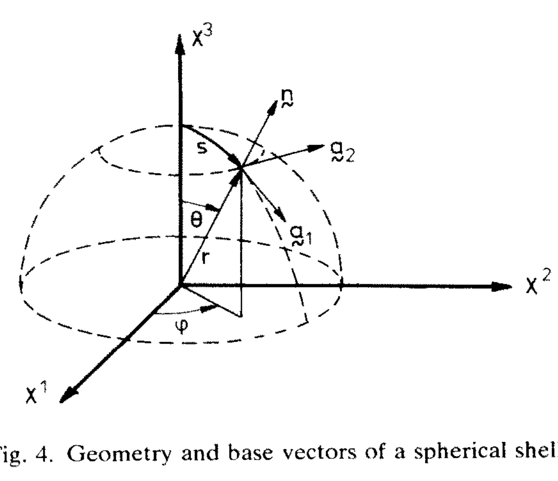


Fig. 4. Geometry and base vectors of a spherical shell.

and the curvature coefficients are

$$b_1^1 = b_2^2 = -\frac{1}{r}, \qquad b_1^2 = b_2^1 = 0. \tag{3.29}$$

The linearized components of the strain tensors (3.10) are

$$\theta_{11} = u_{1,1} + \frac{w}{r}, \tag{3.30a}$$

$$\theta_{22} = r u_1 \sin\frac{s}{r}\cos\frac{s}{r} + r\sin^2\frac{s}{r}\, w \tag{3.30b}$$

and

$$\kappa_{11} = -\left(w_{,11} - \frac{2}{r}u_{1,1} - \frac{w}{r^2}\right), \tag{3.31a}$$

$$\kappa_{22} = -\left(r\sin\frac{s}{r}\cos\frac{s}{r}\, w_{,1} - 2\sin\frac{s}{r}\cos\frac{s}{r}\, u_1 - \sin^2\frac{s}{r}\, w\right). \tag{3.31b}$$

The components of the linearized rotations (3.11) are

$$\varphi_1 = w_{,1} - \frac{u_1}{r}, \qquad \varphi_2 = 0. \tag{3.32}$$

Then, the components of the nonlinear strain-tensor within the moderate rotation theory are

$$\gamma_{11} = \theta_{11} + \tfrac{1}{2}\varphi_1\varphi_1, \tag{3.33a}$$

$$\gamma_{22} = \theta_{22}. \tag{3.33b}$$

For the linear and nonlinear terms of the variated strain increments, (3.22), we get

$$\delta\Delta\gamma_{11}^{\mathrm{L}} = \delta\Delta\theta_{11} + {}^{1}\varphi_1\Delta\varphi_1, \tag{3.34a}$$

$$\delta\Delta\gamma_{22}^{\mathrm{L}} = \delta\Delta\theta_{22} \tag{3.34b}$$

and

$$\delta\Delta\gamma_{11}^{\mathrm{NL}} = \Delta\varphi_1\delta\Delta\varphi_1, \tag{3.35a}$$

$$\delta\Delta\gamma_{22}^{\mathrm{NL}} = 0. \tag{3.35b}$$

The relevant components of the elasticity tensor of the isotropic shell are in the axisymmetric case

$$H^{1111} = \frac{E}{1-\nu^2}\,, \qquad H^{1122} = \frac{\nu E}{1-\nu^2}\frac{1}{r^2 \sin^2 s/r}\,, \qquad H^{2222} = \frac{E}{1-\nu^2}\frac{1}{r^4 \sin s/r}\,. \tag{3.36}$$

4. Finite-element formulation of the axisymmetric problem

The discretization is realized by cubic polynomials for the displacements.

4.1. Element model

Between the normed coordinates ξ_i, Fig. 5, and the arc length, we have the relation

$$s = R\theta_k + s_i = R\theta_k + \Delta s_i \xi_i = R(\theta_k + \Delta\theta_i \xi_i)\,. \tag{4.1}$$

The displacements are discretized by means of Hermitian interpolation polynomials

$$\begin{aligned} H_1 &= 1 - 3\xi^2 + 2\xi^3\,, & H_2 &= \xi - 2\xi^2 + \xi^3\,,\\ H_3 &= \quad 3\xi^2 - 2\xi^3\,, & H_4 &= \quad -\xi^2 + \xi^3\,. \end{aligned}$$

For the element i we get the displacement model

$$\begin{bmatrix} u \\ w \end{bmatrix}^{(i)} = \left[\begin{array}{c|c|c|c|c|c|c|c} H_1 & \Delta s_i H_2 & 0 & 0 & H_3 & \Delta s_i H_4 & 0 & 0 \\ \hline 0 & 0 & H_1 & \Delta s_i H_2 & 0 & 0 & H_3 & \Delta s_i H_4 \end{array}\right]^{(i)} \begin{bmatrix} u_{(k)} \\ u_{,s(k)} \\ w_{(k)} \\ w_{,s(k)} \\ u_{(k+1)} \\ u_{,s(k+1)} \\ w_{(k+1)} \\ w_{,s(k+1)} \end{bmatrix}^{(i)} \tag{4.2}$$

or

$$\boldsymbol{u}^{(i)} = \boldsymbol{\Omega}^{(i)} \hat{\boldsymbol{v}}^{(i)}\,.$$

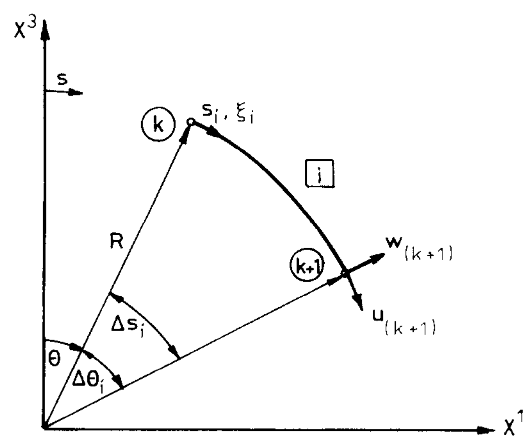


Fig. 5. Axisymmetric spherical shell element.

