Probability tails

COMS 4721
Binomial distribution

Number of heads when a coin with heads bias $p \in [0, 1]$ is tossed $n$ times:

**binomial distribution**

$$S \sim \text{Bin}(n, p)$$
Binomial distribution

Number of heads when a coin with heads bias $p \in [0, 1]$ is tossed $n$ times:

\[ S \sim \text{Bin}(n, p) \]

Basic combinatorics: for any $k \in \{0, 1, 2, \ldots, n\}$,

\[ \mathbb{P}(S = k) = \binom{n}{k} p^k (1 - p)^{n-k}. \]
The outcome of a coin toss with heads bias $p \in [0, 1]$:

**Bernoulli distribution**

$$X \sim \text{Bern}(p) = \text{Bin}(1, p)$$

$$\mathbb{P}(X = 1) = p, \quad \mathbb{P}(X = 0) = 1 - p.$$
The outcome of a coin toss with heads bias $p \in [0, 1]$:

**Bernoulli distribution**

$$X \sim \text{Bern}(p) = \text{Bin}(1, p)$$

$$P(X = 1) = p, \quad P(X = 0) = 1 - p.$$ 

**Mean:**

$$\mathbb{E}(X) = P(X = 0) \cdot 0 + P(X = 1) \cdot 1 = p.$$
The outcome of a coin toss with heads bias $p \in [0, 1]$:

**Bernoulli distribution**

$$X \sim \text{Bern}(p) = \text{Bin}(1, p)$$

$$\mathbb{P}(X = 1) = p, \quad \mathbb{P}(X = 0) = 1 - p.$$ 

**Mean:**

$$\mathbb{E}(X) = \mathbb{P}(X = 0) \cdot 0 + \mathbb{P}(X = 1) \cdot 1 = p.$$ 

**Variance:**

$$\text{var}(X) = \mathbb{E}\left[ (X - \mathbb{E}(X))^2 \right] = p(1 - p).$$

(Standard deviation is $\sqrt{\text{var}(X)}$; more convenient to use than $\mathbb{E}\left[ |X - \mathbb{E}(X)| \right]$.)
Let $X_1, X_2, \ldots, X_n$ be iid Bern($p$) random variables, and let $S \sim \text{Bin}(n, p)$. Then $S$ has the same distribution as $X_1 + X_2 + \cdots + X_n$. 
Binomial = sums of iid Bernoullis

Let $X_1, X_2, \ldots, X_n$ be iid Bern($p$) random variables, and let $S \sim \text{Bin}(n, p)$. Then $S$ has the same distribution as $X_1 + X_2 + \cdots + X_n$.

**Mean:** By linearity of expectation,

$$
E(S) = E\left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} E(X_i) = np.
$$
Binomial = sums of iid Bernoullis

Let $X_1, X_2, \ldots, X_n$ be iid Bern($p$) random variables, and let $S \sim \text{Bin}(n, p)$. Then $S$ has the same distribution as $X_1 + X_2 + \cdots + X_n$.

**Mean:** By linearity of expectation,

$$
\mathbb{E}(S) = \mathbb{E}\left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \mathbb{E}(X_i) = np.
$$

**Variance:** Since $X_1, X_2, \ldots, X_n$ are independent,

$$
\text{var}(S) = \text{var}\left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \text{var}(X_i) = np(1 - p).
$$
**Deviations from the mean**

**Question:** What are the “typical” values (i.e., non-tail event) of $S \sim \text{Bin}(n, p)$?
**Deviations from the mean**

**Question:** What are the “typical” values (i.e., non-tail event) of $S \sim \text{Bin}(n, p)$?

How do we rigorously quantify the probability mass in the tails?
Let $S \sim \text{Bin}(n, p)$, and define

$$\text{RE}(a \| b) := a \ln \frac{a}{b} + (1 - a) \ln \frac{1 - a}{1 - b} \geq 0 \quad (= 0 \text{ iff } a = b)$$

(relative entropy between Bernoulli distributions with heads biases $a$ and $b$).
Let $S \sim \text{Bin}(n, p)$, and define

$$\text{RE}(a \parallel b) := a \ln \frac{a}{b} + (1 - a) \ln \frac{1 - a}{1 - b} \geq 0 \quad (= 0 \text{ iff } a = b)$$

(relative entropy between Bernoulli distributions with heads biases $a$ and $b$).

