

HW #3 Solutions

1. Trains arrive at the 116th Street station according to an exponential distribution with rate λ . Whenever a train arrives, the number of people that get off the train is uniformly distributed between 10 and 100. Only one exit gate is opened (so that all people exiting must pass through the same gate), and the line at the gate has 12 people waiting on average. What is the expected time a random person needs to wait to exit the station?

Answer:

For this problem we have to use Little's law but first we need to compute the various components of the law. Let P be the number of the people that get off the bus when it arrives. Also let N be a random variable denoting the queue length (number) of people waiting in the gate to get out and S the service time of the gate. We know that the trains arrive according to an exponential distribution with rate λ . We know the $E[P] = \frac{100-10}{2} + 10 = 55$ as P is drawn from a uniform distribution in $[10, 100]$. Let A be an exponential distributed variable denoting the arrival time of the trains. We know that $E[A] = \frac{1}{\lambda}$ and this is the average time of arrival. The average rate or the rate of arrivals is one over the average time of arrival that is λ . The train arrivals are independent of the people that get off. Then the rate of the people that get off the trains is $\frac{1}{E[A]} \cdot E[P] = 55 \cdot \lambda$. We also know that we have an exit that has a on average a queue of 12 people. Now we apply Little's law to obtain the queue length and from the queue length we will obtain the average service time $E[S]$ We have that:

$$E[N] = 55 \cdot \lambda \cdot E[S] \Rightarrow E[S] = \frac{E[N]}{55 \cdot \lambda} = \frac{12}{55 \cdot \lambda} \text{ where } \lambda \text{ is a rate so } E[S] \text{ is in time units}$$

2. In class, we started the computation for the configuration shown above of the probability that device 1 makes a successful transmission, given that device 1 transmits, where
 - each device transmits during each clock tick with probability p (where the decision to transmit is a Bernoulli process), and the recipient device of the transmission is chosen uniformly from the set of remaining devices (other than the transmitter).
 - two frames collide on a link ℓ if they are transmitted during the same clock tick and both traverse ℓ
 - a transmission from device i to device j fails if the transmission of the frame from i to j collides with another frame along any link on the path from i to j (collisions that happen off this path do not affect the transmission from i to j).
 - the switch at the top "knows" which hub to transmit to in order for a particular device to be reached. Therefore, when delivering a frame to device j , it forwards the frame only on the link that leads to device j .
 - the other hubs do not have these suppression capabilities and must forward all transmissions on all outgoing links.

Recall that we looked at a particular clock tick t , and we let T_i be a random variable that equals 1 when device i transmits at time t and 0 otherwise. We let d_i be a random variable that equals the destination of any transmission from device i , and we defined X_i to be a random variable that equals 1 if device i 's transmission at time t is successful and 0 otherwise. We therefore wish to compute $\Pr(X_i = 1 | T_i = 1)$. We did this as follows:

$$\begin{aligned}
\Pr(X_1 = 1 | T_1 = 1) &= \Pr(X_1 = 1, T_1 = 1) / \Pr(T_1 = 1) \\
&= \left(\sum_{j=2}^n \Pr(X_1 = 1, T_1 = 1, d_1 = j) + \sum_{j=n+1}^{kn} \Pr(X_1 = 1, T_1 = 1, d_1 = j) \right) / p \\
&= \frac{(n-1)p(1-p)^{n-1} \frac{1}{kn-1} ((1-p) + p \frac{(k-1)n-1}{kn-1})^{(k-1)n} + \sum_{j=n+1}^{kn} \Pr(X_1 = 1, T_1 = 1, d_1 = j)}{p}
\end{aligned}$$

To complete the computation, you need to compute $\sum_{j=n+1}^{kn} \Pr(X_1 = 1, T_1 = 1, d_1 = j)$.

Answer:

In this question the key concept is to understand what we are trying to compute and to break it up into different parts. Then we can compute the probabilities for the different cases and compose the total probability. We have to compute $\sum_{j=n+1}^{kn} \Pr(X_1 = 1, T_1 = 1, d_1 = j)$ that is nothing more than the probability of success for device D1 if it transmits to a group outside its group. We will break the probability into different parts:

$$\underbrace{\left(\frac{kn-n}{kn-1} \right) \cdot \underbrace{p}_{\substack{\text{probability} \\ \text{that device 1} \\ \text{transmits}}} \cdot \underbrace{(1-p)^{2n-1}}_{\substack{\text{no other device in group} \\ \text{1 and its destination} \\ \text{group transmits}}}}_{\substack{\text{devices in group 1 and its selected destination}}} \cdot \underbrace{\left[\underbrace{(1-p)}_{\substack{\text{no transmission}}} + p \cdot \underbrace{\left(\frac{kn-2n-1}{kn-1} \right)}_{\substack{\text{transmission to other} \\ \text{groups except} \\ \text{1 and device 1's target group}}} \right]^{kn-2n}}_{\substack{\text{devices outside group 1 and its selected destination}}}$$