Then the discretized strains follow by differentiation according to the geometric relations. For linear strain measures, the matrix representation for an element i is

$$\begin{bmatrix} \theta_{11} \\ \theta_{22} \\ \kappa_{11} \\ \kappa_{22} \end{bmatrix}^{(i)} = \begin{bmatrix} \boldsymbol{h}_1^{\mathrm{t}} \\ \boldsymbol{h}_2^{\mathrm{t}} \\ \boldsymbol{h}_3^{\mathrm{t}} \\ \boldsymbol{h}_4^{\mathrm{t}} \end{bmatrix}^{(i)} \hat{\boldsymbol{v}}^{(i)}, \qquad \boldsymbol{e}^{(i)} = \boldsymbol{H}^{(i)}\hat{\boldsymbol{v}}^{(i)}. \tag{4.3}$$

For the nonlinear strain measures we get with (3.32) and (4.2)

$$\varphi_1 = \left[-\frac{1}{R} H_1 \;\Big|\; -\frac{\Delta s_i}{R} H_2 \;\Big|\; \frac{1}{\Delta s_i} \dot{H}_1 \;\Big|\; \dot{H}_2 \;\Big|\; -\frac{1}{R} H_3 \;\Big|\; -\frac{\Delta s_i}{R} H_4 \;\Big|\; \frac{1}{\Delta s_i} \dot{H}_3 \;\Big|\; \dot{H}_4 \right] \hat{\boldsymbol{v}}^{(i)} \tag{4.4}$$

or $\varphi_1 = \boldsymbol{\varphi}_1^{\mathrm{t}} \hat{\boldsymbol{v}}^{(i)}$, for the terms (3.33a), (3.34a) and (3.35a)

$${}^1\gamma_{11} = [\boldsymbol{h}_1^{\mathrm{t}} + \tfrac{1}{2}(\boldsymbol{\varphi}_1^{\mathrm{t}}\, {}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}}]^{(i)}\, {}^1\hat{\boldsymbol{v}}^{(i)}, \tag{4.5a}$$

$$\delta\Delta\gamma_{11}^{\mathrm{L}} = [\boldsymbol{h}_1^{\mathrm{t}} + (\boldsymbol{\varphi}_1^{\mathrm{t}}\, {}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}}]^{(i)} \Delta\hat{\boldsymbol{v}}^{(i)}, \tag{4.5b}$$

$$\delta\Delta\gamma_{11}^{\mathrm{NL}} = [(\boldsymbol{\varphi}_1^{\mathrm{t}} \Delta\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}}]^{(i)} \Delta\hat{\boldsymbol{v}}^{(i)}. \tag{4.5c}$$

Thus we have got the following matrices for the variation of linear strain increments

$$\delta\Delta\boldsymbol{e}^{\mathrm{L}(i)} = \begin{bmatrix} \delta\Delta\gamma_{11}^{\mathrm{L}} \\ \delta\Delta\gamma_{22}^{\mathrm{L}} \\ \delta\Delta\kappa_{11} \\ \delta\Delta\kappa_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{h}_1^{\mathrm{t}} + (\boldsymbol{\varphi}_1^{\mathrm{t}}\, {}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}} \\ \boldsymbol{h}_2^{\mathrm{t}} \\ \boldsymbol{h}_3^{\mathrm{t}} \\ \boldsymbol{h}_4^{\mathrm{t}} \end{bmatrix}^{(i)} \delta\Delta\hat{\boldsymbol{v}}^{(i)} = \bar{\boldsymbol{H}}^{\mathrm{L}(i)} \delta\Delta\hat{\boldsymbol{v}}^{(i)}$$

with

$$\bar{\boldsymbol{H}}^{\mathrm{L}(i)} = \boldsymbol{H}^{(i)} + \begin{bmatrix} (\boldsymbol{\varphi}_1^{\mathrm{t}}\, {}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}} \\ \boldsymbol{O}^{\mathrm{t}} \\ \boldsymbol{O}^{\mathrm{t}} \\ \boldsymbol{O}^{\mathrm{t}} \end{bmatrix}^{(i)} = \boldsymbol{H}^{(i)} + \boldsymbol{H}^{\mathrm{L}(i)}, \tag{4.7}$$

and for the variation of nonlinear strain increments

$$\delta\Delta\boldsymbol{e}_\gamma^{\mathrm{NL}(i)} = \begin{bmatrix} \delta\Delta\gamma_{11}^{\mathrm{NL}} \\ \delta\Delta\gamma_{22}^{\mathrm{NL}} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{\varphi}_1^{\mathrm{t}}\Delta\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}} \\ \boldsymbol{O}^{\mathrm{t}} \end{bmatrix}^{(i)} \delta\Delta\hat{\boldsymbol{v}}^{(i)} = \bar{\boldsymbol{H}}_\gamma^{\mathrm{NL}(i)} \delta\Delta\hat{\boldsymbol{v}}^{(i)}. \tag{4.8}$$

