Homework 0

1. Suppose a 6-sided die is weighted so that, for each $k \in \{1, 2, 3, 4, 5, 6\}$, the side showing k pips is k times as likely to show up as the side showing 1 pip. What is the probability that a roll of this die shows an even number of pips?

2. Suppose A and B are events from a probability space such that $\Pr(A \cap B) = 1/4$, $\Pr(A^{\mathsf{c}}) = 1/3$, and $\Pr(B) = 1/2$. What is $\Pr(A \cup B)$?

3. (Hard problem.) You repeatedly roll a fair 6-sided die and stop after seeing 6 pips face up. Suppose only even numbers of pips show up in all rolls. What is the probability that the number of rolls is 1, given this information?

4. If X is the number of heads in 5 tosses of a fair coin, and Y is number of pips shown in the roll of a fair 6-sided die, what are the variances of X and Y?

5. Roll a fair 6-sided die; let X indicate if number of pips is at most 4, and let Y indicate if number of pips is even. Are X and Y independent?

6. Toss a fair coin 10 times, and let X be the number times HTH appears as a substring of the outcome. What is the expected value of X? (Hint: Write X as a sum of 8 indicator random variables, and use the linearity of expectation.)

7. If $\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y)$, then what can you say about $\mathrm{cov}(X, Y)$?

8. If $X = Y$, then what is the value of $\mathrm{cor}(X, Y)$?

9. Is it possible to have $\mathrm{cor}(X, Y) > 1$? What about $\mathrm{cor}(X, Y) < -1$?

10. Toss a fair coin two times; let

    - $X =$ number of heads
    - $Y = \begin{cases} 1 & \text{if first toss is heads and second toss is tails} \\ 0 & \text{if both tosses are the same} \\ -1 & \text{if first toss is tails and second toss is heads} \end{cases}$

    For each $x \in \mathrm{range}(X)$, what is the expected value of Y given $X = x$?

11. What is the distribution function for $X \sim \mathrm{Unif}([0, 1])$?

12. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

    Determine whether each of the following is a valid or invalid algebraic expression. If you answer "valid", carry-out the matrix-matrix multiplication and write the result.

(a) $A + B$ (b) AA (c) AB (d) CA (e) CBA

13. Determine the rank of the matrix A from the previous problem.

14. (Hard problem.) Suppose unit vectors $v^{(1)}, \dots, v^{(n)}$ satisfy $|\langle v^{(i)}, v^{(j)} \rangle| \le 1/n$ for all $i \ne j$. Show that these vectors must be linearly independent.

15. Let A and B be 3×2 matrices defined as follows:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 5 & 9 \end{bmatrix}.$$

The column spaces of A and B are not orthogonal. Explain how to find a unit vector in $\mathsf{CS}(A) \cap \mathsf{CS}(B)$, and then write such a unit vector.

16. Let A be the 4×3 matrix obtained by starting with the identity matrix and then removing the last column. Let $v = (1, 2, 3, 4)$. Write the orthogonal projection operator for $\mathsf{CS}(A)$, and compute the orthogonal projection of v onto $\mathsf{CS}(A)$.

17. Find the gradient of $f_1 \colon \mathbb{R}^3 \to \mathbb{R}$ given by $f_1(x) = x_1^3 + x_2^2 + x_3 + 1$.

18. Find the gradient of $f_2 \colon \mathbb{R}^3 \to \mathbb{R}$ given by $f_2(x) = x_1 x_2 x_3$.

19. Find the gradient of $f_3 \colon \mathbb{R}^3 \to \mathbb{R}$ given by $f_3(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$.

20. Find the gradient of $f_4 \colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$f_4(x) = \frac{1}{2} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

21. Find the gradient of $f_5 \colon \mathbb{R}^3 \to \mathbb{R}$ given by $f_5(x) = \exp(-(x_1^2 + x_2^2 + x_3^2))$.

22. Find the gradient of $f_6 \colon \mathbb{R}^3 \to \mathbb{R}$ given by $f_6(x) = f_2(x) f_5(x)$.

23. Write a Python function that returns a Numpy array containing all odd integers from a given Numpy array `x` of numbers.

```python
def extract_odd_numbers(x):
  # your code here
```

24. Write a Python function that, given a positive integer `n`, returns an $n \times n$ Numpy array where the first column is all ones, the second column is all twos, and so on.

```python
def make_funny_matrix(n):
  # your code here
```

25. Write a Python function that returns an $m \times (n + 1)$ Numpy array that appends a column of all-ones to the end of a given $m \times n$ Numpy array `A`.

```python
def append_ones_column(A):
  # your code here
```

Solutions

https://drive.google.com/file/d/15AnOOQ9-azmLcH0c_zJ89s2GoTQ4-QYV/view?usp=sharing