**Upper tail bound**: For any $u > p$,

$$\mathbb{P}(S \geq n \cdot u) \leq \exp\left(-n \cdot \text{RE}(u \parallel p)\right).$$

**Lower tail bound**: For any $\ell < p$,

$$\mathbb{P}(S \leq n \cdot \ell) \leq \exp\left(-n \cdot \text{RE}(\ell \parallel p)\right).$$
Let $S \sim \text{Bin}(n, p)$, and define

$$\text{RE}(a\|b) := a \ln \frac{a}{b} + (1 - a) \ln \frac{1 - a}{1 - b} \geq 0 \quad (= 0 \text{ iff } a = b)$$

(relative entropy between Bernoulli distributions with heads biases $a$ and $b$).

**Upper tail bound:** For any $u > p$,

$$\mathbb{P}(S \geq n \cdot u) \leq \exp\left(-n \cdot \text{RE}(u\|p)\right).$$

**Lower tail bound:** For any $\ell < p$,

$$\mathbb{P}(S \leq n \cdot \ell) \leq \exp\left(-n \cdot \text{RE}(\ell\|p)\right).$$

Both exponentially small in $n$. 
Let $S \sim \text{Bin}(n, p)$, and define

$$\text{RE}(a\|b) := a \ln \frac{a}{b} + (1 - a) \ln \frac{1 - a}{1 - b} \geq 0 \quad (=0 \text{ iff } a = b)$$

(relative entropy between Bernoulli distributions with heads biases $a$ and $b$).

**Upper tail bound**: For any $u > p$,

$$P(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u\|p)).$$

**Lower tail bound**: For any $\ell < p$,

$$P(S \leq n \cdot \ell) \leq \exp(-n \cdot \text{RE}(\ell\|p)).$$

Both exponentially small in $n$.

Large deviations from mean $p \cdot n$ (e.g., $(u - p) \cdot n$) are exponentially unlikely.
Illustration of large deviations

\[ p = 1/3, \quad u = 1/3 + 0.05, \quad n = 100 \]
\[ \exp(-\text{RE}(u||p)) \approx 0.995 \]
Illustration of large deviations

\[ p = \frac{1}{3}, \quad u = \frac{1}{3} + 0.05, \quad n = 200 \]

\[ \exp(-RE(u||p)) \approx 0.995 \]
Illustration of large deviations

\[ p = \frac{1}{3}, \quad u = \frac{1}{3} + 0.05, \quad n = 300 \]

\[ \exp(-RE(u\|p)) \approx 0.995 \]
$p = 1/3, \quad u = 1/3 + 0.05, \quad n = 400$

$\exp(-RE(u||p)) \approx 0.995$
$p = 1/3, \quad u = 1/3 + 0.05, \quad n = 500 \quad \exp(-\text{RE}(u\|p)) \approx 0.995$
Illustration of Large Deviations

\[ p = \frac{1}{3}, \quad u = \frac{1}{3} + 0.05, \quad n = 600 \]
\[ \exp(-\text{RE}(u\|p)) \approx 0.995 \]
Illustration of large deviations

\[ p = \frac{1}{3}, \quad u = \frac{1}{3} + 0.05, \quad n = 700 \]

\[ \exp(-\text{RE}(u||p)) \approx 0.995 \]
Illustration of large deviations

$p = 1/3, \ u = 1/3 + 0.05, \ n = 800$

$\exp(- \operatorname{RE}(u \parallel p)) \approx 0.995$
Illustration of large deviations

\[ p = \frac{1}{3}, \quad u = \frac{1}{3} + 0.05, \quad n = 900 \]
\[ \exp(-\text{RE}(u\|p)) \approx 0.995 \]
Illustration of large deviations

\[
p = \frac{1}{3}, \quad u = \frac{1}{3} + 0.05, \quad n = 1000
\]
\[
\exp(-\text{RE}(u\|p)) \approx 0.995
\]
Proof of Chernoff bound (upper tail bound)

**Theorem:** For $S \sim \text{Bin}(n, p)$, $\mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p))$ for $u > p$. 

Consider $n$ iid Bernoulli random variables: $X_1, X_2, \ldots, X_n$. Let $E \subseteq \{0, 1\}^n$ be all outcomes $x = (x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^n x_i \geq n \cdot u$.

