

PS0 due tonight

PS1 out now, due 9/29

- Please submit on Gradescope as zip file containing

0.pdf, 1.pdf, 2.pdf, 3.pdf, etc.

↳ for declaring collaboration, AI use, etc.

- This will make peer evaluation easier

- We'll "distribute" peer evaluation packets

on 10/1 in lecture (or come to my office)

I have been posting some typeset lecture notes, which you should also read (perhaps after the lectures).

ONLINE ALLOCATION in $\mathbb{R}^N$

For $t=1, 2, \dots$

- Learner picks allocation $p_t \in \Delta^{N-1}$

- Learner observes loss vector $l_t \in \mathbb{R}_+^N$

- Learner incurs loss $\langle p_t, l_t \rangle$

Hedge algorithm $\eta > 0$

Init: $W_1 = (1, \dots, 1)$

In round t :

- Choose $p_t = (p_{t,1}, \dots, p_{t,N}) \in \Delta^{N-1}$

$$\text{where } p_{t,i} = \frac{W_{t,i}}{Z_t} \quad \forall i \in [N]$$

$$\text{and } Z_t = \sum_{i=1}^N W_{t,i}$$

- Get $l_t = (l_{t,1}, \dots, l_{t,N}) \in \mathbb{R}_+^N$

- Update:

$$W_{t+1,i} = W_{t,i} \cdot \exp(-\eta l_{t,i})$$

$$\beta = e^{-\eta}$$

Theorem In Online Allocation for $\mathbb{R}^N$, for any T and any $l_1, \dots, l_T \in \mathbb{R}_+^N$, Hedge guarantees for all $q \in \Delta^{N-1}$,

$$\underbrace{\sum_{t=1}^T \langle p_t - q, l_t \rangle}_{R_{T,q}} \leq \frac{\ln N}{\eta} + \frac{\eta}{2} \sum_{t=1}^T \langle p_t, l_t^2 \rangle$$

$\uparrow l_t^2 = (l_{t,1}^2, \dots, l_{t,N}^2)$

Entropy $H(q) = -\sum_{i=1}^N q_i \ln\left(\frac{1}{q_i}\right)$

Average "surprise" for random outcome $\sim q$

Cross-entropy $CE(q, p) = -\sum_{i=1}^N q_i \ln\left(\frac{1}{p_i}\right)$

Using possibly different surprise measure p even though random outcome $\sim q$

Relative entropy $RE(q, p) = CE(q, p) - H(q)$

Excess surprise ...

Fact (Gibbs' inequality)

$$RE(q, p) \geq 0, \text{ equality iff } q = p.$$

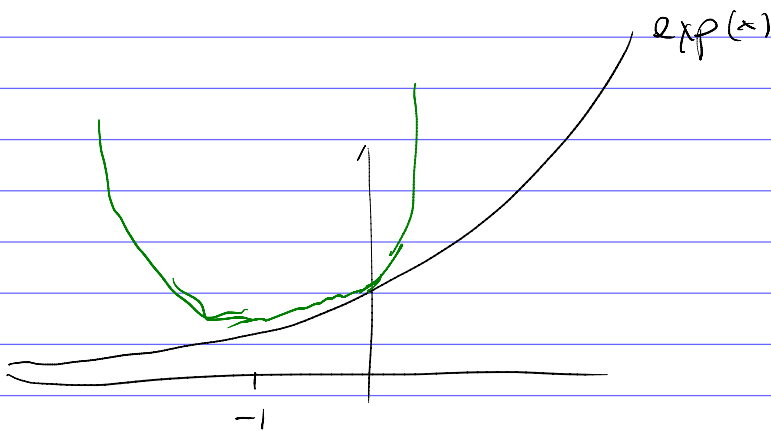
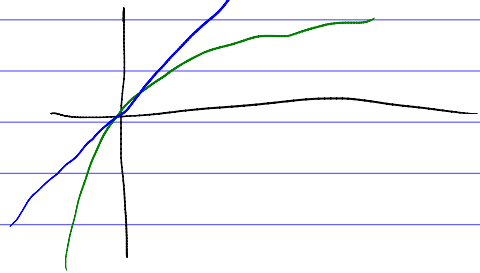
$$RE(q, p) = \sum_{i=1}^N q_i \ln\left(\frac{q_i}{p_i}\right)$$

