9.1 Prony’s method

Suppose a vector \( x^* \in \mathbb{R}^d \) is sparse in the canonical basis—i.e., \( \text{nnz}(x^*) =: k \ll d \). The problem of sparse signal recovery is to recover \( x^* \) from a small number of linear measurements of \( x^* \), without knowing beforehand the indices of these non-zero values. What should these measurements be, and how can they be used to recover \( x^* \)?

It is possible to recover \( x^* \) just from \( 2k \) linear measurements, using a method of Gaspard Riche de Prony from 1795. (Here we present a variant of Prony’s method.) The linear measurements required are the lowest \( 2k \) frequencies of the discrete Fourier transform of \( x^* \). Let \( M \in \mathbb{C}^{2k \times d} \) be the partial discrete Fourier transform matrix, where

\[
M_{f,t} := \omega^{ft}, \quad \text{for } f \in [2k]_0, \ t \in [d]_0,
\]

where \( \omega := \exp(-2\pi i / d) \) and \([n]_0 := \{0,1,2,\ldots,n-1\}\). (In this section, we use zero-indexing of vectors and matrices for notational convenience.) Let \( y = (y_0,y_1,\ldots,y_{2k-1}) := Mx^* \). The claim is that if \( k \leq d/2 \), then \( x^* \) can be recovered from \( y \). This is remarkable because the system of equations \( Mx = y = Mx^* \) (where \( x \) is viewed as a vector of variables) is an under-determined linear system (i.e., fewer constraints than variables). Provided that the system is feasible, there are infinitely many solutions to the linear system. However, there is only one \( k \)-sparse solution, and it can be recovered using Prony’s method.

Denote the time indices of the non-zero entries in \( x^* \) by \( t_1,t_2,\ldots,t_k \in [d]_0 \), and define the following \( k \times k \) matrices:

\[
A := \begin{bmatrix} y_0 & y_1 & \cdots & y_{k-1} \\
y_1 & y_2 & \cdots & y_k \\
\vdots & \vdots & \ddots & \vdots \\
y_{k-1} & y_k & \cdots & y_{2k-2} \end{bmatrix}, \quad B := \begin{bmatrix} y_1 & y_2 & \cdots & y_k \\
y_2 & y_3 & \cdots & y_{k+1} \\
\vdots & \vdots & \ddots & \vdots \\
y_k & y_{k+1} & \cdots & y_{2k-1} \end{bmatrix}.
\]

Note that for \( f,g \in [k]_0 \),

\[
A_{f,g} = y_{f+g} = \sum_{t=0}^{d-1} \omega^{(f+g)t} x^*_t = \sum_{i=1}^{k} x^*_i \omega^{ft_i} \omega^{gt_i},
\]

and, similarly,

\[
B_{f,g} = y_{f+g+1} = \sum_{t=0}^{d-1} \omega^{(f+g+1)t} x^*_t = \sum_{i=1}^{k} (x^*_i \omega^{t_i}) \omega^{ft_i} \omega^{gt_i}.
\]
Therefore, we may write $A = VDVT$ and $B = VSDVT$, where

$$V := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \omega_{t1} & \omega_{t2} & \cdots & \omega_{tk} \\ \omega_{2t1} & \omega_{2t2} & \cdots & \omega_{2tk} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{(k-1)t1} & \omega_{(k-1)t2} & \cdots & \omega_{(k-1)tk} \end{bmatrix},$$

$$D := \text{diag}(x_{t1}^*, x_{t2}^*, \ldots, x_{tk}^*),$$

$$S := \text{diag}(\omega_{t1}, \omega_{t2}, \ldots, \omega_{tk}).$$

The matrix $V$ is a special kind of matrix called a Vandermonde matrix, and it is well-known that such matrices are invertible—the determinant of $V$ is $\prod_{1 \leq i < j \leq k} (\omega_{ji} - \omega_{ij}) \neq 0$. It follows that $A$ is also invertible, and

$$BA^{-1} = VS V^{-1}.$$  

This means that $BA^{-1}$ is uniquely diagonalizable, and its eigenvalues $\omega_{t1}, \omega_{t2}, \ldots, \omega_{tk}$ reveal the time indices on which $x^*$ is non-zero. To recover the actual non-zero values of $x^*$, we can form $V$ from these eigenvalues, and compute the diagonal entries of $V^{-1}AV^{-T}$.

Unfortunately, this procedure (like Prony’s original method) is numerically unstable because it involves inverting the matrix $V$.

### 9.2 Incoherence

Define a set of $n$ linear measurements with a matrix $A \in \mathbb{R}^{n \times d}$: i.e., for a signal $x^* \in \mathbb{R}^d$, the $n$ measurements are $y := Ax^*$. How many non-zero entries can $x^*$ have and still be recoverable from $y$? The answer depends on the matrix $A$.

