

## HW #2 Solutions

1. You work in a department store that has a Santa Claus who sits children on his lap and asks them what they want for Christmas. Parents complain that on average, they wait 50 minutes to see Santa and on average spend only 30 seconds with Santa. You are unable to determine the distribution on the time that Santa spends with children. Can you still calculate the expected length of the line? If so, how and what is it? In not, why not?

Answer:

This exercise cannot be solved with the information that was given

2. Let  $X_1, X_2, \dots, X_k$  be independent, exponentially distributed random variables with rate  $\lambda$ . Let  $Y = \min\{X_1, \dots, X_k\}$ . Show that  $Y$  is exponentially distributed. What is the rate of  $Y$ ?

Answer:

We have that:

$$P[Y > t] = P[\min\{X_1, X_2, \dots, X_k\} > t] = P[X_1 > t, X_2 > t, \dots, X_k > t]$$

but we know that  $X_1, X_2, \dots, X_k$  are independent, exponentially distributed random variables with rate  $\lambda$  so:

$$P[Y > t] = P[X_1 > t, X_2 > t, \dots, X_k > t] = P[X_1 > t] \cdot P[X_2 > t] \cdot \dots \cdot P[X_k > t] \Rightarrow$$

$$P[Y > t] = P[X_1 > t] \cdot P[X_2 > t] \cdot \dots \cdot P[X_k > t] = e^{-\lambda t} \cdot e^{-\lambda t} \cdot \dots \cdot e^{-\lambda t} \Rightarrow$$

$$P[Y > t] = e^{-k\lambda t}, \text{ therefore the rate is } k \cdot \lambda$$

3. You arrive at a bus stop at 3:15 pm. A bus arrives according to a uniform distribution between 3 and 4 pm.

(a) What is the probability that you missed the bus?

Answer:

Let  $X$  be random variable uniformly distributed on  $[0, 60]$  denoting the arrival of the bus in minutes after 3:00pm. Because  $X$  is a uniform random variable we know that:

$$X \in [a, b], P[X < t] = \frac{t - a}{b - a}$$

Then the probability that you missed when you arrive at 3:15pm is:  $P[X < 15]$ . In our case this gives us:

$$P[X < 15] = \frac{15 - 0}{60 - 0} = \frac{1}{4}$$

- (b) Somebody informs you that the bus has not yet arrived. Given this information, what is the probability that it comes within the next 15 minutes?

Answer:

For this part we will use conditional probability.

$$P[15 < X < 30 \mid X > 15] = \frac{P[15 < X < 30, X > 15]}{P[X > 15]}$$

$$\frac{P[15 < X < 30, X > 15]}{P[X > 15]} = \frac{P[15 < X < 30]}{P[X > 15]}$$

$$\frac{P[15 < X < 30]}{P[X > 15]} = \frac{\frac{15}{60}}{\frac{45}{60}} = \frac{15}{45} = \frac{1}{3}$$

4. You arrive at a bus stop at 3:15 pm. A bus arrives according to an exponential distribution with rate  $\lambda$ , starting at 3pm (it will not arrive before then).

- (a) What is the probability that you missed the bus?

Answer:

Let  $X$  be random variable exponentially distributed on  $[0, \infty]$  denoting the arrival of the bus in minutes after 3:00pm. Because  $X$  is an exponential random variable we know that:

$$X \in [a, \infty], P[X < t] = 1 - e^{-(t-a) \cdot \lambda}$$

Then the probability that you missed the bus is:

$$P[X < 15] = e^{-(15-0) \cdot \lambda} = 1 - e^{-15 \cdot \lambda}$$

- (b) Somebody informs you that the bus has not yet arrived. Given this information, what is the probability that it comes within the next 15 minutes?

