Margins

Let $S$ be a collection of labeled examples from $\mathbb{R}^d \times \{-1,+1\}$. We say $S$ is linearly separable if there exists $w \in \mathbb{R}^d$ such that
\[
\min_{(x,y) \in S} y\langle w, x \rangle > 0,
\]
and we call $w$ a linear separator for $S$.

The (minimum) margin of a linear separator $w$ for $S$ is the minimum distance from $x$ to the hyperplane orthogonal to $w$, among all $(x,y) \in S$. Note that this notion of margin is invariant to positive scaling of $w$. If we rescale $w$ so that
\[
\min_{(x,y) \in S} y\langle w, x \rangle = 1,
\]
then this minimum distance is $1/\|w\|_2$. Therefore, the linear separator with the largest minimum margin is described by the following mathematical optimization problem:
\[
\min_{w \in \mathbb{R}^d} \frac{1}{2} \|w\|_2^2 \\
\text{s.t. } y\langle w, x \rangle \geq 1, \quad (x,y) \in S.
\]

Perceptron algorithm

The Perceptron algorithm is given as follows. The input to the algorithm is a collection $S$ of labeled examples from $\mathbb{R}^d \times \{-1,+1\}$.

- Begin with $\hat{w}_1 := 0 \in \mathbb{R}^d$.
- For $t = 1, 2, \ldots$
  - If there is a labeled example in $S$ (call it $(x_t, y_t)$) such that $y_t\langle \hat{w}_t, x_t \rangle \leq 0$, then set $\hat{w}_{t+1} := \hat{w}_t + y_t x_t$.
  - Else, return $\hat{w}_t$.

**Theorem.** Let $S$ be a collection of labeled examples from $\mathbb{R}^d \times \{-1,+1\}$. Suppose there exists a vector $w_* \in \mathbb{R}^d$ such that
\[
\min_{(x,y) \in S} y\langle w_*, x \rangle = 1.
\]
Then Perceptron on input $S$ halts after at most $\|w_*\|_2^2 L^2$ loop iterations, where $L := \max_{(x,y) \in S} \|x\|_2$.

**Proof.** Suppose Perceptron does not exit the loop in the $t$-th iteration. Then there is a labeled example $(x_t, y_t) \in S$ such that
\[
y_t\langle w_*, x_t \rangle \geq 1, \quad y_t\langle \hat{w}_t, x_t \rangle \leq 0.
\]
We bound $\langle w_*, \hat{w}_{t+1} \rangle$ from above and below to deduce a bound on the number of loop iterations. First, we bound $\langle w_*, \hat{w}_t \rangle$ from below:
\[
\langle w_*, \hat{w}_{t+1} \rangle = \langle w_*, \hat{w}_t \rangle + y_t \langle w_*, x_t \rangle \geq \langle w_*, \hat{w}_t \rangle + 1.
\]
Since \( \hat{w}_1 = 0 \), we have
\[
\langle w_*, \hat{w}_t \rangle \geq t.
\]
We now bound \( \langle w_*, \hat{w}_{t+1} \rangle \) from above. By Cauchy-Schwarz,
\[
\langle w_*, \hat{w}_{t+1} \rangle \leq \|w_*\|_2 \|\hat{w}_{t+1}\|_2.
\]
Also,
\[
\|\hat{w}_{t+1}\|_2^2 = \|\hat{w}_t\|_2^2 + 2y_t\langle \hat{w}_t, x_t \rangle + y_t^2 \|x_t\|_2^2 \leq \|\hat{w}_t\|_2^2 + L^2.
\]
Since \( \|\hat{w}_1\|_2 = 0 \), we have
\[
\|\hat{w}_{t+1}\|_2^2 \leq L^2 t,
\]
so
\[
\langle w_*, \hat{w}_{t+1} \rangle \leq \|w_*\|_2 L \sqrt{t}.
\]
Combining the upper and lower bounds on \( \langle w_*, \hat{w}_{t+1} \rangle \) shows that
\[
t \leq \langle w_*, \hat{w}_{t+1} \rangle \leq \|w_*\|_2 L \sqrt{t},
\]
which in turn implies the inequality \( t \leq \|w_*\|_2^2 L^2 \).

**Online Perceptron algorithm**

The Online Perceptron algorithm is given as follows. The input to the algorithm is a sequence \((x_1, y_1), (x_2, y_2), \ldots\) of labeled examples from \( \mathbb{R}^d \times \{-1, +1\} \).

- Begin with \( \hat{w}_1 := 0 \in \mathbb{R}^d \).
- For \( t = 1, 2, \ldots \):
  - If \( y_t \langle \hat{w}_t, x_t \rangle \leq 0 \), then set \( \hat{w}_{t+1} := \hat{w}_t + y_t x_t \).
  - Else, \( \hat{w}_{t+1} := \hat{w}_t \).

We say that Online Perceptron makes a **mistake** in round \( t \) if \( y_t \langle \hat{w}_t, x_t \rangle \leq 0 \).

**Theorem.** Let \((x_1, y_1), (x_2, y_2), \ldots\) be a sequence of labeled examples from \( \mathbb{R}^d \times \{-1, +1\} \) such that there exists a vector \( w_* \in \mathbb{R}^d \) satisfying
\[
\min_{t=1,2,\ldots} y_t \langle w_*, x_t \rangle = 1.
\]
Then Online Perceptron on input \((x_1, y_1), (x_2, y_2), \ldots\) makes at most \( \|w_*\|_2^2 L^2 \) mistakes, where \( L := \max_{t=1,2,\ldots} \|x_t\|_2 \).