Some shorthand notation:

▶ $p[x] :=$ probability mass of outcome $x$ when $X_i$ has heads bias $p$.

▶ $u[x] :=$ probability mass of outcome $x$ when $X_i$ has heads bias $u$.

Core of the proof: Consider any outcome $x \in E$ with, say, $k \geq n \cdot u$ heads:

$$p[x] \cdot u[x] = p^k (1 - p)^{n-k} \leq (p u)^k (1 - p - u)^{n-k} \leq (p u)^n \cdot (1 - p - u)^{n-k} \leq \exp(-n \cdot \text{RE}(u||p)).$$
**Proof of Chernoff Bound (Upper Tail Bound)**

**Theorem:** For $S \sim \text{Bin}(n, p)$, $\mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p))$ for $u > p$.

Consider $n$ iid Bernoulli random variables: $X_1, X_2, \ldots, X_n$. Let $\mathcal{E} \subseteq \{0, 1\}^n$ be all outcomes $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^{n} x_i \geq n \cdot u$. 

Theorem: For $S \sim \text{Bin}(n, p)$, $\mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p))$ for $u > p$.

Consider $n$ iid Bernoulli random variables: $X_1, X_2, \ldots, X_n$. Let $\mathcal{E} \subseteq \{0, 1\}^n$ be all outcomes $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^{n} x_i \geq n \cdot u$.

Some shorthand notation:

- $p[\mathbf{x}] :=$ probability mass of outcome $\mathbf{x}$ when $X_i$ has heads bias $p$.
- $u[\mathbf{x}] :=$ probability mass of outcome $\mathbf{x}$ when $X_i$ has heads bias $u$.
**Proof of Chernoff Bound (Upper Tail Bound)**

**Theorem:** For $S \sim \text{Bin}(n, p)$, $\mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u\|p))$ for $u > p$.

Consider $n$ iid Bernoulli random variables: $X_1, X_2, \ldots, X_n$. Let $\mathcal{E} \subseteq \{0, 1\}^n$ be all outcomes $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^n x_i \geq n \cdot u$.

**Some shorthand notation:**

- $p[\mathbf{x}] :=$ probability mass of outcome $\mathbf{x}$ when $X_i$ has heads bias $p$.
- $u[\mathbf{x}] :=$ probability mass of outcome $\mathbf{x}$ when $X_i$ has heads bias $u$.

**Core of the proof:** Consider any outcome $\mathbf{x} \in \mathcal{E}$ with, say, $k \geq n \cdot u$ heads:
Theorem: For $S \sim \text{Bin}(n, p)$, $\mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p))$ for $u > p$.

Consider $n$ iid Bernoulli random variables: $X_1, X_2, \ldots, X_n$. Let $\mathcal{E} \subseteq \{0, 1\}^n$ be all outcomes $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^{n} x_i \geq n \cdot u$.

Some shorthand notation:

- $p[\mathbf{x}] := \text{probability mass of outcome } \mathbf{x} \text{ when } X_i \text{ has heads bias } p$.
- $u[\mathbf{x}] := \text{probability mass of outcome } \mathbf{x} \text{ when } X_i \text{ has heads bias } u$.

Core of the proof: Consider any outcome $\mathbf{x} \in \mathcal{E}$ with, say, $k \geq n \cdot u$ heads:

\[
\frac{p[\mathbf{x}]}{u[\mathbf{x}]} \leq \exp(-n \cdot \text{RE}(u||p)).
\]
Proof of Chernoff bound (upper tail bound)

**Theorem**: For $S \sim \text{Bin}(n, p)$, $\Pr(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u \parallel p))$ for $u > p$.

