HW Solutions #3

ELEN E4710 - Intro to Network Engineering — Due 10/25/2004
Fall 2004 — Prof. Rubenstein

Homework must be turned in at the beginning of class on the due date indicated above. CVN students have one additional day. Late assignments will not be accepted.

1. Suppose that n devices share a LAN, where each device sends frames that take L microseconds to transmit onto the wire, with $L > 2\tau$ where τ is the maximum propagation delay on the LAN. k of these n devices use ALOHA, where the backoff occurs at rate λ, the other $n-k$ devices use slotted ALOHA with slots of size L and backoff at rate λ. What is the probability of successful transmission for

   (a) A device using ALOHA (w/o carrier sensing)

   Ans) When an unslotted ALOHA device sends a frame successfully at time T, there is no other frame transmitted on the wire in the time interval $[T-L, T+L]$ with length $2L$.
   The probability that an unslotted ALOHA device doesn't transmit any frame within $2(L+\tau)$ amount of time is $P_u = e^{-\lambda 2L}$. The probability that a slotted ALOHA device doesn't transmit any frame within $2(L+\tau)$ amount of time is also $P_s = e^{-\lambda 2L}$. Because $2(L+\tau) < 3L$, we only need the random backoff larger than $2L$ to guarantee that no collusion will happen. Because all devices are independent. $P\{$succ. transmission for unslotted ALOHA$\} = P_1 = P_u^{k-1} P_s^{n-k} = e^{-2(n-1)\lambda L}$. ∎

   (b) A device using slotted ALOHA

   Ans) When a slotted ALOHA device sends a frame successfully at time T, there is no other frame transmitted in the time interval $[T-L, T+L]$ by any unslotted ALOHA device and no other frame transmitted in the time interval $[T, T+L]$ by any slotted ALOHA device.
   The probability that an unslotted ALOHA device doesn't transmit any frame within $2(L+\tau)$ amount of time is $P_u = e^{-\lambda 2L}$. The probability that a slotted ALOHA device doesn't transmit any frame within $2L$ amount of time is $P_s = e^{-\lambda L}$. Because all devices are independent. $P\{$succ. transmission for unslotted ALOHA$\} = P_2 = P_u^k P_s^{n-k-1} = e^{-(n+k-1)\lambda L}$. ∎

   (c) A device drawn at uniformly at random from the set of devices.

   Ans) Uniformly select a device, it will be an unslotted device with probability k/n.
   $P\{$ succ. transmission $\} = (k/n)P_1 + [(n-k)/n]P_2$ ∎

2. Suppose there are two types of devices. Type s transmits packets of size L, the other type t transmits packets of size $2L$. A slotted ALOHA collision avoidance mechanism is used, where slots accommodate size L packets. Suppose there is one of each type of device attempting to transmit upon the same medium (i.e., 2 devices total), where it takes time T to output a frame of size L.

   (a) Suppose carrier sensing is not used. What should the backoff time λ_t be in terms of λ_s so that each device is as likely as the other to successfully transmit a packet?

   Ans)
   $P\{$succ. transmission for s $\} = P\{$no transmission from device t in both the previous and the current time-slots $\} = e^{-\lambda_t 2L}$.
   $P\{$succ. transmission for t $\} = P\{$no transmission from device s in both the current and the next time-slots $\} = e^{-\lambda_s 2L}$. Making two probabilities equal, we have $\lambda_t = \lambda_s$.
   ∎

   (b) Suppose the goal is to equalize the expected amount of bandwidth a device transmits within an interval. What should λ_t be in terms of λ_s in this case?

   Ans) Define p_s, p_t to be the successful transmission probability for each device. Define W_s, W_t to be the backoff time for each device.

We have $E[W_s] = 1/\lambda_s$, $E[W_t] = 1/\lambda_t$. The transmission time for s is one time-slot; the transmission time for t is two time-slots. Therefore, the average times between each transmission are $1/\lambda_s + L$ and $1/\lambda_t + 2L$ for s and t.

The successful transmission probabilities are $p_s = e^{-\lambda_t 2L}$ and $p_t = e^{-\lambda_s 2L}$ respectively.

Finally, to make the bandwidth equal, we have:

$$p_s L/(E[W_s] + L) = p_t 2L/(E[W_t] + 2L)$$
$$\frac{Le^{-\lambda_t 2L}}{\frac{1}{\lambda_s} + L} = \frac{2Le^{-\lambda_s 2L}}{\frac{1}{\lambda_t} + 2L}$$
$$2(\frac{1}{\lambda_s} + L)e^{-\lambda_s 2L} = (\frac{1}{\lambda_t} + 2L)e^{-\lambda_t 2L}$$

The reason why the bandwidth in a loss-free system is $T/(E[W] + T)$ where T is the transmission time of the pkt and W is the waiting time between transmissions is the following:

Bandwidth will be $\sum T_j / \sum (W_j + T_j)$ where T_j is transmission time and W_j is waiting time.