The strains (membrane strains and changes of curvature) in the current configuration (index 1) are

$${}^1\boldsymbol{e}^{\mathrm{NL}(i)} = \begin{bmatrix} {}^1\gamma_{11} \\ {}^1\gamma_{22} \\ {}^1\kappa_{11} \\ {}^1\kappa_{22} \end{bmatrix} = {}^1\bar{\boldsymbol{H}}^{\mathrm{NL}(i)}\, {}^1\hat{\boldsymbol{v}}^{(i)} = (\boldsymbol{H} + \tfrac{1}{2}\boldsymbol{H}^{\mathrm{L}})^{(i)}\, {}^1\hat{\boldsymbol{v}}^{(i)}. \tag{4.9}$$

The constitutive relations (3.3) and (3.4) for an element i can be written in matrix notation

$$\begin{bmatrix} N^{11} \\ N^{22} \\ M^{11} \\ M^{22} \end{bmatrix}^{(i)} = \left[\begin{array}{c|c} \boldsymbol{C}_\gamma & \boldsymbol{O} \\ \hline \boldsymbol{O} & \frac{h^2}{12}\boldsymbol{C}_\gamma \end{array}\right]^{(i)} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ K_{11} \\ K_{22} \end{bmatrix}^{(i)} \tag{4.10}$$

with the submatrix

$$\boldsymbol{C}_\gamma = h\left[\begin{array}{c|c} H^{1111} & H^{1122} \\ \hline & H^{2222} \end{array}\right]. \tag{4.11}$$

The principle of virtual work (3.27) can now be written for an element i in the form

$$\begin{aligned} &\int_{\mathcal{M}^i} (\delta\Delta\boldsymbol{e}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\Delta\boldsymbol{e}^{\mathrm{L}}\,\mathrm{d}a + \int_{\mathcal{M}^i} (\delta\Delta\boldsymbol{e}^{\mathrm{NL}})^{\mathrm{t}}\boldsymbol{C}\,{}^{1}\boldsymbol{e}^{\mathrm{NL}}\,\mathrm{d}a \\ &\quad = \int_{\mathcal{M}^i} \delta\Delta\boldsymbol{v}^{\mathrm{t}}\boldsymbol{\Omega}^{\mathrm{t}}\boldsymbol{t}\,\mathrm{d}a + \int_{\mathcal{M}^i} \delta\Delta\boldsymbol{u}^{\mathrm{t}}\boldsymbol{\Omega}^{\mathrm{t}}\boldsymbol{\Omega}\ddot{\boldsymbol{v}}\,\mathrm{d}a - \int_{\mathcal{M}^i} (\delta\Delta\boldsymbol{e}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\,{}^{1}\boldsymbol{e}^{\mathrm{NL}}\,\mathrm{d}a\,. \end{aligned} \tag{4.12}$$

With the previously introduced strains we get

$$\begin{aligned} &\int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}(\bar{\boldsymbol{H}}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\bar{\boldsymbol{H}}^{\mathrm{L}}\Delta\hat{\boldsymbol{v}}\,\mathrm{d}a + \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}(\bar{\boldsymbol{H}}^{\mathrm{NL}})^{\mathrm{t}}\boldsymbol{C}_\gamma\,{}^{1}\bar{\boldsymbol{H}}^{\mathrm{NL}}\,{}^{1}\hat{\boldsymbol{v}}\,\mathrm{d}a \\ &\quad = \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}\boldsymbol{\Omega}^{\mathrm{t}}\boldsymbol{t}\,\mathrm{d}a + \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}\boldsymbol{\Omega}^{\mathrm{t}}\boldsymbol{\Omega}\ddot{\boldsymbol{v}}\,\mathrm{d}a - \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}(\bar{\boldsymbol{H}}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}^{1}\boldsymbol{H}^{\mathrm{NL}}\,{}^{1}\hat{\boldsymbol{v}}\,\mathrm{d}a\,, \end{aligned} \tag{4.13}$$

$$\delta W_1 + \delta W_2 = \delta W_3 + \delta W_4 + \delta W_5\,.$$

In the following we give a more detailed form of the terms in (4.13).

(i) Calculation of the term

$$\delta W_1 = \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}(\bar{\boldsymbol{H}}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\bar{\boldsymbol{H}}^{\mathrm{L}}\Delta\hat{\boldsymbol{v}}\,\mathrm{d}a$$

with $\bar{\boldsymbol{H}}^{\mathrm{L}} = (\boldsymbol{H} + \boldsymbol{H}^{\mathrm{L}})$ gives

$$\delta W_1 = \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}\{\boldsymbol{H}^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H} + (\boldsymbol{H}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H} + \boldsymbol{H}^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H}^{\mathrm{L}} + (\boldsymbol{H}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H}^{\mathrm{L}}\}\Delta\hat{\boldsymbol{v}}\,\mathrm{d}a\,. \tag{4.14}$$