We’ll start with an example with $d = 2n$ for $n$ that is a power of two:

$$A := \left[ I \frac{1}{\sqrt{n}}H \right] \in \mathbb{R}^{n \times 2n}. \quad (9.1)$$

Here, $H \in \{\pm 1\}^{n \times n}$ is the $n \times n$ Hadamard matrix, which is a real-valued analogue of the discrete Fourier transform matrix. Notably, the matrix $n^{-1/2}H$ is orthogonal (and symmetric).

These two orthogonal bases, $I$ and $n^{-1/2}H$ (the “spike” and “sine” bases), are said to be incoherent, as any column of $I$ and any column of $n^{-1/2}H$ are almost orthogonal: their inner product has magnitude $n^{-1/2}$. This incoherence property makes it possible to recover sparse signals $x^*$ from the measurements $Ax^*$. This hinges on the following uncertainty principle of Donoho and Stark (1989), which says that a signal cannot be very sparse with respect to two incoherent bases.

**Theorem 9.1.** Let $Q, R \in \mathbb{R}^{n \times n}$ be any orthogonal matrices such that $\|\langle Qe_i, Re_j \rangle\| \leq \mu$ for all $i, j \in [n]$. Suppose $y = Q\alpha = R\beta \neq 0$. Then $\text{nnz}(\alpha) \text{nnz}(\beta) \geq 1/\mu^2$ and $\text{nnz}(\alpha) + \text{nnz}(\beta) \geq 2/\mu$.

**Proof.** Since $y = \sum_{i=1}^n \alpha_i Qe_i = \sum_{j=1}^n \beta_j Re_j$, we have

$$\|y\|_2^2 = \sum_{i=1}^n \sum_{j=1}^n \langle \alpha_i Qe_i, \beta_j Re_j \rangle \leq \sum_{i=1}^n \sum_{j=1}^n \|\langle Qe_i, Re_j \rangle\| |\alpha_i||\beta_j| \leq \mu \sum_{i=1}^n |\alpha_i| \sum_{j=1}^n |\beta_j|.$$

By Cauchy-Schwarz, we have $\sum_{i=1}^n |\alpha_i| \leq \sqrt{\text{nnz}(\alpha)}\|\alpha\|_2$ and $\sum_{j=1}^n |\beta_j| \leq \sqrt{\text{nnz}(\beta)}\|\beta\|_2$. Since $\|y\|_2 = \|\alpha\|_2 = \|\beta\|_2$, it follows that $1 \leq \mu \sqrt{\text{nnz}(\alpha) \text{nnz}(\beta)} \leq \mu (\text{nnz}(\alpha) + \text{nnz}(\beta))/2$ where the second inequality is due to the AM/GM inequality.
The “spike and sine” bases, $I$ and $n^{-1/2}H$, satisfy the precondition of Theorem 9.1 with $\mu = 1/\sqrt{n}$. This means that if a signal $y$ has at most $k$ non-zero entries with respect to the “spike” (coordinate) basis, then it must have at least $2\sqrt{n} - k$ non-zero entries when written in terms of the “sine” (Hadamard) basis. Indeed, the vector $y$ with $\sqrt{n}$ non-zero entries $y\sqrt{n} = y_1\sqrt{n} = \cdots = y_n = 1$ (in the “spike” basis) has exactly $\sqrt{n}$ non-zero entries with respect to the “sine” basis.

**Theorem 9.2.** Let $A = [Q \mid R] \in \mathbb{R}^{n \times 2n}$ for some orthogonal matrices $Q, R \in \mathbb{R}^{n \times n}$ such that $|\langle Q e_i, R e_j \rangle| \leq \mu$ for all $i, j \in [n]$. Furthermore, suppose $x^* \in \mathbb{R}^{2n}$ satisfies $\text{nnz}(x^*) < 1/\mu$. Then $x^*$ is the unique sparsest solution to the linear system $Ax = y$ for $y := Ax^*$.

**Proof.** Suppose $x \in \mathbb{R}^d$ satisfies $Ax = y = Ax^*$ and $\text{nnz}(x) < 1/\mu$. Then $A(x - x^*) = 0$. Write $x - x^*$ as $(z_0, -z_1)$, where $z_0, z_1 \in \mathbb{R}^n$, so $Az = Qz_0 - Rz_1 = 0$. Suppose for sake of contradiction that $x - x^* \neq 0$. By Theorem 9.1, $\text{nnz}(z) = \text{nnz}(z_0) + \text{nnz}(z_1) \geq 2/\mu$. However, since $\text{nnz}(x^*) < 1/\mu$ and $\text{nnz}(x) < 1/\mu$, we also have $\text{nnz}(z) = \text{nnz}(x - x^*) < 2/\mu$. This is a contradiction, so we must have $x - x^* = 0$, i.e., $x = x^*$.

