Homework 0

Problem 1. Let $A$ be the $6 \times 6$ matrix defined as follows (by a product of three $6 \times 6$ matrices):

$$A := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}.$$

(a) Let $R$ be the span of $A$'s rows. What is the dimension of $R$?

(b) True or false: \[ \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
1 \\
2 \\
\end{bmatrix} \] is an eigenvector of $A$.

(Oops—the entry in the Google Form for this part is “Problem 2(b)”, even though we really mean “Problem 1(b)”.)

(c) True or false: Every eigenvector of $A$ corresponding to eigenvalue $1/2$ has the form \[ \begin{bmatrix}
c \\
c \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \text{ or } \begin{bmatrix}
c \\
-c \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \] for some real number $c$.

(d) Let $V$ be the subspace spanned by eigenvectors of $A$ corresponding to the eigenvalue $1/4$. What is the dimension of $V$?

(e) What is the largest eigenvalue of $A^3$?

Problem 2. Let $X$ and $Y$ be independent real-valued random variables defined on the same probability space, each with variance equal to 1. Define a third random variable $Z := (1 + X + Y)/2$. What is the variance of $Z$?

Problem 3. Suppose a permutation $\pi$ is picked uniformly at random from the space of all permutations on a set $T$ with cardinality $|T| = 10$. (Think of $\pi$ as a function that maps $T$ to itself.) Let $X$ be the number of fixed points of $\pi$—i.e., the number of $t \in T$ such that $\pi(t) = t$. What is the expected value of $X^2$?