Here, the first term contains the linear stiffness matrix. The following terms are caused by the linearized nonlinear strain components. As $\boldsymbol{H}^{\mathrm{L}}$ is a sparsed (4×8)-matrix, we can simplify the first term in (4.14):

$$
\boldsymbol{H}^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H}^{\mathrm{L}} =
\begin{array}{|c|c|c|}
\cline{3-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & (\boldsymbol{\varphi}_1^{\mathrm{t}}\,{}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}} \\
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \boldsymbol{O}^{\mathrm{t}} \\
\cline{3-3}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \boldsymbol{O}^{\mathrm{t}} \\
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \boldsymbol{O}^{\mathrm{t}} \\
\hline
\boldsymbol{C}_\gamma & \boldsymbol{O} & H^{1111}(\boldsymbol{\varphi}_1^{\mathrm{t}}\,{}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}} \\
\cline{3-3}
 & & H^{1122}(\boldsymbol{\varphi}_1^{\mathrm{t}}\,{}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}} \\
\hline
\boldsymbol{O} & & \boldsymbol{O} \\
\hline
\boldsymbol{h}_1\boldsymbol{h}_2 & & \begin{array}{c} H^{1111}\boldsymbol{h}_1(\boldsymbol{\varphi}_1^{\mathrm{t}}\,{}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}} \\ + H^{1122}\boldsymbol{h}_2(\boldsymbol{\varphi}_1^{\mathrm{t}}\,{}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}} \end{array} \\
\hline
\end{array}
\qquad (4.15)
$$

In an analogous way, the terms $(\boldsymbol{H}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H}$ and $(\boldsymbol{H}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H}^{\mathrm{L}}$ are treated. So we finally get for δW_1,

$$
\begin{aligned}
\delta W_1 &= \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}(\bar{\boldsymbol{H}}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\bar{\boldsymbol{H}}^{\mathrm{L}}\Delta\hat{\boldsymbol{v}}\,\mathrm{d}a \\
&= \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}\{\boldsymbol{H}^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H} + H^{1111}\boldsymbol{\varphi}_1^{\mathrm{t}}\,{}^1\hat{\boldsymbol{v}}[(\boldsymbol{h}_1 + (\boldsymbol{\varphi}_1^{\mathrm{t}}\,{}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1)\boldsymbol{\varphi}_1^{\mathrm{t}} + \boldsymbol{\varphi}_1\boldsymbol{h}_1^{\mathrm{t}} \\
&\qquad + H^{1122}\boldsymbol{\varphi}_1^{\mathrm{t}}\,{}^1\hat{\boldsymbol{v}}(\boldsymbol{h}_2\boldsymbol{\varphi}_1^{\mathrm{t}} + \boldsymbol{\varphi}_1\boldsymbol{h}_2^{\mathrm{t}})\}\Delta\hat{\boldsymbol{v}}\,\mathrm{d}a \\
&= (\delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}})^{(i)}(\boldsymbol{K} + {}^1\boldsymbol{K}^{\mathrm{L}})^{(i)}\Delta\hat{\boldsymbol{v}}^{(i)} .
\end{aligned}
\qquad (4.16)
$$

In the last line of (4.16), the condensation into the linear stiffness matrix and a linearized geometrical matrix is introduced.

(ii) Calculation of the term

$$
\delta W_2 = \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}(\bar{\boldsymbol{H}}^{\mathrm{NL}})^{\mathrm{t}}\boldsymbol{C}_\gamma\,{}^1\bar{\boldsymbol{H}}_\gamma^{\mathrm{NL}}\,{}^1\hat{\boldsymbol{v}}\,\mathrm{d}a
$$

$$
\begin{aligned}
\delta W_2 &= \int_{\mathcal{M}^i} N^{11}\,\delta\Delta\gamma_{11}^{\mathrm{NL}}\,\mathrm{d}a = \int_{\mathcal{M}^i} N^{11}(\boldsymbol{\varphi}_1^{\mathrm{t}}\Delta\hat{\boldsymbol{v}})^{\mathrm{t}}\boldsymbol{\varphi}_1^{\mathrm{t}}\delta\Delta\hat{\boldsymbol{v}}\,\mathrm{d}a \\
&= \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}\,N^{11}\,\boldsymbol{\varphi}_1\boldsymbol{\varphi}_1^{\mathrm{t}}\Delta\hat{\boldsymbol{v}}\,\mathrm{d}a
\end{aligned}
\qquad (4.17)
$$

with

$$
N^{11} = \boldsymbol{C}_\gamma\,{}^1\bar{\boldsymbol{H}}_\gamma^{\mathrm{NL}}\,{}^1\hat{\boldsymbol{v}} = H^{1111}(\boldsymbol{h}_1^{\mathrm{t}} + \tfrac{1}{2}(\boldsymbol{\varphi}_1^{\mathrm{t}}\,{}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}})\,{}^1\hat{\boldsymbol{v}} + H^{1122}(\boldsymbol{h}_2^{\mathrm{t}}\,{}^1\hat{\boldsymbol{v}}) . \qquad (4.18)
$$

We finally get

$$\delta W_2 = \int_{\mathcal{M}^i} \delta\Delta \boldsymbol{v}^{\mathrm{t}}(\bar{\boldsymbol{H}}^{\mathrm{NL}})^{\mathrm{t}}\boldsymbol{C}_\gamma\, {}^1\boldsymbol{H}_\gamma^{\mathrm{NL}}\, {}^1\hat{\boldsymbol{v}}\, \mathrm{d}a$$

$$= \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}\{H^{1111}[\boldsymbol{h}_1^{\mathrm{t}} + \tfrac{1}{2}(\boldsymbol{\varphi}_1^{\mathrm{t}}\, {}^1\hat{\boldsymbol{v}})\boldsymbol{\varphi}_1^{\mathrm{t}}]\, {}^1\hat{\boldsymbol{v}} + H^{1122}\boldsymbol{h}_2^{\mathrm{t}}\, {}^1\hat{\boldsymbol{v}}\}\boldsymbol{\varphi}_1\boldsymbol{\varphi}_1^{\mathrm{t}}\Delta\hat{\boldsymbol{v}}\, \mathrm{d}a$$

$$= (\delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}})^{(i)}\, {}^1\bar{\boldsymbol{K}}^{\mathrm{NL}(i)}\Delta\hat{\boldsymbol{v}}^{(i)}\,. \tag{4.19}$$

The last line in (4.19) shows the condensation into a non-linear matrix ${}^1\boldsymbol{K}^{\mathrm{NL}}$.