Consider $n$ iid Bernoulli random variables: $X_1, X_2, \ldots, X_n$. Let $\mathcal{E} \subseteq \{0, 1\}^n$ be all outcomes $x = (x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^{n} x_i \geq n \cdot u$.

**Some shorthand notation**:
- $p[x] :=$ probability mass of outcome $x$ when $X_i$ has heads bias $p$.
- $u[x] :=$ probability mass of outcome $x$ when $X_i$ has heads bias $u$.

**Core of the proof**: Consider any outcome $x \in \mathcal{E}$ with, say, $k \geq n \cdot u$ heads:

$$
\frac{p[x]}{u[x]} = \frac{p^k(1-p)^{n-k}}{u^k(1-u)^{n-k}}
$$
**Proof of Chernoff bound (Upper tail bound)**

**Theorem:** For $S \sim \text{Bin}(n, p)$, $\mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p))$ for $u > p$.

Consider $n$ iid Bernoulli random variables: $X_1, X_2, \ldots, X_n$. Let $E \subseteq \{0, 1\}^n$ be all outcomes $x = (x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^{n} x_i \geq n \cdot u$.

**Some shorthand notation:**

- $p[x] :=$ probability mass of outcome $x$ when $X_i$ has heads bias $p$.
- $u[x] :=$ probability mass of outcome $x$ when $X_i$ has heads bias $u$.

**Core of the proof:** Consider any outcome $x \in E$ with, say, $k \geq n \cdot u$ heads:

$$\frac{p[x]}{u[x]} = \frac{p^k (1-p)^{n-k}}{u^k (1-u)^{n-k}} = \left(\frac{p}{u}\right)^k \left(\frac{1-p}{1-u}\right)^{n-k}$$
**Theorem:** For \( S \sim \text{Bin}(n, p) \), \( \mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u\|p)) \) for \( u > p \).

Consider \( n \) iid Bernoulli random variables: \( X_1, X_2, \ldots, X_n \). Let \( \mathcal{E} \subseteq \{0, 1\}^n \) be all outcomes \( x = (x_1, x_2, \ldots, x_n) \) where \( \sum_{i=1}^{n} x_i \geq n \cdot u \).

**Some shorthand notation:**

- \( p[x] := \) probability mass of outcome \( x \) when \( X_i \) has heads bias \( p \).
- \( u[x] := \) probability mass of outcome \( x \) when \( X_i \) has heads bias \( u \).

**Core of the proof:** Consider any outcome \( x \in \mathcal{E} \) with, say, \( k \geq n \cdot u \) heads:

\[
\frac{p[x]}{u[x]} = \frac{p^k (1-p)^{n-k}}{u^k (1-u)^{n-k}} = \left( \frac{p}{u} \right)^k \left( \frac{1-p}{1-u} \right)^{n-k} \leq \left( \frac{p}{u} \right)^{n \cdot u} \left( \frac{1-p}{1-u} \right)^{n \cdot (1-u)} .
\]
**Proof of Chernoff Bound (Upper Tail Bound)**

**Theorem:** For $S \sim \text{Bin}(n, p)$, $\mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p))$ for $u > p$.

Consider $n$ iid Bernoulli random variables: $X_1, X_2, \ldots, X_n$. Let $\mathcal{E} \subseteq \{0, 1\}^n$ be all outcomes $x = (x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^n x_i \geq n \cdot u$.

**Some shorthand notation:**

- $p[x] :=$ probability mass of outcome $x$ when $X_i$ has heads bias $p$.
- $u[x] :=$ probability mass of outcome $x$ when $X_i$ has heads bias $u$.

**Core of the proof:** Consider any outcome $x \in \mathcal{E}$ with, say, $k \geq n \cdot u$ heads:

$$\frac{p[x]}{u[x]} = \frac{p^k (1 - p)^{n-k}}{u^k (1 - u)^{n-k}} = \left(\frac{p}{u}\right)^k \left(\frac{1 - p}{1 - u}\right)^{n-k} \leq \left(\frac{p}{u}\right)^{n \cdot u} \left(\frac{1 - p}{1 - u}\right)^{n \cdot (1 - u)}.$$

$$\mathbb{P}(S \geq n \cdot u) = \sum_{x \in \mathcal{E}} p[x]$$
**Proof of Chernoff bound (upper tail bound)**

**Theorem:** For \( S \sim \text{Bin}(n, p) \), \( \mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p)) \) for \( u > p \).

