Problem 1. Suppose $n$ data points $S \subset \mathbb{R}^d$ are partitioned into $k$ subsets $S_1, S_2, \ldots, S_k$; let $\mu_j := \text{mean}(S_j)$ for each $j \in [k]$. Let $A \in \mathbb{R}^{n \times d}$ be the data matrix (whose rows are the points in $S$), and let $M \in \mathbb{R}^{n \times d}$ be the matrix whose $i$-th row is $\mu_j^\top$ when the $i$-th data point belongs to $S_j$. Let $\Pi$ denote the orthogonal projector to the top $k$-dimensional (uncentered) PCA subspace $W$ for $S$; let $\tilde{S} \subset W$ be the $\Pi$-projected data points (i.e., the rows of $A\Pi$).

Assume, for some $c_0 \geq 1$, that $C \subset W$ is a set of representatives with $k$-means cost on $\tilde{S}$ at most $c_0$ times optimal (among representatives in $W$):

$$\text{cost}(\tilde{S}, C) \leq c_0 \cdot \min_{C \subset W: |C| = k} \text{cost}(\tilde{S}, C).$$

Instructions: Do at least two parts among (a), (b), and (c); and also do part (d).

(a) Modify the proof from the $k$-means notes to deduce that for each $i \in [k]$,

$$\|\mu_i - \hat{C}(\mu_i)\|_2^2 \leq \frac{1}{|S_i|} \left( 1 + \sqrt{\frac{c_0 \cdot \|A - M\Pi\|_F^2}{\|A\Pi - M\|_F^2}} \right)^2 \|A\Pi - M\|_F^2.$$

You can just explain, precisely, how the argument from the notes should be modified. Explain why $\|A - M\Pi\|_F / \|A\Pi - M\|_F$ is at most one (assuming the denominator is non-zero).

(b) Prove that there is an absolute constant $c \geq 1$ such that

$$\|A\Pi - M\|_F^2 \leq c \cdot \min \left\{ \|A - M\|_F^2, k \|A - M\|_2^2 \right\}.$$

(c) Suppose each $x \in S_i$ is actually a random vector with distribution $N(\mu_i, \sigma^2 I)$. We’ll use this assumption to prove a bound on $\|A - M\|_2^2$. (Here, $M$ remains the matrix whose rows are the $\mu_i^\top$.) Prove that with probability 0.99, $\|A - M\|_2^2 \leq O(\sigma^2(n + d))$. Since this is fairly routine, you can just give a rough sketch of the argument.

Hint: There are many ways to do this, but one simple way is to use $\chi^2$ tail bounds together with a covering argument.

(d) Continuing from (c), assume further that $|S_i| = n/k$ for each $i \in [k]$. Deduce an upper-bound on $\max_{i \in [k]} \|\mu_i - \hat{C}(\mu_i)\|_2^2$ that holds with probability 0.99. It should be stated in terms of $c_0$, $d$, $k$, $n$, and $\sigma$. Here, you can use the results from parts (a), (b), and (c), as well as big-$O$ notation.

Problem 2. Let $T$ be an undirected tree on $n$ vertices $V$. Consider the metric space $(V, \rho)$, where $\rho$ is the shortest path metric on $T$ (i.e., $\rho(x, y) =$ length of shortest path from $x$ to $y$). Note that because $T$ is a tree, the shortest path from $x$ to $y$ is actually the only path!

Instructions: Do at least two parts among (a), (b), and (c).

(a) Show how to construct an embedding $f: (V, \rho) \to \ell_1^n$ with no distortion.

Hint: Remove a single leaf $v \in T$, recursively construct the embedding for $T \setminus \{v\}$ (into $\ell_1^{n-1}$), and then modify the embedding to also work with $v$. 

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(b) Show how to find subtrees $T'$ and $T''$ of $T$ such that: (i) $T'$ and $T''$ share a single vertex $v_0 \in T$ and no edges, (ii) $T = T' \cup T''$, and (iii) $\max\{|T'|, |T''|\} \leq 1 + 3n/4$. (It is possible to do this with $1 + 2n/3$ instead of $1 + 3n/4$.)

(Here, $|\cdot|$ denotes number of vertices.)

The following lemma is helpful:

**Lemma 1.** For any tree $T$ on $n$ vertices, there is a special vertex $v_0 \in T$ such that $T \setminus \{v_0\}$ is a forest of trees, each of which has at most $n/2$ vertices.

You can use the lemma without proof, and also assume a blackbox algorithm for finding this special vertex $v_0 \in T$.

(c) Show how to construct an embedding $f : (V, \rho) \to \ell^k_\infty$ with no distortion and $d = O(\log n)$.

*Hint:* Use divide and conquer, as suggested in (b). Here is a rough sketch of a possible approach.

1. If $n$ is smaller than some absolute constant, just let $f$ be the Fréchet embedding.
2. Otherwise, obtain subtrees $T'$ and $T''$ as in (b), and recursively construct embeddings $f' : T' \to \ell^k_\infty$ and $f'' : T'' \to \ell^k_\infty$.
3. Modify $f'$ and $f''$ so that the special vertex $v_0 \in T' \cap T''$ gets mapped to the same point (so $f'(v_0) = f''(v_0)$). (Can this be done without incurring any distortion?)
4. Construct a new embedding $f : T \to \ell^{k+1}_\infty$:

$$f(x) := \begin{cases} (f'(x), \rho(x, v_0)) & \text{if } x \in T', \\ (f''(x), -\rho(x, v_0)) & \text{if } x \in T''. \end{cases}$$

If you follow this sketch, fill in the details, and prove that it is correct.