If the sum is over n intervals, this is $(\sum T_j/n)/(\sum W_j/n + \sum T_j/n)$, which as n goes to infinity, becomes $E[T]/(E[W] + E[T])$.

∎

(c) Suppose device t has the ability to sense the line. Explain (in a sentence or two) why this makes no difference.

Ans) Device t has to wait until the next time-slot to transmit frames. But by that time, the probability of collision is the same. Because device s will finish its transmission within the current time-slot and start a backoff timer. Moreover, backoff time is exponential and therefore memoryless.

∎

(d) Suppose device s has the ability to sense the line. Show (mathematically) why this does make a difference.

Ans)

Let I_k be an indicator that equals 1 if device t transmits during round k.

We want to show that $P(I_k = 1 | I_{k-1} = 1) > P(I_k = 1)$.

Let $J_k = 1$ if device t starts a packet transmission during round k.

In the steady state, clearly, $P(J_k = 1 | I_k = 1) = 1/2$, i.e., randomly choose a slot, and if device t is transmitting, then half the time, it's the first half of the packet. Also, in the steady state $P(I_k = 1) = P(I_{k-1} = 1)$ for all k.

Let $p = P(J_k = 1 | J_{k-1} = 0)$, i.e., that device t starts its transmission during the kth slot given it isn't continuing another transmission.

Then $P(I_k = 1 | I_{k-1} = 1) = 0.5 + 0.5p$ (half the time, it's a continuing packet, the other half of the time, it chooses to start a transmission during this next slot).

$q = P(I_k = 1) = P(I_k = 1 | I_{k-1} = 1)P(I_{k-1} = 1) + P(I_k = 1 | I_{k-1} = 0)P(I_{k-1} = 0) = (0.5 + 0.5p)q + p(1 - q)$.

Solving for q yields $q = 2p/(p + 1)$.

So we must show that $(p + 1)/2 > 2p/(p + 1)$, or that

$(p + 1)^2 - 4p > 0$, or $(p - 1)^2 > 0$, but clearly this is true for all p.

∎

3. N devices share a LAN and frame transmission times are segmented into slots such that two transmissions during the same slot always cause transmission failures, whereas two transmissions during different slots do not cause transmission failures. Assume that each device has a frame to send every time-slot, but only performs the transmission with a probability p.

(a) Suppose this slotted collision avoidance mechanism is implemented using slotted ALOHA, where slots last for time T. What is the rate of the backoff timer as a function of T and p?

Ans) By using slotted ALOHA, $P\{\text{send in the next slot}\} = P\{X < T\} = 1 - e^{-\lambda T} = p$.
Therefore, $\lambda = -\frac{1}{T} ln(1-p)$ ∎

(b) What is the probability of a successful transmission in a given timetick (in terms of N and p).

Ans) $P\{\text{exactly one of the device transmits}\} = \binom{N}{1} p(1-p)^{N-1} = Np(1-p)^{N-1}$. ∎

(c) What is the probability that device 1's transmission is successful, given device 1 attempts a transmission?

Ans) $P\{\text{all other devices don't transmit}\} = (1-p)^{N-1}$. ∎

(d) What is the expected number of successful transmissions per time-slot?

Ans) Define X to be the number of successful transmissions in a time slot.
$E[X] = \sum_{i=1}^{\infty} i\ P\{\text{i succ. transmissions}\} = P\{\text{1 succ. transmissions}\} = P\{\text{exactly one of the device transmits}\} = Np(1-p)^{N-1}$. ∎

(e) What value of p (in terms of N) maximizes the probability in part 3b.

Ans) Define $f = Np(1-p)^{N-1}$.
Set $\partial f/\partial p = N(1-p)^{N-1} - Np(N-1)(1-p)^{N-2} = 0$
$(1-p) - p(N-1) = 0$
$p = 1/N$ ∎

4. $N+1$ devices numbered $1, \cdots, N+1$ share a LAN and frame transmission times are segmented into slots such that two transmissions during the same slot always cause transmission failures, whereas two transmissions during different slots do not cause transmission failures. Odd numbered slots are used to send new transmissions. If the transmission fails because of a collision, then the subsequent even slot is used to attempt a retransmission. The probability that a device transmits on an odd-numbered slot is p, where the decision to transmit is independent from the device's previous transmission history and from the transmission decisions of the other devices. Given the device's transmission in an odd slot collides with other transmissions, the device reattempts the transmission during the even round with probability q. The decision to retransmit during the even round is independent of the decisions by the other devices that also experienced a collision in the previous odd round.

Let $X_{i,j}$ be an indicator random variable that equals one if device i attempts a transmission during round $2j+1$ and 0 otherwise. Let $Y_{i,j}$ be an indicator random variable that equals 1 if device i succeeds in completing its transmission (if one is made) in either round $2j+1$ or in round $2j+2$.