The formula above equals $\sum_{j=n+1}^{kn} \Pr(X_1 = 1, T_1 = 1, d_1 = j)$ and the answer to the question is:

$$\underbrace{(n-1)p(1-p)^{n-1} \frac{1}{kn-1} ((1-p) + p \frac{(k-1)n-1}{kn-1})^{(k-1)n} + \left(\frac{kn-n}{kn-1} \right) p(1-p)^{2n-1} \left[(1-p) \left(\frac{kn-2n-1}{kn-1} \right) \right]^{kn-2n}}_{\substack{p \\ \Pr(X_i=1|T_i=1)}}$$

3. Suppose a switch is used in place of hub 1 (the hub immediately above device 1) and the switch at the top is replaced by a hub.

- (a) Give the set of transmission scenarios where device 1's transmission would fail under the original configuration but succeeds here.

Answer:

The scenarios are:

- i. a device i in group 1 transmits to another device j , ($i \neq j$) in group 1 with the condition that no one else transmits to i or j .
 - ii. a device i in group 1 transmits to another device j , ($i \neq j$) in group k , ($k \neq 1$) with the condition that no other device from group 1 transmits to a device outside of group 1 and no device outside of group 1 transmits at all.
- (b) Give the set of transmission scenarios where devices 1's transmission would succeed under the original configuration but would fail here.

Answer:

The scenarios are:

- i. a device in group i transmits to another device in group j , ($i \neq j \neq 1$) with the condition that no other device transmits from and to groups i, j

(c) Compute $\Pr(X_1 = 1 | T_1 = 1)$

Answer:

This problem is like the one in question two so we will have the same approach.

$$\begin{aligned} \Pr(X_1 = 1 | T_1 = 1) &= \Pr(X_1 = 1, T_1 = 1) / \Pr(T_1 = 1) \\ &= \left(\sum_{j=2}^n \Pr(X_1 = 1, T_1 = 1, d_1 = j) + \sum_{j=n+1}^{kn} \Pr(X_1 = 1, T_1 = 1, d_1 = j) \right) / p \end{aligned}$$

We will compute the two sums independently then we have:

$$\sum_{j=2}^n \Pr(X_1 = 1, T_1 = 1, d_1 = j) = \underbrace{\left(\frac{n-1}{nk-1} \right)}_{\text{number of destinations in group 1}} \cdot \underbrace{p}_{\text{device 1 transmits}} \cdot \underbrace{(1-p)}_{\text{device's 1 destination (in group 1) does not transmit}} \cdot \left(\underbrace{(1-p)}_{\text{no transmission}} + \underbrace{p \cdot \left(\frac{nk-3}{nk-1} \right)}_{\text{transmit to devices other than 1 and its destination excluding also itself.}} \right)^{nk-2}$$

all the rest devices in the system other than device 1 and its destination (nk - 2)

and $\sum_{j=n+1}^{kn} \Pr(X_1 = 1, T_1 = 1, d_1 = j) =$

$$= \underbrace{\left(\frac{nk-n}{nk-1} \right)}_{\text{number of different destinations outside group 1}} \cdot \underbrace{p}_{\text{device 1 transmits}} \cdot \left(\underbrace{(1-p)}_{\text{no transmission}} + \underbrace{p \cdot \left(\frac{n-2}{nk-1} \right)}_{\text{transmit to devices other than 1 excluding also themselves.}} \right)^{n-1} \cdot \underbrace{(1-p)}_{\text{all devices outside group 1 do not transmit}}^{nk-n}$$

all the rest devices in group 1 other than device 1 can transmit inside group one except to 1

4. N devices share a LAN and frame transmission times are segmented into slots such that two transmissions during the same slot always cause transmission failures, whereas two transmissions during different slots do not cause transmission failures. Assume that each device has a frame to send every time-slot, but only performs the transmission with a probability p .

(a) What is the probability of a successful transmission in a given timetick (in terms of N and p).

