

$$1) \quad E[\# \text{ rolls}] = 1 \cdot \left(\frac{1}{n} \cdot 1 + 1 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} \right) + (1 + E[\# \text{ rolls}]) (1-x)$$

$$[P(A \text{ and } B) = P(A) + P(B) - P(A \cup B)]$$

$$E = x + 1 - x + E - E \cdot x$$



$$E \cdot x = 1$$

$$E[\# \text{ rolls}] = 1/x = 1/\frac{1}{2n-1} = \frac{2n-1}{1} = 2n-1 \quad (\text{assuming } n \geq 10)$$

(Or, a simpler way: this is a Geometric r.v. where $P(\text{success}) = x \therefore E[\text{Geo}] = 1/x$)



$$\text{assuming } n \geq 9: P(\text{sum} = 10) = P(1,9) + P(2,8) + \dots + P(9,1) = \frac{1}{n} \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + \dots + \frac{1}{n} \cdot \frac{1}{n} = \frac{9}{n^2}$$

$$E[\# \text{ rolls}] = E[\text{a Geometric r.v. with } P(\text{success}) = 9/n^2] = 1/(9/n^2) = n^2/9$$

$$2) \quad P(\text{HHHHH} \text{ ???}) + P(\text{HHH} \text{ ???TTTT}) - P(\text{HHHHHTTTTT})$$

$$= \left(\frac{1}{2}\right)^5 \cdot 1^5 + \left(\frac{1}{2}\right)^4 1^3 \left(\frac{1}{2}\right)^5 - \left(\frac{1}{2}\right)^{10} = \frac{1}{2^5} + \frac{1}{2^7} - \frac{1}{2^{10}}$$

$P(\text{you pick larger than 2})$
 $P(\text{you pick "2"})$
 $P(\text{you pick smaller than 2, then pick larger than 2})$ } you can interpret the rules differently and get a different answer (this interpretation strictly follows what's written, but others are O.K.)

$$3) \quad P(\text{you win}) = \frac{1}{13} \cdot \left(\frac{12}{13} + \frac{0}{13} \cdot \frac{12}{13} \right) + \frac{1}{13} \cdot \left(\frac{11}{13} + \frac{1}{13} \cdot \frac{11}{13} \right) + \dots + \frac{1}{13} \cdot \left(\frac{0}{13} + \frac{12}{13} \cdot \frac{0}{13} \right)$$

$$= \frac{1}{13} \sum_{i=1}^{12} \left(\frac{13-i}{13} + \frac{i-1}{13} \cdot \frac{13-i}{13} \right)$$

$$= .59$$

$$E[\text{money in 1 game}] = 1 \cdot P(\text{you win}) + (-1) \cdot P(\text{you lose})$$

$$= .59 - (1 - .59) = .18$$

$$E[\text{money in 100 games}] \approx \$18$$

$$[E(\Sigma) = \Sigma E]$$

4) a) $\boxed{n \text{ red } m \text{ blue}}$ $P(X=0) = P(\text{you picked blue } k \text{ times}) = \left(\frac{m}{n+m}\right)^k$

b) $P(X=0) = \left(\frac{m}{n+m}\right) \left(\frac{m-1}{n+m-1}\right) \left(\frac{m-2}{n+m-2}\right) \dots \left(\frac{m-(k-1)}{n+m-(k-1)}\right) = \frac{m!}{(m-k)!} \frac{(n+m-k)!}{(n+m)!}$ (assuming enough blue and red)

c) $\sum_{i=1}^k I_i$

$$P(\text{same color}) = P(rr \text{ or } bb) = \frac{n}{n+m} \frac{n}{n+m} + \frac{m}{n+m} \frac{m}{n+m} = \frac{n^2+m^2}{(n+m)^2} = \alpha$$

Let $I_i = \begin{cases} 1 & \text{if } i, i-1 \text{ same color (with probability } \alpha) \\ 0 & \text{if not same color (1-\alpha)} \end{cases}$

$$E[I_i] = \sum x p_x(x) = 0 \cdot (1-\alpha) + 1 \cdot \alpha = \alpha$$

$$Z = \sum_{i=1}^k I_i \quad E[Z] = E[\sum I_i] = \sum E[I_i] = \sum \alpha = (k-1)\alpha = (k-1) \frac{n^2+m^2}{(n+m)^2}$$

d) Let $I_i = \begin{cases} 1 & \text{if } i, i-1 \text{ same color (with probability } \frac{n}{n+m} \frac{n-1}{n-1+m} + \frac{m}{n+m} \frac{m-1}{n+m-1} = \alpha) \\ 0 & \text{if not same color (1-\alpha)} \end{cases}$

$$E[I_i] = \alpha$$

$$W = \sum_{i=1}^k I_i \quad E[W] = E[\sum I_i] = \sum E[I_i] = (k-1)\alpha$$

not knowing what was picked before i ,
we can say that $P(i\text{th pick is red}) = \frac{n}{n+m}$
and $P(i+1\text{st pick is red} \mid i\text{th was red}) = \frac{n-1}{n-1+m}$

5) a) $X = \# \text{ on die 1}$ $Y = \# \text{ on die 2}$ $Z = (X+Y) \bmod 6 + 1$



These are pairwise independent (eg. knowing $X=3$, Z still has equal probability of being any value from 1 to 6)

but $P(X=1, Y=1, Z=1) = 0 \neq P(X=1)P(Y=1)P(Z=1) = \left(\frac{1}{6}\right)^3$

b) $P(X=l, Y=n, Z=m) = P(X=l)P(Y=n)P(Z=m)$

$[P(AB) = P(A)P(B|A)]$

$P(X=l, Y=n) \cdot P(Z=m | X=l, Y=n) = P(X=l)P(Y=n)P(Z=m)$

$P(Y=n)P(Z=m) = P(X=l)P(Y=n)P(Z=m)$

$P(X=l, Y=n) = P(X=l)P(Y=n)$

6)
$$P(F \text{ did not} | H \text{ did not}) = \frac{P(F \text{ did not}, H \text{ did not})}{P(H \text{ did not})} = \frac{1-p_1 + p_1(1-p_2)(1-p_3)}{1-p_1 + p_1(1-p_2)}$$

$$= \frac{1-p_1p_2-p_1p_3+p_1p_2p_3}{1-p_1p_2}$$