Consider \( n \) iid Bernoulli random variables: \( X_1, X_2, \ldots, X_n \).

Let \( E \subseteq \{0, 1\}^n \) be all outcomes \( x = (x_1, x_2, \ldots, x_n) \) where \( \sum_{i=1}^n x_i \geq n \cdot u \).

**Some shorthand notation:**
- \( p[x] := \text{probability mass of outcome } x \text{ when } X_i \text{ has heads bias } p \).
- \( u[x] := \text{probability mass of outcome } x \text{ when } X_i \text{ has heads bias } u \).

**Core of the proof:** Consider any outcome \( x \in E \) with, say, \( k \geq n \cdot u \) heads:

\[
\frac{p[x]}{u[x]} = \frac{p^k(1-p)^{n-k}}{u^k(1-u)^{n-k}} = \left( \frac{p}{u} \right)^k \left( \frac{1-p}{1-u} \right)^{n-k} \leq \left( \frac{p}{u} \right)^{n\cdot u} \left( \frac{1-p}{1-u} \right)^{n\cdot (1-u)}.
\]

\[
\mathbb{P}(S \geq n \cdot u) = \sum_{x \in E} p[x] \leq \sum_{x \in E} u[x] \left( \frac{p}{u} \right)^{n\cdot u} \left( \frac{1-p}{1-u} \right)^{n\cdot (1-u)}.
\]
**Proof of Chernoff bound (upper tail bound)**

**Theorem**: For $S \sim \text{Bin}(n, p)$, $\mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u \| p))$ for $u > p$.

Consider $n$ iid Bernoulli random variables: $X_1, X_2, \ldots, X_n$.
Let $\mathcal{E} \subseteq \{0, 1\}^n$ be all outcomes $x = (x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^n x_i \geq n \cdot u$.

Some shorthand notation:
- $p[x] :=$ probability mass of outcome $x$ when $X_i$ has heads bias $p$.
- $u[x] :=$ probability mass of outcome $x$ when $X_i$ has heads bias $u$.

**Core of the proof**: Consider any outcome $x \in \mathcal{E}$ with, say, $k \geq n \cdot u$ heads:

$$
\frac{p[x]}{u[x]} = \frac{p^k(1-p)^{n-k}}{u^k(1-u)^{n-k}} = \left(\frac{p}{u}\right)^k \left(\frac{1-p}{1-u}\right)^{n-k} \leq \left(\frac{p}{u}\right)^{n \cdot u} \left(\frac{1-p}{1-u}\right)^{n \cdot (1-u)}.
$$

$$
\mathbb{P}(S \geq n \cdot u) = \sum_{x \in \mathcal{E}} p[x] \leq \sum_{x \in \mathcal{E}} u[x] \left(\frac{p}{u}\right)^{n \cdot u} \left(\frac{1-p}{1-u}\right)^{n \cdot (1-u)}
$$

$$
\leq \left(\frac{p}{u}\right)^{n \cdot u} \left(\frac{1-p}{1-u}\right)^{n \cdot (1-u)}.
$$
**Proof of Chernoff bound (upper tail bound)**

**Theorem:** For $S \sim \text{Bin}(n, p)$, $\mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p))$ for $u > p$.

Consider $n$ iid Bernoulli random variables: $X_1, X_2, \ldots, X_n$. Let $\mathcal{E} \subseteq \{0, 1\}^n$ be all outcomes $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where $\sum_{i=1}^n x_i \geq n \cdot u$.

**Some shorthand notation:**
- $p[\mathbf{x}] :=$ probability mass of outcome $\mathbf{x}$ when $X_i$ has heads bias $p$.
- $u[\mathbf{x}] :=$ probability mass of outcome $\mathbf{x}$ when $X_i$ has heads bias $u$.