Answer:

We have a total of N devices, we denote with T a random variable that equals 1 when we have a successful transmission and 0 otherwise. Then:

$$\Pr(T = 1) = \binom{N}{1} \cdot p \cdot (1-p)^{N-1} \Rightarrow \Pr(T = 1) = N \cdot p \cdot (1-p)^{N-1}$$

because we chose one device to transmit (probability p) out of N and all the rest not to (probability $(1-p)$ for each).

- (b) What is the probability that device 1's transmission is successful, given device 1 attempts a transmission?

Answer:

Let S_i be a random variable that is 1 when device i makes a successful transmission 0 otherwise. Also let T_i be a random variable that is 1 when device i makes a transmission 0 otherwise. We have to compute the following conditional probability:

$$\Pr(S_1 = 1 \mid T_1 = 1) = \frac{\Pr(S_1 = 1, T_1 = 1)}{\Pr(T_1 = 1)} = \frac{p \cdot (1-p)^{N-1}}{p} = (1-p)^{N-1}$$

$$\Pr(S_1 = 1 \mid T_1 = 1) = (1-p)^{N-1}$$

- (c) What is the expected number of successful transmissions?

Answer:

We keep the notation of the previous part. Also let N_s be the number of successful transmissions. We also know that $\forall i, 0 < i \leq N : E[S_i] = 1 \cdot P[S_i = 1] + 0 \cdot P[S_i = 0] \Rightarrow E[S_i] = p \cdot (1-p)^{N-1}$ by the definition of S_i to be the probability of success for a single device i and the fact that all devices are the same. Then:

$$N_s = \sum_{i=1}^N S_i \Rightarrow E[N_s] = E\left[\sum_{i=1}^N S_i\right] = \sum_{i=1}^N E[S_i] = N \cdot E[S_i] = N \cdot p \cdot (1-p)^{N-1} \Rightarrow$$

$$E[N_s] = N \cdot p \cdot (1-p)^{N-1}$$

- (d) What value of p (in terms of N) maximizes the above probability.

Answer:

We use the answer from part (a) and we define the function $f(p) = N \cdot p \cdot (1-p)^{N-1}$. Now we take the first derivative and we get:

$$\frac{df(p)}{dp} = N \cdot (1-p)^{N-1} - N \cdot (N-1) \cdot p \cdot (1-p)^{N-2}$$

$$\frac{df(p)}{dp} = 0 \Leftrightarrow N \cdot (1-p)^{N-1} - N \cdot (N-1) \cdot p \cdot (1-p)^{N-2} = 0 \Leftrightarrow$$

$$p = 1 \text{ or } N \cdot (1-p) - N \cdot (N-1) \cdot p = 0 \Leftrightarrow p = 1 \text{ or } 1-p-N \cdot p+p=0 \Leftrightarrow$$

$$p = 1 \text{ or } N \cdot p = 1 \Leftrightarrow p = 1 \text{ or } p = \frac{1}{N} \text{ but } f(1) < f\left(\frac{1}{N}\right) \text{ because } 0 < \left(1 - \frac{1}{N}\right)^{N-1}, N > 1$$

So the answer is:

$$p_{max} = \frac{1}{N}$$

5. Suppose that n devices share a LAN, where each device sends frames that take L microseconds to transmit onto the wire, with $L > 2\tau$ where τ is the maximum propagation delay on the LAN. k of these n devices use ALOHA, where the backoff occurs at rate λ , the other $n - k$ devices use slotted ALOHA with slots of size L and backoff at rate λ . What is the probability of successful transmission for

- (a) A device using ALOHA

Answer:

Let U_i , $1 \leq i \leq k$ denote a random variable that is 1 when device i that uses unslotted ALOHA does a successful transmission 0 otherwise. We define TU_i , $1 \leq i \leq k$ be a random variable that is 1 when device i that uses unslotted ALOHA does a transmission 0 otherwise. Let S_j , $1 \leq j \leq n - k$ denote a random variable that is 1 when device j that uses slotted ALOHA does a successful transmission 0 otherwise. We also define TS_j , $1 \leq j \leq n - k$ be a random variable that is 1 when device j that uses slotted ALOHA does a transmission 0 otherwise. For a device using ALOHA the probability of a successful transmission is the probability that no other device both unslotted ALOHA and slotted ALOHA will do a transmission in a $2 \cdot L$ length time interval. This happens because for the unslotted ALOHA we can transmit at a random point in time, X so we have to guard a $[X - L, X + L]$ or a time interval of length $2 \cdot L$. Here the point X does not play any role as the poisson distribution is stationary. Therefore when the device that transmits is unslotted ALOHA we have that:

$$P[U_k = 1] = P[TU_i = 0, TS_j = 0 \quad \forall i, j \quad 1 \leq i \leq k, \quad 1 \leq j \leq n - k] \quad (1)$$