(iii) Calculation of the third term on the right-hand side gives

$$\delta W_5 = \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}(\bar{\boldsymbol{H}}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\, {}^1\boldsymbol{H}^{\mathrm{NL}}\, {}^1\hat{\boldsymbol{v}}\, \mathrm{d}a$$

$$= \int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}\{\boldsymbol{H}^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H} + (\boldsymbol{H}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H} + \tfrac{1}{2}\boldsymbol{H}^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H}^{\mathrm{L}} + \tfrac{1}{2}(\boldsymbol{H}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H}^{\mathrm{L}}\}\, {}^1\hat{\boldsymbol{v}}\, \mathrm{d}a\,. \tag{4.20}$$

Again, the first term contains the linear stiffness matrix.

$$\int_{\mathcal{M}^i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}(\bar{\boldsymbol{H}}^{\mathrm{L}})^{\mathrm{t}}\boldsymbol{C}\, {}^1\boldsymbol{H}^{\mathrm{NL}}\, {}^1\hat{\boldsymbol{v}}\, \mathrm{d}a$$

$$= \int_{\mathcal{M}_i} \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}}\{\boldsymbol{H}^{\mathrm{t}}\boldsymbol{C}\boldsymbol{H} + H^{1111}\boldsymbol{\varphi}_1^{\mathrm{t}}\, {}^1\hat{\boldsymbol{v}}[\boldsymbol{\varphi}_1\boldsymbol{h}_1^{\mathrm{t}} + \tfrac{1}{2}\boldsymbol{h}_1\boldsymbol{\varphi}_1^{\mathrm{t}} + \tfrac{1}{2}\boldsymbol{\varphi}_1^{\mathrm{t}}\, {}^1\hat{\boldsymbol{v}}\boldsymbol{\varphi}_1\boldsymbol{\varphi}_1^{\mathrm{t}}]$$

$$+ H^{1122}\boldsymbol{\varphi}_1^{\mathrm{t}}\, {}^1\hat{\boldsymbol{v}}[\boldsymbol{\varphi}_1\boldsymbol{h}_2^{\mathrm{t}} + \tfrac{1}{2}\boldsymbol{h}_2\boldsymbol{\varphi}_1^{\mathrm{t}}]\}\, {}^1\hat{\boldsymbol{v}}\, \mathrm{d}a$$

$$= \delta\Delta\hat{\boldsymbol{v}}^{\mathrm{t}(i)}(\boldsymbol{K}\, {}^1\hat{\boldsymbol{v}} + {}^1\boldsymbol{k})^{(i)}\,. \tag{4.21}$$

The interelement geometrical continuity conditions and the geometrical boundary conditions are realized by Boolean matrices $\boldsymbol{B}^{(i)}$. So we arrive from element oriented nodal displacement vectors $\hat{\boldsymbol{v}}^{(i)}$ at the global reduced nodal displacement vector $\boldsymbol{v}$

$$\hat{\boldsymbol{v}}^{(i)} = \boldsymbol{B}^{(i)}\boldsymbol{v}\,. \tag{4.22}$$

The principle of virtual work leads to the assembled linear stiffness matrix

$$\boldsymbol{K} = \sum_{i=1}^{n_e} \boldsymbol{B}^{(i)\mathrm{t}}\boldsymbol{K}^{(i)}\boldsymbol{B}^{(i)}\,, \tag{4.23}$$

the linearized geometrical stiffness matrix

$${}^1\boldsymbol{K} = \sum_{i=1}^{n_e} \boldsymbol{B}^{(i)\mathrm{t}}({}^1\boldsymbol{K}^{\mathrm{L}} + {}^1\boldsymbol{K}^{\mathrm{NL}})^{(i)}\boldsymbol{B}^{(i)}\,, \tag{4.24}$$

the mass matrix

$$M = \sum_{i=1}^{n_e} B^{(i)t} \int_{\mathcal{M}^i} \rho^{(i)} h^{(i)} \Omega^{(i)t} \Omega^{(i)} \, da B^{(i)} \tag{4.25}$$

and the right-hand side vectors

$${}^2f = \sum_{i=1}^{n_e} B^{(i)t} \int_{\mathcal{M}^i} \Omega^{(i)t} t \, da \,, \qquad {}^1k = \sum_{i=1}^{n_e} B^{(i)t} \, {}^1k^{(i)} \,. \tag{4.26}$$

From d'Alembert's principle we get the kinetic equilibrium conditions

$$M \, {}^2\ddot{v} + (K + {}^1K) \Delta v = {}^2f - {}^1k - K \, {}^1v \,. \tag{4.27}$$

In the linear case ${}^1K$ and ${}^1k$ disappear.

The step-by-step solution over the time with a numerical integration method—here the Newmark-method [16]—leads, in the linear case, to the linear matrix equations

$${}^1\tilde{K} \Delta v = {}^2\tilde{f} \tag{4.28a}$$

with

$${}^1\tilde{K} = a_0 M + K \tag{4.28b}$$

and

$${}^2\tilde{f} = {}^2f - K \, {}^1v + M(a_1 \, {}^1\dot{v} + a_2 \, {}^1\ddot{v}) \,. \tag{4.28c}$$

Applying substructure technique, the out-of-contact-surfaces can be pre-eliminated during the whole contact process so that the computation effort is decisively reduced [4].