What is $P(Y_{1,5} = 1 | X_{1,5} = 1)$?

Ans)

$P(Y_{1,5} = 1 | X_{1,5} = 1) = P(Y_{1,5} = 1, X_{1,5} = 1)/P(X_{1,5} = 1) = P(Y_{1,5} = 1, X_{1,5} = 1)/p$.
$P(Y_{1,5} = 1, X_{1,5} = 1) = P(\text{succ. in the odd slot}) + P(\text{succ. in the even slot})$.
$P(\text{succ. in the odd slot}) = (1-p)^N p$.
$P(\text{succ. in the even slot}) = \sum_{n=1}^{N} P(\text{succ. in the even slot}|\text{n other devices transmitted in the odd slot}) P(\text{n other devices transmitted in the odd slot}) = \sum_{n=1}^{N} (\binom{N}{n} p^N (1-p)^{N-n} p)((1-q)^n q) = pq(1-pq)^N$.
Therefore, $P(Y_{1,5} = 1 | X_{1,5} = 1) = P(Y_{1,5} = 1, X_{1,5} = 1)/p = (1-p)^N + q(1-pq)^N$.

An easier way to solve the problem is the following:

Let $Z_{i,j}$ be an indicator RV that is Bernoulli and equals 1 with prob q. Device i transmits during the even round only if its attempt at transmitting in the preceding odd round fails, and $Z = 1$. Clearly, this describes the same behavior.

Then $P(\text{success}) = P(Z = 0 \text{and success in odd round}) + P(Z = 1 \text{and success in odd or even round})$.

The first term is $p(1-q)(1-p)^N$. The second term is $pq(1-p+p(1-q))^N$ (each other device either doesn't transmit in the odd round, or it does does transmit, but then doesn't in the even round). ∎

5. Let C_1, C_2, C_3 be three connections that use CDMA to transmit upon the same channel using chipping signals (1,1,1,1,1,1,1,1), (1,1,-1,-1,1,1,-1,-1), and (1,1,-1,-1,-1,-1,1,1) respectively. If the received chipping signal is (1,1,1,1,-1,-1,3,3), what was the value of the bit transmitted by connections C_1, C_2, C_3 (where the value is either 1 or -1)?

   Ans) $(1,1,1,1,1,1,1,1) \cdot (1,1,1,1,-1,-1,3,3) = 8 > 0$
   $(1,1,-1,-1,1,1,-1,-1) \cdot (1,1,1,1,-1,-1,3,3) = -8 < 0$
   $(1,1,-1,-1,-1,-1,1,1) \cdot (1,1,1,1,-1,-1,3,3) = 8 > 0$
   Therefore, $(C_1, C_2, C_3) = (1,-1,1)$. ∎

6. In the bridged LAN pictured above, k LANs are bridged together using switches. Each LAN contains n devices such that there are kn devices on the bridged LAN. The devices participate in a slotted protocol, with each device transmitting a frame in a given slot with probability p, the destination location is uniformly distributed among the other devices.

   (a) An additional device D is added to LAN 1 (so that there are $n+1$ devices on this LAN). If X_i is a random variable that equals 1 if device D tries to transmit during the ith slot, and S_i is a random variable that equals 1 when D's transmission during the ith slot is successful, what is $\Pr(S_i = 1 | X_i = 1)$?

   Ans) Define Y to be the LAN where the destination of D's transmission locates. $P\{Y = i\} = 1/K$.
   $P\{S_i = 1 | X_i = 1\} = P\{S_i = 1, X_i = 1\}/P\{X_i = 1\}$
   $P\{S_i = 1, X_i = 1\} = \sum_{n=1}^{K} P\{S_i = 1, X_i = 1 | Y = n\} P\{Y = n\} = \frac{1}{K}\sum_{n=1}^{K} P\{S_i = 1, X_i = 1 | Y = n\}$
   $P\{S_i = 1, X_i = 1 | Y = n\} = P\{\text{no devices transmit in the first } n \text{ LANs }\} P\{X_i = 1\} P\{\text{no devices in the last } K-n \text{ LANs transmit packets to the first n LANs }\} = (1-p)^{jn} P\{X_i = 1\}[1-p+p\frac{n(k-j)-1}{nk}]^{n(k-j)}$
   So $P\{S_i = 1 | X_i = 1\} = \frac{1}{K}\sum_{n=1}^{K}(1-p)^{jn}[1-p+p\frac{n(k-j)-1}{nk}]^{n(k-j)}$. ∎

   (b) Suppose device D reduces the probability of transmission to p/n. How does this affect $\Pr(S_i = 1 | X_i = 1)$?

   Ans) From the above procedure, we see that $P\{X_i = 1\}$ doesn't affect this conditional probability. ∎