But we know that $P[TU_i = 0] = e^{-2 \cdot L \cdot \lambda}$ and $P[TS_j = 0] = e^{-2 \cdot L \cdot \lambda}$ for a $2 \cdot L$ time interval. We also know that each device is independent of the rest so (1) now becomes:

$$P[U_k = 1] = P[TU_1 = 0] \cdot \dots \cdot P[TU_k = 0] \cdot P[TS_1 = 0] \cdot \dots \cdot P[TS_{n-k} = 0] \Rightarrow$$

$$P[U_k = 1] = (e^{-2 \cdot L \cdot \lambda})^k \cdot (e^{-2 \cdot L \cdot \lambda})^{n-k} = (e^{-2 \cdot L \cdot \lambda})^n \Rightarrow P[U_k = 1] = e^{-n \cdot 2 \cdot L \cdot \lambda}$$

- (b) A device using slotted ALOHA

Answer:

We keep the same notation as in part (a). Here we have that our device is slotted ALOHA so the difference with the unslotted case is that there are specific points in time that our device can transmit and these points are common for all slotted devices. On the other hand the unslotted devices can transmit arbitrarily with no restrictions. Thus we have to compute the probability that no slotted device is going to transmit in a time interval of length L ($[X - L, X]$) and no unslotted device is going to transmit in a time interval of length $2 \cdot L$ ($[X - L, X + L]$). Using this fact we have that $P[TU_i = 0] = e^{-2 \cdot L \cdot \lambda}$ and $P[TS_j = 0] = e^{-L \cdot \lambda}$. Moreover each device is independent of the rest. Therefore we get:

$$P[S_m = 1] = P[TU_i = 0, TS_j = 0 \quad \forall i, j \quad 1 \leq i \leq k, \quad 1 \leq j \leq n - k] \quad (1)$$

$$P[S_m = 1] = P[TU_1 = 0] \cdot \dots \cdot P[TU_k = 0] \cdot P[TS_1 = 0] \cdot \dots \cdot P[TS_{n-k} = 0] \Rightarrow$$

$$P[S_k = 1] = (e^{-2 \cdot L \cdot \lambda})^k \cdot (e^{-L \cdot \lambda})^{n-k} \Rightarrow P[S_k = 1] = e^{-(k \cdot 2 \cdot L \cdot \lambda + (n-k) \cdot L \cdot \lambda)} = e^{-(n+k) \cdot L \cdot \lambda} \Rightarrow$$

$$P[S_k = 1] = e^{-(n+k) \cdot L \cdot \lambda}$$

- (c) A device drawn uniformly at random from the set of devices.

Answer:

Let D be a random variable denoting a device drawn uniformly at random from the set of devices. Also let TD be 1 when D does a successful transmission 0 otherwise. We combine the results from parts (a),(b) and we get:

$$P[TD = 1] = \sum_{i=1}^k (P[D = U_i] \cdot P[TU_i = 1]) + \sum_{j=1}^{n-k} (P[D = S_j] \cdot P[TS_j = 1])$$

$$P[TD = 1] = \frac{k}{n} \cdot e^{-n \cdot 2 \cdot L \cdot \lambda} + \frac{n-k}{n} \cdot e^{-(n+k) \cdot L \cdot \lambda}$$

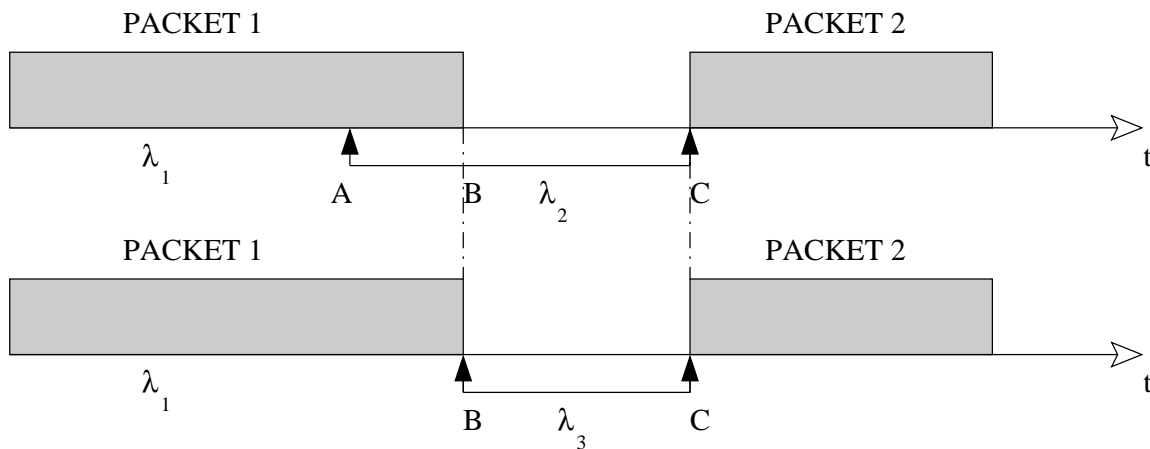
6. A set of devices on a LAN transmit frames whose lengths are exponentially distributed with rate λ_1 . The system used to use non-persistent CSMA where a device would check if the wire was available at the instant it wanted to transmit and send the frame if the wire was free. If not, the device would back off for a time t that was exponentially distributed with rate λ_2 . When time t completed, the process would repeat until the frame was sent.