Answer:

Here we can use the memoryless property of the exponential distribution. Remember that only two distributions have that property: the geometric distribution(discrete time) and the exponential distribution(continuous time). Let  $X$  be an exponential random variable. The memoryless property states that:

$$P[X > t + s | X > t] = P[X > s].$$

Proof:

$$P[X > t + s | X > t] = \frac{P[X > t + s, X > t]}{P[X > t]} = \frac{P[X > t + s]}{P[X > t]} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = \frac{e^{-\lambda t} \cdot e^{-\lambda s}}{e^{-\lambda t}} = e^{-\lambda s}$$

$$P[X > t + s | X > t] = e^{-\lambda s} = P[X > s].$$

Now we will use the memoryless property we can see that:

$$P[X \leq 30 | X > 15] = 1 - P[X > 30 | X > 15] = 1 - P[X > 15] = 1 - e^{-15 \cdot \lambda} \Rightarrow$$

$$P[X \leq 30 | X > 15] = 1 - e^{-15 \cdot \lambda}$$

- (c) Suppose there are 3 buses numbered 1,2,3 where the first bus arrives at a time after 3pm that is exponentially distributed with rate  $\lambda$ . The second bus arrives at a time after the arrival of the first bus that is exponentially distributed with rate  $\lambda$ , and the third bus arrives at a time after the second bus that is exponentially distributed with rate  $\lambda$ . What is the probability that the third bus takes more than an hour to arrive?

Answer:

Let  $X_i$  denote the arrival time of the  $i^{th}$  bus. Then the probability that the third bus will take more than an hour to arrive is  $P[X_1 + X_2 + X_3 > 1]$  (we assume that the rate  $\lambda$  is measured in  $hours^{-1}$ ). Then we have that:

$$\begin{aligned}
 P[X_1 + X_2 + X_3 > 1] &= P[X_1 > 1] + P[X_1 + X_2 > 1, X_1 \leq 1] + \\
 &\quad + P[X_1 + X_2 + X_3 > 1, X_1 + X_2 \leq 1] \\
 P[X_1 + X_2 + X_3 > 1] &= e^{-\lambda} + \int_{t_1=0}^1 P[X_2 > 1 - t_1] \cdot \lambda \cdot e^{-\lambda \cdot t_1} \cdot dt_1 + \\
 &\quad + \int_{t_1=0}^1 \int_{t_2=0}^{1-t_1} P[X_3 > 1 - t_1 - t_2] \cdot \lambda \cdot e^{-\lambda \cdot t_1} \cdot \lambda \cdot e^{-\lambda \cdot t_2} \cdot dt_2 \cdot dt_1 \quad (1)
 \end{aligned}$$

Where  $\lambda e^{-\lambda \cdot t_1}$  is  $Dens(X_1 = t_1)$ . Also we know that:

$$\int_{t_1=0}^1 P[X_2 > 1 - t_1] \cdot \lambda \cdot e^{-\lambda \cdot t_1} \cdot dt_1 = \int_{t_1=0}^1 e^{-\lambda \cdot (1-t_1)} \cdot \lambda \cdot e^{-\lambda \cdot t_1} \cdot dt_1 = \int_{t_1=0}^1 \lambda \cdot e^{-\lambda} \cdot dt_1 = \lambda \cdot e^{-\lambda} \quad (2)$$

and:

$$\begin{aligned}
 &\int_{t_1=0}^1 \int_{t_2=0}^{1-t_1} P[X_3 > 1 - t_1 - t_2] \cdot \lambda \cdot e^{-\lambda \cdot t_1} \cdot \lambda \cdot e^{-\lambda \cdot t_2} \cdot dt_2 \cdot dt_1 = \\
 &\int_{t_1=0}^1 \int_{t_2=0}^{1-t_1} e^{-\lambda \cdot (1-t_1-t_2)} \cdot \lambda \cdot e^{-\lambda \cdot t_1} \cdot \lambda \cdot e^{-\lambda \cdot t_2} \cdot dt_2 \cdot dt_1 = \\
 &\int_{t_1=0}^1 \int_{t_2=0}^{1-t_1} e^{-\lambda} \cdot e^{\lambda \cdot t_1} \cdot e^{\lambda \cdot t_2} \cdot \lambda \cdot e^{-\lambda \cdot t_1} \cdot \lambda \cdot e^{-\lambda \cdot t_2} \cdot dt_2 \cdot dt_1 = \\
 &\lambda^2 \cdot \int_{t_1=0}^1 \int_{t_2=0}^{1-t_1} e^{-\lambda} \cdot dt_2 \cdot dt_1 = \lambda^2 \cdot e^{-\lambda} \cdot \int_{t_1=0}^1 \int_{t_2=0}^{1-t_1} dt_2 \cdot dt_1 = \lambda^2 \cdot e^{-\lambda} \cdot \int_{t_1=0}^1 (1-t_1) dt_1 = \\
 &\lambda^2 \cdot e^{-\lambda} \cdot (-1) \cdot \left[ \frac{(1-t_1)^2}{2} \right]_0^1 = \frac{\lambda^2 \cdot e^{-\lambda}}{2} \quad (3)
 \end{aligned}$$

Now from (1) using (2),(3) we get:

$$P[X_1 + X_2 + X_3 > 1] = e^{-\lambda} + \lambda \cdot e^{-\lambda} + \frac{\lambda^2 \cdot e^{-\lambda}}{2}$$

5. Suppose that the arrival times of three buses numbered 1,2,3 are exponentially distributed with rates  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  (here, the any bus can come first, i.e., the clocks for all buses start at time 0).