The uncertainty principle suggests the following property of a measurement matrix $A$:

$$
\mu(A) := \max_{1 \leq i < j \leq d} \frac{|\langle A e_i, A e_j \rangle|}{\|A e_i\|_2 \|A e_j\|_2}.
$$

The measurement matrices $A$ satisfying the preconditions of Theorem 9.1 have coherence $\mu$.

Matrices $A$ with low coherence, like those from Theorems 9.1, can also be used as measurement matrices with similar guarantees as from Theorem 9.2. To show this, we first establish an intermediate result.

**Lemma 9.1.** Every set of fewer than $1 + 1/\mu(A)$ non-zero columns of $A$ is linearly independent.

**Proof.** Consider a set of $k \leq 1 + 1/\mu(A)$ non-zero columns of $A$. Without loss of generality, we assume these columns are $\widetilde{A} := [A e_1 | A e_2 | \cdots | A e_k]$, and each such column $A e_i$ has unit length. First observe that for each $i \in [k]$, $\sum_{j \neq i} |\langle A e_i, A e_j \rangle| < 1$:

$$
\sum_{j \neq i} |\langle A e_i, A e_j \rangle| \leq \sum_{j \neq i} \mu(A) \leq (k - 1) \mu(A) < 1.
$$

This means that the $k \times k$ matrix $G := \widetilde{A}^\top \widetilde{A}$ is strictly diagonal dominant: each diagonal entry $G_{i,i}$ is strictly larger than the sum of the absolute values of all off-diagonal entries $\sum_{j \neq i} |G_{i,j}|$ in the same row. It is easy to see that strictly diagonal dominant matrices are non-singular (Claim 9.1 below). Therefore $G$ is positive definite, which implies that $A$ has rank $k$.

**Claim 9.1.** Suppose $M \in \mathbb{R}^{k \times k}$ is strictly diagonal dominant. Then $M$ is non-singular.

**Proof.** Suppose $Mx = 0$. Pick any $i \in \arg\max_{j \in [k]} |x_j|$. We have $0 = \sum_{j=1}^k M_{i,j}x_j$, which implies that

$$
|M_{i,i}| x_i = | - M_{i,i} x_i | = \left| \sum_{j \neq i} M_{i,j} x_j \right|.
$$

By the triangle inequality, the right-hand-side is bounded above by $\sum_{j \neq i} |M_{i,j}| |x_j| \leq \sum_{j \neq i} |M_{i,j}| |x_i|$. Therefore

$$
|M_{i,i}| x_i \leq \sum_{j \neq i} |M_{i,j}| |x_i|.
$$

Since $\sum_{j \neq i} |M_{i,j}| > |M_{i,i}|$, we must have that $|x_i| = 0$, which means that $x = 0$. Thus $M$ is non-singular.
Donoho and Elad (2003) define the *spark* of matrix $A$, written $\text{spark}(A)$, to be the minimum number of columns of $A$ that are linearly dependent. In other words, every collection of $\text{spark}(A) - 1$ columns of $A$ are linearly independent. Lemma 9.1 shows that $\text{spark}(A) \geq 1 + 1/\mu(A)$.

**Theorem 9.3.** Suppose $x^* \in \mathbb{R}^d$ satisfies $\text{nnz}(x^*) < \text{spark}(A)/2$. Then $x^*$ is the unique sparsest solution to the linear system $Ax = y$ for $y := Ax^*$.

Proof. Suppose $x \in \mathbb{R}^d$ satisfies $Ax = y = Ax^*$ and $\text{nnz}(A) < \text{spark}(A)/2$. Then $A(x - x^*) = 0$ and is also a linear combination of fewer than $\text{spark}(A)$ columns of $A$. These columns of $A$ are linearly independent (by the definition of spark), so $x - x^* = 0$, i.e., $x = x^*$.

Theorem 9.3 implies that it is possible to recover $x^*$ from the measurements $Ax^*$ whenever $\text{nnz}(x^*) < \text{spark}(A)/2$, which indeed holds when $\text{nnz}(x^*) < (1 + 1/\mu(A))/2$. Doing this efficiently is another matter, which we discuss next.