**Core of the proof:** Consider any outcome $\mathbf{x} \in \mathcal{E}$ with, say, $k \geq n \cdot u$ heads:

\[
\frac{p[\mathbf{x}]}{u[\mathbf{x}]} = \frac{p^k (1 - p)^{n-k}}{u^k (1 - u)^{n-k}} = \left( \frac{p}{u} \right)^k \left( \frac{1 - p}{1 - u} \right)^{n-k} \leq \left( \frac{p}{u} \right)^{n \cdot u} \left( \frac{1 - p}{1 - u} \right)^{n \cdot (1-u)}.
\]

\[
\mathbb{P}(S \geq n \cdot u) = \sum_{\mathbf{x} \in \mathcal{E}} p[\mathbf{x}] \leq \sum_{\mathbf{x} \in \mathcal{E}} u[\mathbf{x}] \left( \frac{p}{u} \right)^{n \cdot u} \left( \frac{1 - p}{1 - u} \right)^{n \cdot (1-u)} 
\leq \left( \frac{p}{u} \right)^{n \cdot u} \left( \frac{1 - p}{1 - u} \right)^{n \cdot (1-u)} = \exp(-n \cdot \text{RE}(u||p)).
\]
Moderate deviations

How large are “typical” deviations?
How large are “typical” deviations?

“Fact”: $S \sim \text{Bin}(n, p)$ “typically” in $[np - 2\sqrt{np(1-p)}, np + 2\sqrt{np(1-p)}]$. 
How large are “typical” deviations?

“Fact”: $S \sim \text{Bin}(n, p)$ “typically” in $\left[ np - 2\sqrt{np(1-p)}, np + 2\sqrt{np(1-p)} \right]$. 

$\text{Bin}(10, 1/3)$

$np \approx 3.333, \ 2\sqrt{np(1-p)} \approx 2.9814$
How large are “typical” deviations?

“Fact”: $S \sim \text{Bin}(n, p)$ “typically” in $\left[ np - 2\sqrt{np(1-p)}, np + 2\sqrt{np(1-p)} \right]$. 

$np \approx 33.333, \ 2\sqrt{np(1-p)} \approx 9.4281$
How large are “typical” deviations?

**“Fact”**: $S \sim \text{Bin}(n, p)$ “typically” in $\left[ np - 2\sqrt{np(1 - p)}, np + 2\sqrt{np(1 - p)} \right]$.

Bin(1000, 1/3)

$np \approx 333.333, \ 2\sqrt{np(1 - p)} \approx 29.8142$
To rigorously quantify moderate deviations, can again use Chernoff bound

$$
P(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p)),
$$

but ask how small can $u$ be before the bound exceeds some fixed $\delta \in (0, 1)$?
To rigorously quantify moderate deviations, can again use Chernoff bound

$$\mathbb{P}(S \geq n \cdot u) \leq \exp\left(-n \cdot \text{RE}(u\|p)\right),$$

but ask how small can $u$ be before the bound exceeds some fixed $\delta \in (0, 1)$?

By calculus, for $u > p$,

$$\text{RE}(u\|p) \geq \frac{(u - p)^2}{2u}.$$ 

Therefore, for $u > p$,

$$\mathbb{P}(S \geq n \cdot u) \leq \exp\left(-n \cdot \text{RE}(u\|p)\right) \leq \exp\left(-n \cdot \frac{(u - p)^2}{2u}\right).$$
To **rigorously quantify moderate deviations**, can again use Chernoff bound

\[ \mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p)) , \]

but ask how small can \( u \) be before the bound exceeds some fixed \( \delta \in (0, 1) \)?

By calculus, for \( u > p \),

\[ \text{RE}(u||p) \geq \frac{(u - p)^2}{2u} . \]

Therefore, for \( u > p \),

\[ \mathbb{P}(S \geq n \cdot u) \leq \exp(-n \cdot \text{RE}(u||p)) \leq \exp\left(-n \cdot \frac{(u - p)^2}{2u}\right) . \]

By algebra, the RHS is \( \leq \delta \) when

\[ n \cdot u = n \cdot p + \sqrt{2np \ln(1/\delta)} + 2 \ln(1/\delta) = n \cdot p + O(\sqrt{n}) . \]
Similar argument for lower tail.
Similar argument for lower tail.