$$= - \sum_{i=1}^N q_i \ln\left(\frac{p_i}{q_i}\right)$$

$$= - \sum_{i=1}^N q_i \ln\left(1 + \frac{p_i}{q_i} - 1\right)$$

$$\geq - \sum_{i=1}^N q_i \left(\frac{p_i}{q_i} - 1\right) = - \sum_{i=1}^N (p_i - q_i) = 0.$$

$$\ln(1+x) \leq x$$



Key lemma

For any $p \in \Delta^{N-1}$, $l \in \mathbb{R}_+^N$,
let $\tilde{p} \in \Delta^{N-1}$ be defined by

$$\tilde{p}_i = \frac{p_i \exp(-\eta l_i)}{\sum_{j=1}^N p_j \exp(-\eta l_j)}$$

For any $q \in \Delta^{N-1}$,

$$RE(q, \tilde{p}) - RE(q, p) \leq \eta \langle q - p, l \rangle + \frac{\eta^2}{2} \langle p, l^2 \rangle$$

$$RE(q, p_{t+1}) - RE(q, p_t) \leq \eta \langle q - p_t, l_t \rangle + \frac{\eta^2}{2} \langle p_t, l_t^2 \rangle$$

Sum over t :

LHS

$$RE(q, p_{T+1}) - RE(q, p_1)$$

RHS

$$\eta \sum_{t=1}^T \langle q - p_t, l_t \rangle + \frac{\eta^2}{2} \sum_{t=1}^T \langle p_t, l_t^2 \rangle$$

$$-\frac{RHS}{\eta} \leq -\frac{LHS}{\eta} : \quad \sum_{t=1}^T \langle p_t - q, l_t \rangle \leq \underbrace{\frac{RE(q, p_1)}{\eta}}_{\leq \ln N} - \underbrace{\frac{RE(q, p_{T+1})}{\eta}}_{\geq 0} + \frac{\eta}{2} \sum_{t=1}^T \langle p_t, l_t^2 \rangle$$

Proof $RE(q, \tilde{p}) - RE(q, p)$

$$= \sum_{i=1}^N q_i \ln\left(\frac{q_i}{p_i}\right) - q_i \ln\left(\frac{q_i}{\tilde{p}_i}\right)$$

$$= \sum_{i=1}^N q_i \ln\left(\frac{p_i}{\tilde{p}_i}\right)$$

$$= \sum_{i=1}^N q_i \ln\left(\exp(\eta l_i) \cdot \sum_{j=1}^N p_j \exp(-\eta l_j)\right)$$

$$= \underbrace{\sum_{i=1}^N q_i \eta l_i}_{\eta \langle q, l \rangle} + \sum_{i=1}^N q_i \ln\left(\underbrace{\sum_{j=1}^N p_j \exp(-\eta l_j)}_{\leq 1 - \eta l_j + \frac{\eta^2 l_j^2}{2}}\right)$$

$$\leq \eta \langle q, l \rangle + \sum_{i=1}^N q_i \ln\left(\sum_{j=1}^N p_j (1 - \eta l_j + \frac{\eta^2 l_j^2}{2})\right)$$

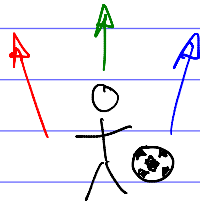
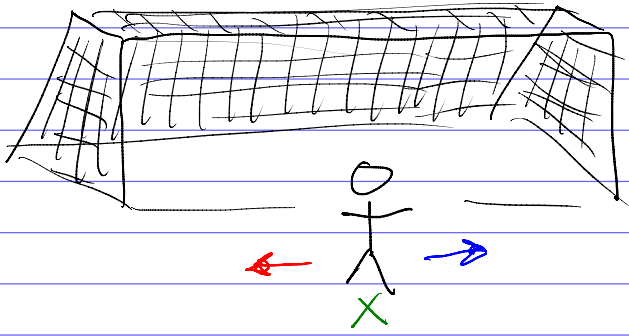
$$= \eta \langle q, l \rangle + \sum_{i=1}^N q_i \ln(1 - \eta \langle p, l \rangle + \frac{\eta^2}{2} \langle p, l^2 \rangle)$$