5. Substructure technique

The numerical effort for nonlinear kinetic problems is considerable because of nonlinear systems of equations in each time step. One can reduce the nonlinear systems if nonlinearities are of more local character, as in the case of the changing contact area and its neighbourhood in our impact problem. Clough and Wilson [4] discussed different substructure techniques for nonlinear subdomains. In this paper a special version is given for step-by-step integration.

5.1. General algorithm

The linear system of equations (4.28a) resulting from the Newmark-method can be condensed like in the static case. Substructuring $\tilde{K}$ yields

$$\begin{bmatrix} \tilde{K}_{aa} & \tilde{K}_{ab} \\ \tilde{K}_{ba} & \tilde{K}_{bb} \end{bmatrix} \begin{bmatrix} \Delta v_a \\ \Delta v_b \end{bmatrix} = \begin{bmatrix} \tilde{f}_a \\ \tilde{f}_b \end{bmatrix} \tag{5.1}$$

where nonlinear terms should only be contained in substructure b. This static condensation can also be applied in the present dynamic calculation with the Newmark-method. From the first equation of (5.1) we get

$$\Delta v_a = \tilde{K}_{aa}^{-1}(\tilde{f}_a - \tilde{K}_{ab}\Delta v_b) \tag{5.2}$$

and from the second, i.e.,

$$\tilde{K}_{ba}\Delta v_a + \tilde{K}_{bb}\Delta v_b = f_b$$

with (5.2), we obtain

$$K^* \Delta v_b = f_b^* , \tag{5.3}$$

with the definitions

$$K^* = \tilde{K}_{bb} - \tilde{K}_{ba}\tilde{K}_{aa}^{-1}\tilde{K}_{ab} \tag{5.4a}$$

and

$$f_b^* = \tilde{f}_b - \tilde{K}_{ba}\tilde{K}_{ba}\tilde{K}_{aa}^{-1}\tilde{f}_a . \tag{5.4b}$$

The nonlinear stiffness properties only have influence on K^*, so we can sum up on the left- and right-hand sides, and we get the system

$${}^1\hat{K}\Delta v_b = \hat{f}_b \tag{5.5a}$$

with

$${}^1\hat{K} = K^* + {}^1K ; \qquad \hat{f}_b = f_b^* - {}^1k_b . \tag{5.5b}$$

This system is much smaller than the whole system in the following examples of impacting bodies. One has to realize that nonlinearities are considered only in a linearized form, so that a post-iteration—as proposed by Bathe, Ramm and Wilson [13]—may be necessary.

The equilibrium equation which has to be used in the 'total Lagrangian' formulation is

$$M\ddot{v}^{(r)} + (K + {}^1K)\,\mathrm{d}v^{(r)} = {}^2f - k^{(r-1)} - kv^{(r-1)} \tag{5.6}$$

where

$$\ddot{v}^{(r)} = a_0(\mathrm{d}v^{(r)} + \Delta v^{(r-1)}) - a_1\,{}^1\dot{v} - a_2\,{}^1\ddot{v} , \tag{5.7a}$$

$$\Delta v^{(r)} = \Delta v^{(r-1)} + \mathrm{d}v^{(r)} , \tag{5.7b}$$

$$\boldsymbol{v}^{(r-1)} = {}^{1}\boldsymbol{v} + \Delta\boldsymbol{v}^{(r-1)} \tag{5.7c}$$

where $\mathrm{d}\boldsymbol{v}^{(r)}$ denotes the postiteration increment. With (4.28) and (5.7), a system of linear equations for $\mathrm{d}\boldsymbol{v}^{(r)}$ can be obtained:

$$({}^{1}\boldsymbol{K} + \tilde{\boldsymbol{K}})\,\mathrm{d}\boldsymbol{v}^{(r)} = {}^{2}\tilde{\boldsymbol{f}} - \tilde{\boldsymbol{K}}\Delta\boldsymbol{v}^{(r-1)} - (\boldsymbol{K}^{(r-1)} - {}^{1}\boldsymbol{K})\,. \tag{5.8}$$

Applying the substructure technique one gets with (5.4)

$$\hat{\boldsymbol{K}}\,\mathrm{d}\boldsymbol{v}_b^{(r)} = \boldsymbol{f}_b^{*} - \boldsymbol{K}^{*}\Delta\boldsymbol{v}_b^{(r-1)} - (\boldsymbol{k}_b^{(r-1)} - {}^{1}\boldsymbol{k})\,. \tag{5.9}$$

5.2. Calculation of the nodal contact forces

The iterative algorithm for the determination of contact stresses has been discussed in Section 2. Now the related matrix formulation can be given in the frame of substructure technique.

Assuming that c nodes are in contact at a given time, the inequality (2.2) has the form

$${}^{2}\boldsymbol{v}_{\mathrm{n}c} = \Delta\boldsymbol{X}_{\mathrm{n}c}\,, \tag{5.10}$$

where n denotes the normal direction to the surface, and $\Delta\boldsymbol{X}_{\mathrm{n}c}$ is a column matrix containing the differences, between coordinates of the surface and the shell in the initial configuration. In the numerical algorithm, the contact surface which has been calculated the time step before, is used as initial contact surface for the next time step. With (3.13b) we get

$$\Delta\boldsymbol{v}_{\mathrm{n}c} = \Delta\boldsymbol{X}_{\mathrm{n}c} - {}^{1}\boldsymbol{v}_{\mathrm{n}c}\,. \tag{5.11}$$