- (a) For the case where  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ , What is the probability that it takes more than an hour for all three buses to arrive?

Answer:

Let  $X_i$  denote the arrival time of the  $i^{th}$  bus. Then the probability that it takes more than an hour for all three buses to arrive is

$$P[X_1 > 1, X_2 > 1, X_3 > 1] = e^{-\lambda} \cdot e^{-\lambda} \cdot e^{-\lambda} = e^{-3 \cdot \lambda}.$$

Another interpretation for the problem is that at least one of the  $X_1, X_2, X_3$  is greater than one hour. Here we use the complement event that all of them arrive within an hour and we get:

$$P[X_1 > 1 \cup X_2 > 1 \cup X_3 > 1] = 1 - P[X_1 \leq 1, X_2 \leq 1, X_3 \leq 1] = 1 - (1 - e^{-\lambda})^3$$

- (b) What is the probability that buses arrive in the order 1,2,3?

Answer:

There are two different ways to solve this problem:

The first solution uses the fact that we have three exponential random variables denoting the arrival times with rates  $\lambda_1, \lambda_2, \lambda_3$ . This means that we have three independent poisson process with rates  $\lambda_1, \lambda_2, \lambda_3$ . The superposition principle for the poisson process allows us to think that we have one poisson process with rate  $\lambda = \lambda_1 + \lambda_2 + \lambda_3$ . Then the probability that the process with rate  $\lambda_1$  (bus 1) will arrive first is:

$$P[X_2 > X_1, X_3 > X_1] = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} (1)$$

The remaining process has a rate  $\lambda = \lambda_2 + \lambda_3$  (this only holds because the process is memoryless). The probability that the process with  $\lambda_2$  (bus 2) will arrive before the process with rate  $\lambda_3$  (bus 3) is:

$$P[X_3 > X_2] = \frac{\lambda_2}{\lambda_2 + \lambda_3} (2)$$

Thus the probability that both independent events (1) and (2) will happen is:

$$P[X_3 > X_2 > X_1] = P[X_2 > X_1, X_3 > X_1] \cdot P[X_3 > X_2] = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \cdot \frac{\lambda_2}{\lambda_2 + \lambda_3}$$

The second solution calculates the probability using density functions. We know that :

$$\begin{aligned} P[X_3 > X_2 > X_1] &= P[X_3 > X_2, X_1 < X_2] = E[P[X_3 > X_2, X_1 < X_2 | X_2]] = \\ P[X_3 > X_2 > X_1] &= \int_{t=0}^{\infty} P[X_3 > t, X_1 < t] \cdot \lambda_2 \cdot e^{-\lambda_2 \cdot t} dt = \int_{t=0}^{\infty} P[X_3 > t] \cdot P[X_1 < t] \cdot \lambda_2 \cdot e^{-\lambda_2 \cdot t} dt \\ P[X_3 > X_2 > X_1] &= \int_{t=0}^{\infty} e^{-\lambda_3 \cdot t} \cdot (1 - e^{-\lambda_1 \cdot t}) \cdot \lambda_2 \cdot e^{-\lambda_2 \cdot t} dt = \lambda_2 \cdot \int_{t=0}^{\infty} e^{-(\lambda_3 + \lambda_2) \cdot t} - e^{-(\lambda_3 + \lambda_2 + \lambda_1) \cdot t} dt \\ P[X_3 > X_2 > X_1] &= \lambda_2 \left[ \frac{1}{\lambda_2 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} \right] \\ P[X_3 > X_2 > X_1] &= \frac{\lambda_1 \cdot \lambda_2}{(\lambda_1 + \lambda_2 + \lambda_3) \cdot (\lambda_2 + \lambda_3)} \end{aligned}$$

- (c) What is the probability that bus 2 arrives more than time  $t$  after bus 1, given that bus 2 arrives after bus 1?