$$\leq \eta \langle q, l \rangle + \sum_{i=1}^N q_i (-\eta \langle p, l \rangle + \frac{\eta^2}{2} \langle p, l^2 \rangle)$$

$$= \eta \langle q, l \rangle - \eta \langle p, l \rangle + \frac{\eta^2}{2} \langle p, l^2 \rangle$$

Hedge is special case of ^{online} "Mirror descent"

MINIMAX THEOREM FOR ZERO-SUM GAMES



		PK			
		left	str.	right	
GK	left	1	1	0	$= A$
	stay	1	0	1	
	right	0	1	1	
		$m \times n$			payoff matrix

If GK chooses action first,
then PK has a best response to that action

"Mixed strategy" : distribution over actions.

If GK chooses $p \in \Delta^{m-1}$

PK chooses $q \in \Delta^{n-1}$

average loss of GK
(gain of PK) = $p^T A q$

For any payoff matrix $A_{m \times n}$

$$\min_{p \in \Delta^{m-1}} \max_{q \in \Delta^{n-1}} p^T A q \geq \max_{q \in \Delta^{n-1}} \min_{p \in \Delta^{m-1}} p^T A q$$

Von Neumann's minimax theorem

$$\text{val}(A) = \min_{p \in \Delta^{m-1}} \max_{q \in \Delta^{n-1}} p^T A q = \max_{q \in \Delta^{n-1}} \min_{p \in \Delta^{m-1}} p^T A q$$

What we'll show:

$$\min_{p \in \Delta^{m-1}} \max_{q \in \Delta^{n-1}} p^T A q \leq \max_{q \in \Delta^{n-1}} \min_{p \in \Delta^{m-1}} p^T A q$$

Payoff matrix $A \in [0, 1]^{m \times n}$

Simulate an online alloc. problem ($N = m$)
learner = Hedge

In round t :

Hedge picks $p_t \in \Delta^{m-1}$

Choose $l_t = A q_t$ where q_t is
best response for P.K.
when GK announces p_t

$$\max_{q \in \Delta^{n-1}} p_t^T A q = p_t^T A q_t$$

So Hedge Theorem implies ($\eta = \sqrt{\frac{\ln N}{T}}$)

$$\frac{1}{T} \sum_{t=1}^T \langle p_t, l_t \rangle \leq \min_{p \in \Delta^{m-1}} \frac{1}{T} \sum_{t=1}^T \langle p, l_t \rangle + \sqrt{\frac{2 \ln(m)}{T}}$$

$$\bar{p} := \frac{1}{T} \sum_{t=1}^T p_t \quad \bar{q} := \frac{1}{T} \sum_{t=1}^T q_t$$

$$\min_{p \in \Delta^{m-1}} \max_{q \in \Delta^{n-1}} p^T A q \leq \max_{q \in \Delta^{n-1}} \bar{p}^T A q$$

$$= \max_{q \in \Delta^{n-1}} \frac{1}{T} \sum_{t=1}^T p_t^T A q$$

$$\leq \frac{1}{T} \sum_{t=1}^T \max_{q \in \Delta^{n-1}} p_t^T A q$$

$$= \frac{1}{T} \sum_{t=1}^T p_t^T A q_t$$

$$= \frac{1}{T} \sum_{t=1}^T p_t^T l_t$$

$$\leq \min_{p \in \Delta^{m-1}} \frac{1}{T} \sum_{t=1}^T p^T A q_t + \sqrt{\frac{2 \ln(m)}{T}}$$

$$= \min_{p \in \Delta^{m-1}} p^T A \bar{q} + \sqrt{\frac{2 \ln(m)}{T}}$$

$$\leq \max_{q \in \Delta^{n-1}} \min_{p \in \Delta^{m-1}} p^T A q + \sqrt{\frac{2 \ln(m)}{T}}$$