## Volumes in high-dimensional space

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COMS 4772

Simple volumes

▶ In  $\mathbb{R}^1$ , line segment

 $[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$ 

has one-dimensional volume (a.k.a. *length*) b - a.

▶ In  $\mathbb{R}^2$ , square

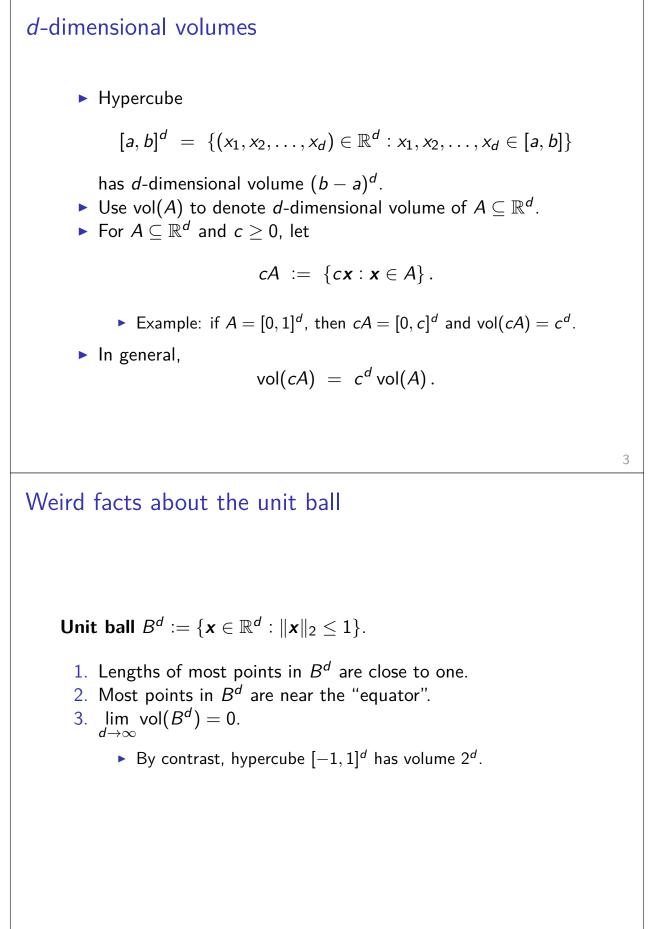
$$[a, b]^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in [a, b]\}$$

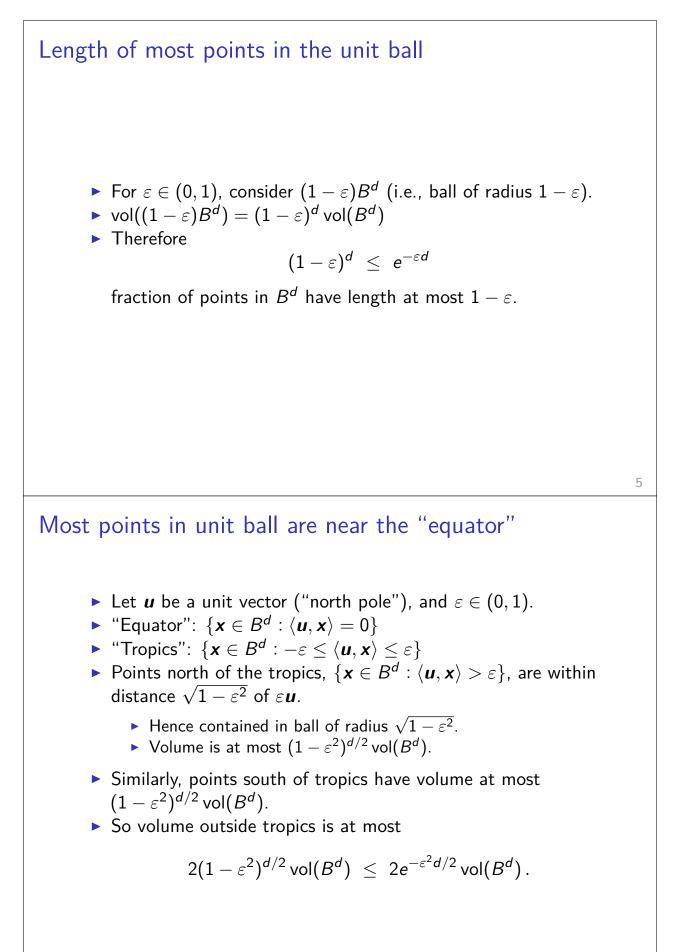
has two-dimensional volume (a.k.a. *area*)  $(b - a)^2$ . In  $\mathbb{R}^3$ , cube

 $[a,b]^3 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1,x_2,x_3 \in [a,b]\}$ 

has three-dimensional volume (a.k.a. volume)  $(b - a)^3$ .

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Volume of unit ball  
• Consider an orthonormal basis 
$$u_1, u_2, \dots, u_d$$
 of  $\mathbb{R}^d$ .  
• Let  $T_i$  be the "tropics" when  $u_i$  is the "north pole".  
• Volume of points in  $\bigcap_{i=1}^d T_i$  is  
 $\operatorname{vol}\left(\bigcap_{i=1}^d T_i\right) \ge \operatorname{vol}(B^d) - \sum_{i=1}^d \operatorname{vol}(T_i^c) \ge (1 - 2de^{-\varepsilon^2 d/2}) \operatorname{vol}(B^d)$ .  
• But  $\operatorname{vol}\left(\bigcap_{i=1}^d T_i\right) = \operatorname{vol}([-\varepsilon, \varepsilon]^d) = (2\varepsilon)^d$ .  
• If  $2de^{-\varepsilon^2 d/2} \le 1$ , then  
 $\operatorname{vol}(B^d) \le \frac{(2\varepsilon)^d}{1 - 2de^{-\varepsilon^2 d/2}}$ .  
• For  $\varepsilon = \sqrt{2\ln(4d)/d}$ , bound is  
 $\operatorname{vol}(B^d) \le 2\left(\frac{8\ln(4d)}{d}\right)^{d/2} \xrightarrow{d \to \infty} 0$ .