By calculus, for $\ell < p \leq 1/2$,

$$\text{RE}(\ell||p) \geq \frac{(p - \ell)^2}{2p}. $$

Therefore, for $\ell < p \leq 1/2$,

$$\mathbb{P}(S \leq n \cdot \ell) \leq \exp(-n \cdot \text{RE}(\ell||p)) \leq \exp\left(-n \cdot \frac{(p - \ell)^2}{2p}\right). $$

By algebra, the RHS is $\delta$ when

$$n \cdot \ell = n \cdot p - \sqrt{2np \ln(1/\delta)} = n \cdot p - O(\sqrt{n}).$$
Combining upper and lower tail bounds: for $p \leq 1/2$,

$$\mathbb{P}\left( S \in \left[ np - \sqrt{2np \ln(1/\delta)}, \, np + \sqrt{2np \ln(1/\delta)} + 2 \ln(1/\delta) \right] \right) \geq 1 - 2\delta .$$

**Union bound:** $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$
Another interpretation: estimating heads bias \( p \leq 1/2 \) from iid sample \( X_1, X_2, \ldots, X_n \) with

\[
\hat{p} := \frac{X_1 + X_2 + \cdots + X_n}{n}.
\]

With probability at least \( 1 - 2\delta \),

\[
p - \sqrt{\frac{2p \ln(1/\delta)}{n}} \leq \hat{p} \leq p + \sqrt{\frac{2p \ln(1/\delta)}{n}} + \frac{2 \ln(1/\delta)}{n};
\]

i.e., the estimate \( \hat{p} \) is usually reasonably close to the truth \( p \).
Another interpretation: estimating heads bias $p \leq 1/2$ from iid sample $X_1, X_2, \ldots, X_n$ with

$$\hat{p} := \frac{X_1 + X_2 + \cdots + X_n}{n}.$$ 

With probability at least $1 - 2\delta$,

$$p - \sqrt{\frac{2p \ln(1/\delta)}{n}} \leq \hat{p} \leq p + \sqrt{\frac{2p \ln(1/\delta)}{n}} + \frac{2 \ln(1/\delta)}{n};$$

i.e., the estimate $\hat{p}$ is usually reasonably close to the truth $p$.

How close? Depends on:

- whether you’re asking about how far above $p$ or how far below $p$ (upper and lower tails are somewhat asymmetric);
- the sample size $n$;
- the true heads bias $p$ itself;
- the “confidence” parameter $\delta$. 
Let \( \hat{f} : \mathcal{X} \rightarrow \mathcal{Y} \) be a classifier, and suppose you have iid test data \( T \) (that are independent of \( \hat{f} \)); let \( n := |T| \).
Let \( \hat{f}: \mathcal{X} \to \mathcal{Y} \) be a classifier, and suppose you have iid test data \( T \) (that are independent of \( \hat{f} \)); let \( n := |T| \).

**True error rate** (with \((X, Y) \sim \mathbb{P})\):

\[
\text{err}(\hat{f}) = \mathbb{P}(\hat{f}(X) \neq Y).
\]

**Test error rate**:

\[
\text{err}(\hat{f}, T) = \frac{1}{n} \sum_{(x, y) \in T} 1\{\hat{f}(x) \neq y\}.
\]

**Distribution of test error rate**:

\[
n \cdot \text{err}(\hat{f}, T) \sim \text{Bin}(n, \text{err}(\hat{f})).
\]
APPLICATION: TEST ERROR RATE

Let $\hat{f}: \mathcal{X} \to \mathcal{Y}$ be a classifier, and suppose you have iid test data $T$ (that are independent of $\hat{f}$); let $n := |T|$.

