Bivariate normal distribution

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1 Basic facts

We say that the distribution of the pair of random variables $X = (X_1, X_2)$ is the *bivariate normal distribution* with 2-dimensional mean vector

$$\mu = (\mu_1, \mu_2)$$

and 2×2 covariance matrix

$$\Sigma = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{bmatrix}$$

if the density function for X is

$$\begin{aligned} p_{(X_1,X_2)}(x_1, x_2) &= \frac{1}{\sqrt{\det(\Sigma)}} \cdot \frac{1}{2\pi} \exp\left(-\frac{1}{2} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}^{\mathsf{T}} \Sigma^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \right) \\ &= \frac{1}{\sqrt{\det(\Sigma)}} \cdot \frac{1}{2\pi} \exp\left(-\frac{(x-\mu)^{\mathsf{T}} \Sigma^{-1} (x-\mu)}{2} \right). \end{aligned}$$

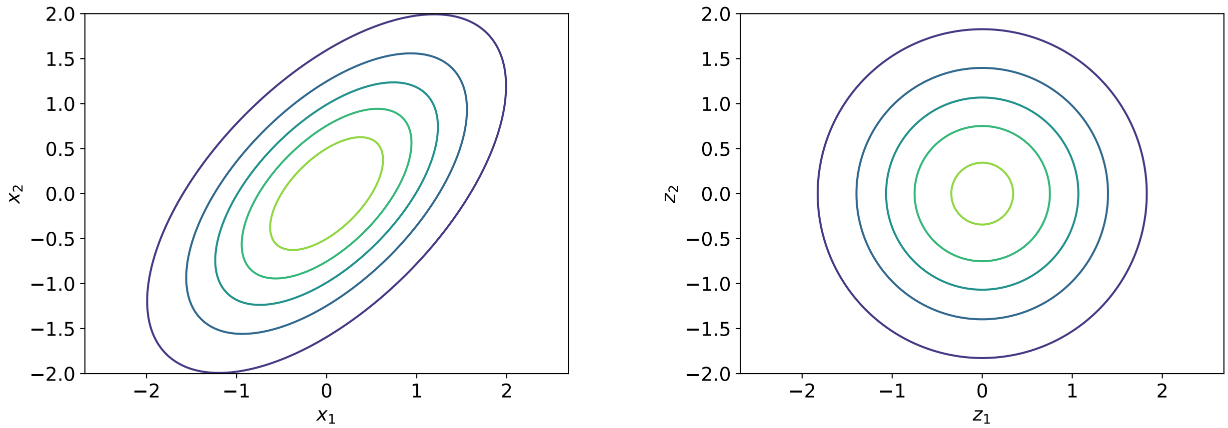
We also use the notation

$$(X_1, X_2) \sim \mathrm{N}(\mu, \Sigma)$$

to declare that (X_1, X_2) has a bivariate normal distribution with mean μ and covariance Σ. The covariance matrix Σ is required to be *positive definite*, which means that Σ is symmetric (i.e., $\Sigma = \Sigma^{\mathsf{T}}$), and $v^{\mathsf{T}} \Sigma v > 0$ for all $v \neq 0$.

The contour lines of equal density value form concentric ellipses centered at (μ_1, μ_2); the density is highest at (μ_1, μ_2), and it falls off quickly away from (μ_1, μ_2). See Figure 1(a).

Using the density function, we can compute the means and variances of X_1 and X_2, as well as the covariance between X_1 and X_2:



(a) (b)

Figure 1: (a) Contour lines of bivariate normal density with $\mu_1 = \mu_2 = 0$, $\Sigma_{1,1} = \Sigma_{2,2} = 10/9$, and $\Sigma_{1,2} = 2/3$. (b) Contour lines of standard bivariate normal density.

- $\mathbb{E}(X_1) = \mu_1$, $\text{var}(X_1) = \Sigma_{1,1}$
- $\mathbb{E}(X_2) = \mu_2$, $\text{var}(X_2) = \Sigma_{2,2}$
- $\text{cov}(X_1, X_2) = \Sigma_{1,2} = \Sigma_{2,1}$

2 Where does bivariate normal distribution come from?

If Z_1 and Z_2 are independent random variables (defined on the same probability space) and each is a standard normal random variable, then distribution of $Z = (Z_1, Z_2)$ is the *standard bivariate normal*, with density function given by the product of the marginal density functions for Z_1 and Z_2 (each being the standard univariate normal density):

$$p_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_1^2}{2}\right) \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_2^2}{2}\right) = \frac{1}{2\pi} \exp\left(-\frac{\|z\|^2}{2}\right).$$

The contours lines of equal density value form concentric circles centered at the origin; the density is highest at origin, and it falls off quickly away from origin. See Figure 1(b).

Define $X = (X_1, X_2)$ to be an affine transformation of Z as follows:

$$X_1 = \mu_1 + A_{1,1}Z_1 + A_{1,2}Z_2$$
$$X_2 = \mu_2 + A_{2,1}Z_1 + A_{2,2}Z_2$$

for some real numbers $\mu_1, \mu_2, A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}$. In matrix form,

$$X = \mu + AZ$$

where

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}.$$

Assume that A is invertible, so

$$Z = A^{-1}(X - \mu).$$

Using the change-of-density rule and some simplifications, we find that the probability density function for X is given by

$$\begin{aligned} p_X(x) &= \det(A^{-1}) \cdot p_Z(A^{-1}(x - \mu)) \\ &= \frac{1}{\det(A)} \cdot \frac{1}{2\pi} \exp\left(-\frac{\|A^{-1}(x-\mu)\|^2}{2} \right) \\ &= \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{(x-\mu)^\intercal \Sigma^{-1}(x-\mu)}{2} \right) \end{aligned}$$

where $\Sigma = AA^\intercal$. This is generically the form of the density function of a bivariate normal distribution, since any positive definite matrix Σ can be written as $AA^\intercal$ for some invertible matrix A.

Upshot: Every (X_1, X_2) that has a bivariate normal distribution is, in fact, an affine transformation of independent standard normal random variables Z_1, Z_2.

All of this naturally generalizes to d-variate normal distributions, for all integers $d \geq 1$.