Volumes in high-dimensional space

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Simple volumes

- In $\mathbb{R}^1$, line segment
  \[ [a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \} \]
  has one-dimensional volume (a.k.a. length) $b - a$.
- In $\mathbb{R}^2$, square
  \[ [a, b]^2 = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in [a, b] \} \]
  has two-dimensional volume (a.k.a. area) $(b - a)^2$.
- In $\mathbb{R}^3$, cube
  \[ [a, b]^3 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1, x_2, x_3 \in [a, b] \} \]
  has three-dimensional volume (a.k.a. volume) $(b - a)^3$. 
$d$-dimensional volumes

- Hypercube

$$[a, b]^d = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_1, x_2, \ldots, x_d \in [a, b]\}$$

has $d$-dimensional volume $(b - a)^d$.

- Use $\text{vol}(A)$ to denote $d$-dimensional volume of $A \subseteq \mathbb{R}^d$.

- For $A \subseteq \mathbb{R}^d$ and $c \geq 0$, let

$$cA := \{cx : x \in A\}.$$

  - Example: if $A = [0, 1]^d$, then $cA = [0, c]^d$ and $\text{vol}(cA) = c^d$.
  - In general,

$$\text{vol}(cA) = c^d \text{vol}(A).$$

Weird facts about the unit ball

Unit ball $B^d := \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$.

1. Lengths of most points in $B^d$ are close to one.
2. Most points in $B^d$ are near the “equator”.
3. $\lim_{d \to \infty} \text{vol}(B^d) = 0$.
   - By contrast, hypercube $[-1, 1]^d$ has volume $2^d$. 
Length of most points in the unit ball

- For $\varepsilon \in (0, 1)$, consider $(1 - \varepsilon)B^d$ (i.e., ball of radius $1 - \varepsilon$).
- $\text{vol}((1 - \varepsilon)B^d) = (1 - \varepsilon)^d \text{vol}(B^d)$
- Therefore
  \[(1 - \varepsilon)^d \leq e^{-\varepsilon d}\]
  fraction of points in $B^d$ have length at most $1 - \varepsilon$.

Most points in unit ball are near the “equator”

- Let $u$ be a unit vector (“north pole”), and $\varepsilon \in (0, 1)$.
- “Equator”: $\{ x \in B^d : \langle u, x \rangle = 0 \}$
- “Tropics”: $\{ x \in B^d : -\varepsilon \leq \langle u, x \rangle \leq \varepsilon \}$
- Points north of the tropics, $\{ x \in B^d : \langle u, x \rangle > \varepsilon \}$, are within distance $\sqrt{1 - \varepsilon^2}$ of $\varepsilon u$.
  - Hence contained in ball of radius $\sqrt{1 - \varepsilon^2}$.
  - Volume is at most $(1 - \varepsilon^2)^{d/2} \text{vol}(B^d)$.
- Similarly, points south of tropics have volume at most $(1 - \varepsilon^2)^{d/2} \text{vol}(B^d)$.
- So volume outside tropics is at most
  \[2(1 - \varepsilon^2)^{d/2} \text{vol}(B^d) \leq 2e^{-\varepsilon^2 d/2} \text{vol}(B^d).\]
Volume of unit ball

- Consider an orthonormal basis \( u_1, u_2, \ldots, u_d \) of \( \mathbb{R}^d \).
- Let \( T_i \) be the “tropics” when \( u_i \) is the “north pole”.
- Volume of points in \( \bigcap_{i=1}^d T_i \) is

\[
\text{vol}\left( \bigcap_{i=1}^d T_i \right) \geq \text{vol}(B^d) - \sum_{i=1}^d \text{vol}(T_i^c) \geq \left(1 - 2de^{-\varepsilon^2d/2}\right)\text{vol}(B^d).
\]

- But \( \text{vol}\left( \bigcap_{i=1}^d T_i \right) = \text{vol}([-\varepsilon, \varepsilon]^d) = (2\varepsilon)^d \).
- If \( 2de^{-\varepsilon^2d/2} \leq 1 \), then

\[
\text{vol}(B^d) \leq \frac{(2\varepsilon)^d}{1 - 2de^{-\varepsilon^2d/2}}.
\]

- For \( \varepsilon = \sqrt{2\ln(4d)/d} \), bound is

\[
\text{vol}(B^d) \leq 2 \left( \frac{8\ln(4d)}{d} \right)^{d/2} \xrightarrow{d \to \infty} 0.
\]