Answer:

The probability that we have to compute is:  $P[X_2 > X_1 + t \mid X_2 > X_1]$  and by the memoryless property of the exponential distribution we obtain:

$$P[X_2 > X_1 + t \mid X_2 > X_1] = P[X_2 > t] = e^{-\lambda_2 \cdot t}$$

- (d) What is the probability that bus 2 takes more than time  $t$  to arrive given that bus 1 comes after bus 2?

Answer:

The probability that bus 2 takes more than time  $t$  to arrive given that bus 1 comes after bus 2 is:

$$P[X_2 > t \mid X_1 > X_2] = \frac{P[X_2 > t, X_1 > X_2]}{P[X_1 > X_2]} \quad (1)$$

Consider the aggregate process with rate  $\lambda_1 + \lambda_2$ , you need to consider the probability that the first arrival of this aggregate comes after time  $t$  and the probability that this first arrival was from process 2. then we get:

$$P[X_1 + X_2 > t, X_1 > X_2] = P[X_1 + X_2 > t] \cdot P[X_1 > X_2] = e^{-(\lambda_1 + \lambda_2) \cdot t} \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

Another way to compute the numerator using the density function:

$$P[X_2 > t, X_1 > X_2] = \int_{x=0}^{\infty} P[x > t, X_1 > x] \cdot \lambda_2 \cdot e^{-\lambda_2 \cdot x} dx = \lambda_2 \cdot \int_{x=0}^{\infty} P[x > t] \cdot P[X_1 > x] \cdot \lambda_2 \cdot e^{-\lambda_2 \cdot x} dx$$

$$P[X_2 > t, X_1 > X_2] = \lambda_2 \cdot \int_{x=t}^{\infty} e^{-\lambda_1 \cdot x} \cdot e^{-\lambda_2 \cdot x} dx = \lambda_2 \cdot \int_{x=t}^{\infty} e^{-(\lambda_1 + \lambda_2) \cdot x} dx = \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot e^{-(\lambda_1 + \lambda_2) \cdot t}$$

$$P[X_2 > t, X_1 > X_2] = \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot e^{-(\lambda_1 + \lambda_2) \cdot t} \quad (2)$$

Also :

$$P[X_1 > X_2] = \frac{\lambda_2}{\lambda_1 + \lambda_2} \quad (3)$$

Now (1) using (2), (3) becomes:

$$P[X_2 > t \mid X_1 > X_2] = e^{-(\lambda_1 + \lambda_2) \cdot t}$$

## 6. Consider a 2-D parity check code.

- (a) Prove that any combination of 1, 2, or 3 bit errors is detectable.

Answer:

In order to have an error that is undetectable we have to have even ( $> 0$ ) number of errors in both rows and columns. So in order to have an error we have to have at least four bit errors. When we have 1, 2, or 3 bit errors we can have only either one column or one row with even number of bit errors.

- (b) For a 16-bit data word (i.e., code bits are extra bits), show a 4 bit error combination that cannot be detected, and show one that can.

Answer:

We denote with X a bit error. The following 16 bit data word we show a 4 bit error combination that cannot be detected (other examples are also valid):

X	X	1	1	1
X	X	0	0	1
1	1	1	1	0
1	0	0	0	1
1	0	0	0	1

Also a 4 bit error combination that can be detected, X denotes a bit error (other examples are also valid):

X	0	1	1	1
0	X	0	0	1
1	1	X	1	0
1	0	0	X	1
1	0	0	0	1

- (c) For a data word with  $k^2$  bits, given that bit flips are a Bernoulli process (independent) with probability  $p$ , compute the probability that 4 bit-errors occur and that the 2-D parity check fails to detect that the word is corrupted.

Answer:

We will have  $(k+1)^2$  bits in total (data+error code). The valid error combinations are the ones that give an even number of errors both in columns and rows. We denote with D a random variable that is 1 when an error is detected, 0 otherwise. Because we only have 4 bit-errors we can only have 2 errors in columns and 2 errors in rows. The total number of ways that we can pick two rows out of  $k+1$  is:

$$\binom{k+1}{2}$$

and the same holds for the  $k+1$  columns. We know also that the probability of an error is  $p$ . Therefore the probability that the parity check will fail to detect the error is:

$$P[D=0] = \binom{k+1}{2} \binom{k+1}{2} p^4 (1-p)^{(k+1)^2-4}$$

1	0	1	1	1
0	1	0	0	1
1	1	1	1	0
1	0	1	0	1
1	0	0	0	1

- (d) Fix the bit error in the above codeword.