Substructuring (5.5), and adding the unknown vector of the nodal contact forces ${}^{2}\boldsymbol{f}_{\mathrm{n}c}$ yields

$$\begin{bmatrix} \hat{\boldsymbol{K}}_{b-c,b-c} & \hat{\boldsymbol{K}}_{b-c,c} \\ \hat{\boldsymbol{K}}_{c,b-c} & \hat{\boldsymbol{K}}_{c,c} \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{v}_{b-c} \\ \Delta\boldsymbol{v}_{\mathrm{n}c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ {}^{2}\boldsymbol{f}_{\mathrm{n}c} \end{bmatrix} + \begin{bmatrix} \hat{\boldsymbol{f}}_{b-c} \\ \hat{\boldsymbol{f}}_{c} \end{bmatrix}. \tag{5.12}$$

Inserting (5.11) into (5.12), leads to an equation for the contact-force vector

$$\hat{\boldsymbol{F}}_{c,c}\,{}^{2}\boldsymbol{f}_{\mathrm{n}c} = \Delta\boldsymbol{X}_{\mathrm{n}c} - {}^{1}\boldsymbol{v}_{\mathrm{n}c} - \hat{\boldsymbol{F}}_{c,b-c}\hat{\boldsymbol{f}}_{b-c} - \hat{\boldsymbol{F}}_{c,c}\hat{\boldsymbol{f}}_{c} \tag{5.13}$$

where $\hat{\boldsymbol{F}} = (\hat{\boldsymbol{K}})^{-1}$. The contact force vector ${}^{2}\boldsymbol{f}_{\mathrm{n}c}$ can also be obtained by solving (5.12) with a modified Gauss elimination method after introducing (5.11) into (5.12). The nodal contact forces must now fulfil inequality (2.1). If this condition is violated for one or more nodal points of the assumed contact surface, the contact is released at these points. After this newly defined contact area, the nodal contact force can be calculated. This iteration has to be carried out until (2.1) is fulfilled for all nodal points being in contact.

Next, the no-penetration-condition (2.2) has to be checked for nodal points outside of the contact area. For this aim, $\Delta\boldsymbol{v}_{b-c}$ must be calculated from (5.12). Then condition (2.2) has to be checked for nodal points near the previously calculated contact area. If condition (2.2) is violated, the points which try to penetrate the surface are assumed to be points of the contact area. Then a new iteration for the determination of the associated nodal contact forces has to be carried out. If condition (2.2) is fulfilled, the contact iteration in the considered time step has converged.

6. Examples

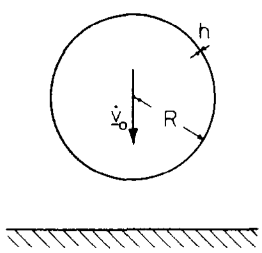
The developed algorithm for contact–impact problems is applied to a spherical shell impacting a rigid plane with a velocity $\dot{v}_0$.

The shell and the system parameters are shown in Fig. 6a. The shell is described by 40 finite elements. Near the contact surface, the number of elements is increased, so that a sufficient number of contact nodes is available.

A parameter study in which the angle β is varied shows that in the cases considered here, a minimum angle $\beta = 35°$ should be used. For larger angles of β, the results do not differ from the solution obtained with $\beta = 35°$ (see Fig. 6b).

In Fig. 7, the total impact force is displayed versus time for the linear and nonlinear cases.

The maximum total impact force decreases in the nonlinear solution, while the time of impact increases. This fact, resulting from the softening behaviour of the shell, is in correspondence with the conservation law of linear momentum for the whole elastic shell.



System parameters:

$E = 2. \cdot 10^8 \text{ kN/m}^2$

$\rho = 10.0 \cdot 10^3 \text{ kg/m}^3$

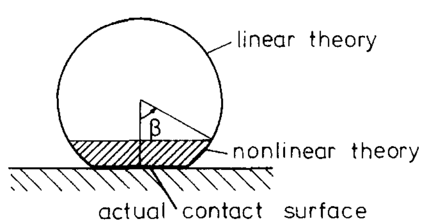
$\nu = 0.3$

$R = .1 \text{ m}$

$\dot{v}_o = 2. \text{ m/s}$

$\frac{R}{h} = 50$

Discretization with
40 ring elements,
160 DOFs.



Computing time

β	CPUs on Cyber 76
linear	12
35°	22
68°	30
90°	35
120°	56

Fig. 6. Spherical shell impacting a rigid plane (a) before impact; (b) during impact.

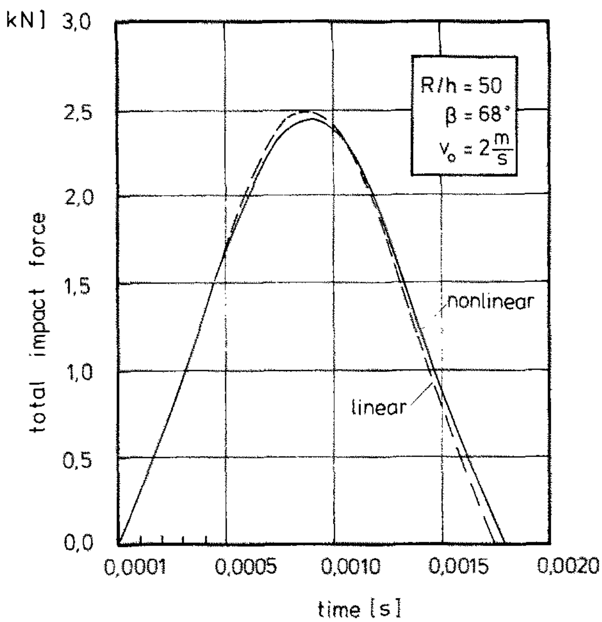


Fig. 7. Total impact force versus time.