The system is being switched to one where the line can be sensed continuously, such that the precise end of the previous transmission can be determined. A backoff whose time is exponentially distributed with rate λ_3 is applied once the wire goes free to prevent synchronized frame transmissions. Compute λ_3 as a function of λ_2 and λ_1 such that the expected transmission rates (i.e., the distribution on how long the backoff is from the end of a transmission frame) are the same in the two systems. Show or explain how you get your result.

Answer:

For this exercise we will use the memoryless property of the exponential distribution.

SCENARIO 1



SCENARIO 2

Figure 1: Different Scenarios.

We can see that for the first scenario we check in a point A and we wait exponentially with rate λ_2 and then we check the line again and we start immediately if the line is sensed empty. On the second scenario we know when

the other transmission has ended and we wait exponentially with rate λ_3 to start transmitting. In this exercise we want the BC interval in the figure 1 to be drawn from the same distribution for both scenarios. Let X_1 a random variable denoting the total time from the time we sensed the line point (A) till the start of transmission of packet 2 for the first scenario point (C). Also let X_2 a random variable denoting the total time from the time that packet 1 ended point (B) till the start of transmission of packet 2 for the second scenario point (C). Due to the memoryless property of the exponential distribution we have that:

$$P(X_1 > AC \mid X_1 > B) = P(X_1 > BC) = e^{-\lambda_2 \cdot t}, \quad P(X_2 > BC) = e^{-\lambda_3 \cdot t}$$

Also we know that we want:

$$P(X_1 > BC) = P(X_2 > BC) \Leftrightarrow e^{-\lambda_2 \cdot t} = e^{-\lambda_3 \cdot t} \Leftrightarrow \lambda_2 = \lambda_3$$

7. Let C_1, C_2, C_3 be three connections that use CDMA to transmit upon the same channel using chipping signals $(1,1,1,1,1,1,1,1)$, $(1,1,-1,-1,1,1,-1,-1)$, and $(1,1,-1,-1,-1,1,1,1)$ respectively. If the received chipping signal is $(1,1,1,1,-1,-1,3,3)$, what was the value of the bit transmitted by connections C_1, C_2, C_3 (where the value is either 1 or -1)?

Answer:

We have to take the dot product of the received signal with the chipping codes and divide by the length of the code. Let $Cd1, Cd2, Cd3$ be $(1,1,1,1,1,1,1,1)$, $(1,1,-1,-1,1,1,-1,-1)$, and $(1,1,-1,-1,-1,1,1,1)$ respectively and R $(1,1,1,1,-1,-1,3,3)$. Then we have:

$$C_1 = (C_{d1} \cdot R) / 8 = ((1, 1, 1, 1, 1, 1, 1, 1) \cdot (1, 1, 1, 1, -1, -1, 3, 3)) / 8 = (1+1+1+1-1-1+3+3) / 8 = 8 / 8 = 1$$

$$C_2 = (C_{d2} \cdot R) / 8 = ((1, 1, -1, -1, 1, 1, -1, -1) \cdot (1, 1, 1, 1, -1, -1, 3, 3)) / 8 = (1+1-1-1-1-1-3-3) / 8 = -8 / 8 = -1$$

$$C_3 = (C_{d3} \cdot R) / 8 = ((1, 1, -1, -1, -1, 1, 1, 1) \cdot (1, 1, 1, 1, -1, -1, 3, 3)) / 8 = (1+1-1-1-1+1+3+3) / 8 = 8 / 8 = 1$$

thus $C_1 = 1$, $C_2 = -1$, $C_3 = 1$