### 9.3 Orthogonal matching pursuit

Orthogonal matching pursuit (OMP), which was proposed by Chen, Billings, and Luo (1989), is a simple greedy algorithm for sparse signal recovery. Tropp (2004) showed that OMP recovers $x^*$ from the measurements $y = Ax^*$ whenever $\text{nnz}(x^*) < (1 + 1/\mu(A))/2$. For simplicity, we assume every column of $A$ is non-zero.

```
input matrix $A = [a_1 | a_2 | \cdots | a_d] \in \mathbb{R}^{n \times d}$, measurement $y \in \mathbb{R}^n$.
1: Let $\Omega_0 := \emptyset$ and $r_0 := y$.
2: for $t = 1, 2, \ldots$ do
3: Pick $i_t \in \arg \max_{j \in [d]} \frac{|\langle a_j, r_{t-1} \rangle|}{\|a_j\|_2}$.
4: Let $\Omega_t := \Omega_{t-1} \cup \{i_t\}$ and $W_t := \text{span}(a_j : j \in \Omega_t)$.
5: Let $r_t := (I - \Pi_{W_t})y$.
6: Break out of loop if $r_t = 0$.
7: end for
output $x := A_{\Omega_t}^\dagger y$.
```

Above, $\Pi_{W_t}$ is the orthogonal projection to the subspace $W_t$, and $A_{\Omega}$ (for any $\Omega \subseteq [d]$) is the matrix $A$ but replacing every column not in $\Omega$ with zeros.

**Theorem 9.4.** Assume every column of $A = [a_1 | a_2 | \cdots | a_d]$ is non-zero. Suppose $x^* \in \mathbb{R}^d$ satisfies $\text{nnz}(x^*) < (1 + 1/\mu(A))/2$. Then on input $A$ and $y$, OMP returns $x^*$.

Proof. Let $\Omega^* \subseteq [d]$ be the indices of the non-zero entries of $x^*$, with $k := |\Omega^*| = \text{nnz}(x^*)$. By Lemma 9.1, the columns of $A$ corresponding to $\Omega^*$ are linearly independent, and therefore $x^* = A_{\Omega^*}^\dagger y$. Therefore, it suffices to prove that $\Omega_k = \Omega^*$.

If OMP picks an index $i_t \in \Omega^*$ in some iteration $t$, then it can never pick it again in later iterations. This is because the residual $r_t$ is orthogonal to $a_{i_t}$. Therefore, it suffices to prove every $i_t$ picked by OMP is in $\Omega^*$. If this holds, then $\Omega_k = \Omega^*$.

We prove that $\Omega_t \subseteq \Omega^*$ by induction. The base case, $t = 0$, is true because $\Omega_0 = \emptyset$. Now suppose for some $t \geq 1$ that $\Omega_{t-1} \subseteq \Omega^*$. We'll show that $i_t \in \Omega^*$. 

It is clear that \( r_{t-1} \) is in the span of \( \{ a_i : i \in \Omega^* \} \). (Indeed, it is the span of \( \{ a_i : i \in \Omega^* \setminus \Omega_{t-1} \} \).) Therefore, we may write \( r_{t-1} \) as

\[
    r_{t-1} = \sum_{i \in \Omega^*} c_i a_i
\]

for some scalars \( c_i, i \in \Omega^* \). By Theorem 9.3, the vector \( x^* \) is the sparest solution to \( Ax = y \), which implies that \( r_{t-1} \neq 0 \). For notational convenience, we re-number the columns so that \( \Omega^* = \{1, 2, \ldots, k\} \) and

\[
    |c_1|\|a_1\|_2 \geq |c_2|\|a_2\|_2 \geq \cdots \geq |c_k|\|a_k\|_2 \geq 0.
\]

Since \( r_{t-1} \neq 0 \), it follows that \( c_1 \neq 0 \). Consider any \( j \geq k + 1 \). We have

\[
    \frac{|\langle r_{t-1}, a_j \rangle|}{\|a_j\|_2} = \frac{\sum_{i=1}^k c_i \langle a_i, a_j \rangle}{\|a_j\|_2} \\
    \leq \sum_{i=1}^k |c_i|\|a_i\|_2 \mu(A) \quad \text{(since } j \neq i \text{ for all } i \in [k]) \\
    \leq |c_1|\|a_1\|_2 k \mu(A).
\]

On the other hand,

\[
    \frac{|\langle r_{t-1}, a_1 \rangle|}{\|a_1\|_2} = \frac{\sum_{i=1}^k c_i \langle a_i, a_1 \rangle}{\|a_1\|_2} \\
    \geq |c_1|\|a_1\|_2 - \frac{\sum_{i=2}^k c_i \langle a_i, a_1 \rangle}{\|a_1\|_2} \\
    \geq |c_1|\|a_1\|_2 - \sum_{i=2}^k |c_i|\|a_i\|_2 \mu(A) \quad \text{(since } i > 1) \\
    \geq |c_1|\|a_1\|_2 (1 - (k - 1) \mu(A)).
\]

Since \( k < (1 + 1/\mu(A))/2 \) and \( |c_1|\|a_1\|_2 > 0 \), these inequalities imply

\[
    \frac{|\langle r_{t-1}, a_1 \rangle|}{\|a_1\|_2} > \frac{|\langle r_{t-1}, a_j \rangle|}{\|a_j\|_2}.
\]

This means that \( j \geq k + 1 \) will not be picked by OMP, and hence \( i_t \in \Omega^* \). \( \square \)