

Asymptotics of the Logarithm of the Number of Threshold Functions in the Algebra of Logic

Yu. A. Zuev

(Presented by Academician G. S. Pospelov on January 13, 1989)

A Boolean function $f: \{-1, 1\}^n \rightarrow \{-1, 1\}$ is called *threshold* if there exist real numbers $a_0, a_1, \dots, a_n$, called weights, such that

$$f(x) = \text{sgn}(a_0 + a_1x_1 + \dots + a_nx_n). \quad (1)$$

Geometrically, $f(x)$ is specified by a hyperplane cutting through the hypercube $[-1, 1]^n$ in such a way that at the vertices $x \in \{-1, 1\}^n$ on one side of the hyperplane $f(x) = 1$, and on the other side $f(x) = -1$. Another interpretation can serve as decision-making by weighted voting among a collective of n members.

The problem of determining the number N_n of threshold functions of n variables arose long ago, and by 1965 the following asymptotic estimates for the logarithm of this number were already known:

$$n^2/2 \leq \log_2 N_n \leq n^2. \quad (2)$$

The derivation of the lower bound in (2) is based on the fact that from each threshold function of n variables, one can obtain exactly 2^n new functions by varying the threshold. For the upper bound, at least two distinct methods of derivation are known. One uses the counting of Chow parameters, while the other goes back to Ludwig Schläfli, who showed in the mid-19th century that m hyperplanes passing through the origin of n -dimensional Euclidean space in general position divide the space into $2 \sum_{i=0}^{n-1} \binom{m-1}{i}$ regions.

The bounds (2), independently obtained by many authors in the late 1950s and early 1960s (see [1, 2]), were not subsequently improved. In [3], a formula was obtained for the number of regions of space under an arbitrary centered arrangement of hyperplanes, which in principle allows answering the question about the value N_n , but its application proved to be too complex a task. In [4], the number $N_n(M)$ of threshold sets of a given cardinality was estimated, which is of interest for threshold representations of Boolean functions [5]. For small $M(n)$ ($\log M(n) = o(n)$), exact asymptotics for $\log N_n(M)$ were obtained, but the estimates for $\log N_n$ were not improved.

Thus, the question regarding the asymptotic value of the ratio $(\log N_n)/n^2$ remained open. Computer calculation of the number of threshold functions for $n \leq 8$ did not allow even a conjectural answer [2].

The present level of development of combinatorial analysis enables the resolution of this question. The purpose of this paper is to establish the following result.

Theorem 1. *For the logarithm of the number N_n of threshold functions of n variables, for sufficiently large n , the inequality*

$$\log_2 N_n > n^2 \left(1 - \frac{10}{\ln n} \right)$$

holds.

Corollary. $\log_2 N_n \sim n^2$ as $n \rightarrow \infty$.

The proof relies on a modern study on space partitioning by hyperplanes [6], in which the idea of using the Euler characteristic [7] is developed on the basis of Möbius inversion [8], and a recent remarkable result on random (± 1) -vectors [9].

We shall consider only threshold functions (1) with $a_0 = 0$, i.e., self-dual functions. In the n -dimensional weight space $(a_1, \dots, a_n)$, consider 2^n hyperplanes of the form

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n = 0,$$

where $x_i = \pm 1$. A one-to-one correspondence between the regions into which the hyperplanes partition the space and the self-dual threshold functions allows us to reduce the problem to counting the number of regions.

Linear subspaces arising from intersections of hyperplanes, ordered by inclusion, form a geometric lattice [10], whose zero element is the origin and whose unit element is the entire space. Each element of the lattice is generated by the intersection of some subset of hyperplanes, i.e., it is the orthogonal complement to the subspace spanned by the direction vectors of these hyperplanes. Inverting the Euler relation for the partition of Euclidean space by hyperplanes, we obtain [6] that the number of regions is equal to $\sum |\mu(s, 1)|$, where $\mu(s, t)$ is the Möbius function and the summation is taken over all elements of the lattice. The Möbius function $\mu(s, t)$ of a geometric lattice for $s \leq t$ is always non-zero [8]. This allows us to use the number of lattice elements as a lower bound for the number of regions.

Select from the set of vectors $\{-1, 1\}^n$ all subsets of cardinality $\lfloor n - 10n \ln^{-1} n \rfloor$. From each pair of mutually opposite subsets, leave one subset. Now, according to [9], almost all subspaces spanned by the selected subsets of vectors do not contain vectors from $\{-1, 1\}^n$ other than their generators. Therefore, the number of distinct subspaces is asymptotically equal to

$$\frac{1}{2} \binom{2^n}{\lfloor n - 10n \ln^{-1} n \rfloor};$$

their orthogonal complements are elements of the lattice. The statement of the theorem follows from this.

Computing Center of the USSR Academy of Sciences, Moscow
Received January 13, 1989

References

- [1] R. O. Winder, "Bounds on threshold gate realizability," *IEEE Trans. Comput.*, vol. C-19, no. 4, pp. 349–353, 1970.
- [2] S. Muroga, *Threshold Logic and Its Applications*, New York: Wiley-Interscience, 1971, 478 pp.

- [3] R. O. Winder, “Partitions of N -space by hyperplanes,” *SIAM J. Appl. Math.*, vol. 14, no. 4, pp. 811–818, 1966.
- [4] Yu. A. Zuev and L. I. Lipkin, “Estimates for the number of threshold sets,” *Izv. Akad. Nauk SSSR. Tekhn. Kibernet.*, 1988, no. 3, pp. 79–85.
- [5] Yu. A. Zuev and L. I. Lipkin, “Threshold representations of Boolean functions,” *Cybernetics*, 1988, no. 6, pp. 29–37.
- [6] T. Zaslavsky, “Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes,” *Mem. Amer. Math. Soc.*, vol. 1, no. 154, pp. 1–102, 1975.
- [7] R. C. Buck, “Partition of space by hyperplanes,” *Amer. Math. Monthly*, vol. 50, no. 9, pp. 541–544, 1943.
- [8] G.-C. Rota, “On the foundations of combinatorial theory I. Theory of Möbius functions,” *Z. Wahrscheinlichkeitstheor. verwandte Geb.*, vol. 2, no. 4, pp. 340–368, 1964.
- [9] A. M. Odlyzko, “On subspace spanned by random (± 1) -vectors,” *J. Combin. Theory, Ser. A*, vol. 47, no. 1, pp. 124–133, 1988.
- [10] G. Birkhoff, *Lattice Theory*, Moscow: Nauka, 1984, 564 pp.

This article was translated from the Russian by Gemini 3.6 Flash.