**Proof.** The proof of this theorem is essentially the same as the proof of the iteration bound for Perceptron.

Online Perceptron may be applied to a collection of labeled examples \( S \) by considering the labeled examples in \( S \) in any (e.g., random) order. If \( S \) is linearly separable, then the number of mistakes made by Online Perceptron can be bounded using the theorem.

However, Online Perceptron is also useful when \( S \) is not linearly separable. This is especially notable in comparison to Perceptron, which never terminates if \( S \) is not linearly separable.

**Theorem.** Let \((x_1, y_1), (x_2, y_2), \ldots\) be a sequence of labeled examples from \( \mathbb{R}^d \times \{-1, +1\} \). Online Perceptron on input \((x_1, y_1), (x_2, y_2), \ldots\) makes at most
\[
\min_{w_* \in \mathbb{R}^d} \left[ \|w_*\|_2^2 L^2 + \|w_*\|_2 L \left( \sum_{t \in M} \ell(\langle w_*, x_t \rangle, y_t) + \sum_{t \in M} \ell(\langle w_*, x_t \rangle, y_t) \right) \right]
\]
mistakes, where \( L := \max_{t=1,2,\ldots} \|x_t\|_2 \), \( M \) is the set of rounds on which Online Perceptron makes a mistake, and \( \ell(\hat{y}, y) := [1 - \hat{y}y]_+ = \max\{0, 1 - \hat{y}y\} \) is the hinge loss of \( \hat{y} \) when \( y \) is the correct label.
Proof. Fix any $w_* \in \mathbb{R}^d$. Consider any round $t$ in which Online Perceptron makes a mistake. Let $\mathcal{M}_t := \{1, \ldots, t\} \cap \mathcal{M}$ and $M_t := |\mathcal{M}_t|$. We will bound $\langle w_*, \hat{w}_{t+1} \rangle$ from above and below to deduce a bound on $M_t$, the number of mistakes made by Online Perceptron through the first $t$ rounds. First we bound $\langle w_*, \hat{w}_{t+1} \rangle$ from above. By Cauchy-Schwarz,

$$\langle w_*, \hat{w}_{t+1} \rangle \leq \|w_*\|_2 \|\hat{w}_{t+1}\|_2.$$ 

Moreover,

$$\|\hat{w}_{t+1}\|_2^2 = \|\hat{w}_t\|_2^2 + 2y_t\langle \hat{w}_t, x_t \rangle + y_t^2 \|x_t\|_2^2 \leq \|\hat{w}_t\|_2^2 + L^2.$$ 

Since $\hat{w}_1 = 0$, we have

$$\|\hat{w}_{t+1}\|_2^2 \leq L^2M_t,$$

and thus

$$\langle w_*, \hat{w}_{t+1} \rangle \leq \|w_*\|_2 L \sqrt{M_t}.$$ 

We now bound $\langle w_*, w_{t+1} \rangle$ from below:

$$\langle w_*, \hat{w}_{t+1} \rangle = \langle w_*, \hat{w}_t \rangle + 1 - [1 - y_t\langle w_*, x_t \rangle]$$
$$\geq \langle w_*, \hat{w}_t \rangle + 1 - [1 - y_t\langle w_*, x_t \rangle]_+$$
$$= \langle w_*, \hat{w}_t \rangle + 1 - \ell(\langle w_*, x_t \rangle, y_t).$$

Since $\hat{w}_1 = 0$,$$

\langle w_*, \hat{w}_{t+1} \rangle \geq M_t - H_t,$$

where

$$H_t := \sum_{t \in \mathcal{M}_t} \ell(\langle w_*, x_t \rangle, y_t).$$

Combining the upper and lower bounds on $\langle w_*, \hat{w}_{t+1} \rangle$ shows that

$$M_t - H_t \leq \langle w_*, \hat{w}_{t+1} \rangle \leq \|w_*\|_2 L \sqrt{M_t},$$

i.e.,

$$M_t - \|w_*\|_2 L \sqrt{M_t} - H_t \leq 0.$$ 

This inequality is quadratic in $\sqrt{M_t}$. By solving it\(^1\), we deduce the bound

$$M_t \leq \frac{1}{2} \|w_*\|_2^2 L^2 + \frac{1}{2} \|w_*\|_2 L \sqrt{\|w_*\|_2^2 L^2 + 4H_t + H_t},$$

which can be further loosened to the following (slightly more interpretable) bound:

$$M_t \leq \|w_*\|_2^2 L^2 + \|w_*\|_2 L \sqrt{H_t + H_t}.$$ 

The claim follows.

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\(^1\)The inequality is of the form $x^2 - bx - c \leq 0$ for some non-negative $b$ and $c$. This implies that $x \leq (b + \sqrt{b^2 + 4c})/2$, and hence $x^2 \leq (b^2 + 2b\sqrt{b^2 + 4c} + b^2 + 4c)/4$. We can then use the fact that $\sqrt{A + B} \leq \sqrt{A} + \sqrt{B}$ for any non-negative $A$ and $B$ to deduce $x^2 \leq b^2 + b\sqrt{b^2 + 4c} + c$. 

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