**True error rate** (with $(X, Y) \sim \mathbb{P}$):

$$\text{err}(\hat{f}) = \mathbb{P}(\hat{f}(X) \neq Y).$$

**Test error rate**:

$$\text{err}(\hat{f}, T) = \frac{1}{n} \sum_{(x, y) \in T} 1\{\hat{f}(x) \neq y\}.$$

**Distribution of test error rate**:

$$n \cdot \text{err}(\hat{f}, T) \sim \text{Bin}(n, \text{err}(\hat{f})).$$

**Applying Chernoff bounds**: with prob. $\geq 1 - 2\delta$ (w.r.t. random draw of $T$),

$$|\text{err}(\hat{f}) - \text{err}(\hat{f}, T)| \leq \sqrt{\frac{2 \text{err}(\hat{f}) \ln(1/\delta)}{n}} + \frac{2 \ln(1/\delta)}{n}.$$
Let \( \hat{f}: \mathcal{X} \rightarrow \mathcal{Y} \) be a classifier, and suppose you have iid test data \( T \) (that are independent of \( \hat{f} \)); let \( n := |T| \).

**True error rate** (with \((X, Y) \sim P\)):

\[
\text{err}(\hat{f}) = \mathbb{P}(\hat{f}(X) \neq Y).
\]

**Test error rate:**

\[
\text{err}(\hat{f}, T) = \frac{1}{n} \sum_{(x, y) \in T} 1\{\hat{f}(x) \neq y\}.
\]

**Distribution of test error rate:**

\[
n \cdot \text{err}(\hat{f}, T) \sim \text{Bin}(n, \text{err}(\hat{f})).
\]

**Applying Chernoff bounds:** with prob. \( \geq 1 - 2\delta \) (w.r.t. random draw of \( T \)),

\[
\left| \text{err}(\hat{f}) - \text{err}(\hat{f}, T) \right| \leq \sqrt{\frac{2 \text{err}(\hat{f}) \ln(1/\delta)}{n}} + \frac{2 \ln(1/\delta)}{n}.
\]

Suggests (very) **rough idea** of the resolution at which you can distinguish classifiers’ error rates, based on size of test set.
Application: confidence intervals

(Estimate of heads bias with \( \hat{p} = (X_1 + \cdots + X_n)/n \).)

With probability at least \( 1 - 2\delta \),

\[
p \in \left[ \hat{p} - \sqrt{\frac{2p \ln(1/\delta)}{n}} - \frac{2 \ln(1/\delta)}{n}, \hat{p} + \sqrt{\frac{2p \ln(1/\delta)}{n}} \right].
\]
Application: confidence intervals

(Estimate of heads bias with $\hat{p} = (X_1 + \cdots + X_n)/n$.)

With probability at least $1 - 2\delta$,

$$p \in \left[ \hat{p} - \sqrt{\frac{2p \ln(1/\delta)}{n}} - \frac{2 \ln(1/\delta)}{n}, \hat{p} + \sqrt{\frac{2p \ln(1/\delta)}{n}} \right].$$

Unfortunately interval also depends on $p$. 
Application: confidence intervals

(Estimate of heads bias with \( \hat{p} = (X_1 + \cdots + X_n)/n \).)

With probability at least \( 1 - 2\delta \),

\[
p \in \left[ \hat{p} - \sqrt{\frac{2p \ln(1/\delta)}{n}} - \frac{2 \ln(1/\delta)}{n}, \hat{p} + \sqrt{\frac{2p \ln(1/\delta)}{n}} \right].
\]

Unfortunately interval also depends on \( p \).

Fix: can “solve” for the largest value of \( q \in [0, 1] \) such that

\[
q \leq \hat{p} + \sqrt{\frac{2q \ln(1/\delta)}{n}}
\]

\( \rightarrow \) Upper limit of confidence interval. (Can similarly get lower limit.)
Recap

- Sums of iid Bernoulli random variables:
  - Deviations from mean of size $\Omega(n)$ are exponentially unlikely.
  - Bulk (e.g., 95%) of probability mass is within $O(\sqrt{n})$ of mean.
  - Applies in many other cases besides sums of iid Bernoulli.

- Tool: Chernoff bound
  - Reason about error rates.
  - Construct confidence intervals.