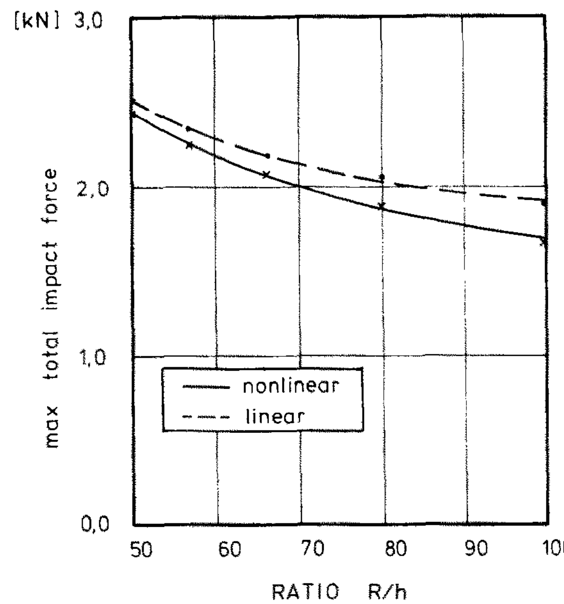


Fig. 8. Total impact force depending on ratio R/h.

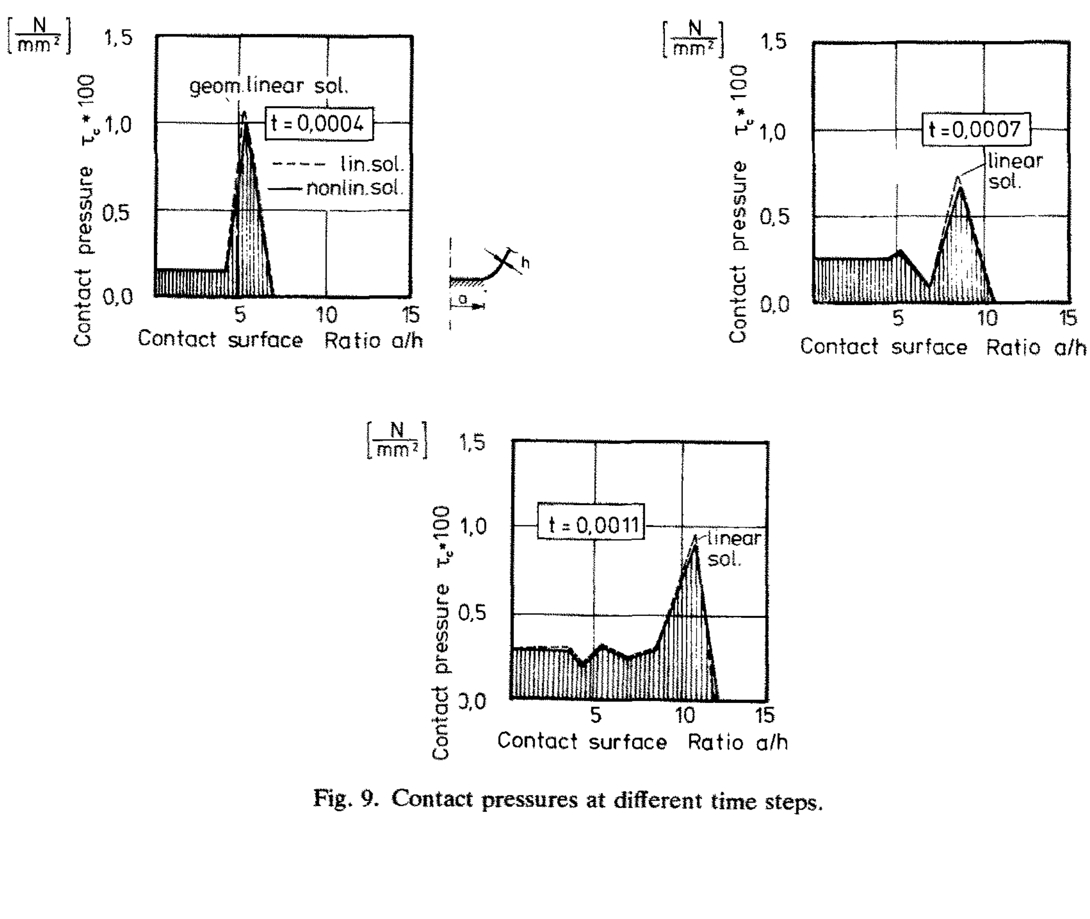


Fig. 9. Contact pressures at different time steps.

In Fig. 8, the maximum total impact forces which have been calculated within linear and nonlinear theory are compared, depending on the ratio R/h. In this parameter study, the total mass of the shell was kept constant.

The differences between linear and nonlinear solutions increase with the ratio R/h. This result shows the growing influence of the nonlinear membrane strains and stresses in comparison with the bending stresses, which remain linear in the moderate rotation shell theory. Therefore, the nonlinearly and linearly computed total impact forces for a ratio $R/h = 10$ are equal. The contact pressures are shown in Fig. 9 at different times for a shell with $h = 0.001$. The peak of the contact pressures appears at the boundary of the contact surfaces—a shell-like phenomenon—which is in contrast to the contact pressure distribution of solid elastic bodies. This was also shown by Updike and Kalnins [17] for the case of static contact of a spherical shell with a rigid plate. In this paper, the influence of transverse shear deformations was studied numerically.

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