Answer:

The error is in position (4,3), the one should become zero (see below):

$$\begin{array}{cccc|c}
 1 & 0 & 1 & 1 & 1 \\
 0 & 1 & 0 & 0 & 1 \\
 1 & 1 & 1 & 1 & 0 \\
 1 & 0 & \mathbf{0} & 0 & 1 \\
 \hline
 1 & 0 & 0 & 0 & 1
 \end{array}$$

7. Consider the (7,4) Linear code whose code bits are generated as follows:

- $c_1 = b_1 \oplus b_3 \oplus b_4$
- $c_2 = b_1 \oplus b_2 \oplus b_4$
- $c_3 = b_2 \oplus b_3 \oplus b_4$

Suppose you receive the codeword 1110111 which was generated using the above linear code. What codeword was most likely transmitted?

Answer:

The check matrix is : 
$$\begin{bmatrix}
 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
 0 & 1 & 1 & 1 & 0 & 0 & 1
 \end{bmatrix}$$
 and 
$$W = \begin{bmatrix}
 1 \\
 1 \\
 1 \\
 0 \\
 1 \\
 1 \\
 1
 \end{bmatrix}$$

$$S = C \cdot W = \begin{bmatrix}
 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
 0 & 1 & 1 & 1 & 0 & 0 & 1
 \end{bmatrix} \cdot \begin{bmatrix}
 1 \\
 1 \\
 1 \\
 \mathbf{0} \\
 1 \\
 1 \\
 1
 \end{bmatrix} = \begin{bmatrix}
 1 \\
 1 \\
 1
 \end{bmatrix}$$

We have a match with the output matrix in the fourth column so the fourth bit should be changed from zero to one.

8. The above codes work best when  $p$ , the probability of a bit being flipped, is very small. What if  $p$  were very large (e.g.,  $1 - \epsilon$  for some very small  $\epsilon$ ) and Bernoulli. How would you modify the coding technique to get guarantees that were as good as if  $p$  were very small (i.e.,  $p = \epsilon$ )?

Answer:

In this problem we know almost surely that a bit is going to be flipped (that assumption is for all bits both code and data). So in order to be able to detect the error we have to flip the bits back and use the fact that:

- $\overline{c_1} = \overline{b_1} \oplus \overline{b_3} \oplus \overline{b_4}$

- $\overline{c_2} = b_1 \oplus \overline{b_2} \oplus \overline{b_4}$
- $\overline{c_3} = \overline{b_2} \oplus \overline{b_3} \oplus \overline{b_4}$

Then the linear code is as effective as it was when the probability of error was very small.

9. Suppose we switch to the linear code:

- $c_1 = b_1 \oplus b_3$
- $c_2 = b_2 \oplus b_4$
- $c_3 = b_2 \oplus b_3$

(a) Can this code always detect single-bit errors? Explain why or why not.

Answer:

This code can always detect single-bit errors because each of the bits  $b_1, b_2, b_3, b_4$  exist as single terms in one of the parity bits  $c_1, c_2, c_3$ . That means that a change in a data bit will cause inconsistency with the parity bit and the error will be detected. The same holds for a single error in the code.

(b) List the sets of single-bit errors that should be detected, but not repaired (because 2 possible repairs are equally likely).

Answer:

The set of single-bit errors that can be detected but not repaired is  $S_d = b_1, b_4, c_1, c_2$ . We can obtain this result if we notice that there are two pairs of the same columns in the check matrix (see below):

$S = C \cdot W = \begin{bmatrix} \mathbf{1} & 0 & 1 & \underline{0} & \mathbf{1} & \underline{0} & 0 \\ \mathbf{0} & 1 & 0 & \underline{1} & \mathbf{0} & \underline{1} & 0 \\ \mathbf{0} & 1 & 1 & \underline{0} & \mathbf{0} & \underline{0} & 1 \end{bmatrix}$  So we can detect an error in one of the  $b_1, b_4, c_1, c_2$  but we cannot correct it because we don't know exactly which bit is between  $(b_1, c_1)$  and  $(b_4, c_2)$ .