HW Solutions #2

ELEN E4710 - Intro to Network Engineering — Due Oct. 4, 2004
Fall 2004 — Prof. Rubenstein

Homework must be turned in at the beginning of class on the due date indicated above. CVN students have one additional day. Late assignments will not be accepted.

1. You work in a department store that has a Santa Claus who sits children on his lap and asks them what they want for Christmas. Parents complain that on average, they wait 50 minutes to see Santa. Santa claims that he spends only 2 minutes with each child. Can you calculate the expected length of the line? If so, how and what is it? If not, why not?

   Ans) By Little's Law, the average length of the line equals the average arrival rate times the average waiting time. Since the average arrival rate must be less than or equal to the service rate, which is $1/2$, the average length of the line is less than or equal to $(1/2) * 50 = 25$. ■

2. Let $X_1, X_2, \cdots, X_k$ be independent, exponentially distributed random variables with rate λ.
   Let $Y = \min\{X_1, \cdots, X_k\}$. Show that Y is exponentially distributed. What is the rate of Y?

   Ans) $P\{Y > y\} = P\{min\{X_1, \ldots, X_k\} > y\} = P\{X_1 > y \cap \cdots \cap X_k > y\} = P\{X_1 > y\} \ldots P\{X_k > y\}$
   $P\{X_i > x\} = e^{-\lambda x}$ for all i. $P\{Y > y\} = \prod_{i=1}^{k} e^{-\lambda y} = e^{-k\lambda y}$
   Therefore, we know Y is an exponential random variable with rate $k\lambda$. ■

3. You arrive at a bus stop at 3:15 pm. A bus arrives according to a uniform distribution between 3 and 4 pm.

   (a) What is the probability that you missed the bus?

   Ans) Define T to be the amount of time (hours) from 3:00pm to the time when the bus comes.
   $P\{\text{miss the bus}\} = P\{T \leq 1/4\} = 1/4$. ■

   (b) Somebody informs you that the bus has not yet arrived. Given this information, what is the probability that it comes within the next 15 minutes?

   Ans) $P\{1/4 < T \leq 1/2 | T > 1/4\} = P\{1/4 < T \leq 1/2 \cap 1/4 < T\}/P\{T > 1/4\} = P\{1/4 < T \leq 1/2\}/P\{T > 1/4\} = \frac{1/4}{3/4} = 1/3$. ■

4. You arrive at a bus stop at 3:15 pm. A bus arrives according to an exponential distribution with rate λ, starting at 3pm (it will not arrive before then).

   (a) What is the probability that you missed the bus?

   Ans) Define T to be the same as in the previous question.
   $P\{T \leq t\} = 1 - e^{-\lambda t}$
   $P\{\text{miss the bus}\} = P\{T \leq 1/4\} = 1 - e^{-\frac{1}{4}\lambda}$.

   ■

   (b) Somebody informs you that the bus has not yet arrived. Given this information, what is the probability that it comes within the next 15 minutes?

   Ans) By the memoryless property, $P\{T \leq 1/2 | T > 1/4\} = P\{T \leq 1/2 - 1/4\} = P\{T \leq 1/4\} = 1 - e^{-\frac{1}{4}\lambda}$. ■

   (c) Suppose there are 3 buses numbered 1,2,3 where the first bus arrives at a time after 3pm that is exponentially distributed with rate λ. The second bus arrives at a time after the arrival of the first bus that is exponentially distributed with rate λ, and the third bus arrives at a time after the second bus that is exponentially distributed with rate λ. What is the probability that the third bus takes more than an hour to arrive?

Ans) Define T_1 to be the amount of time (hours) from 3:00pm to the time when the first bus comes. Define X_2 to be the amount of time (hours) between the arrival of the first bus and the second bus. Similarly, we define X_3 to be the amount of time (hours) between the arrival of the second bus and the third bus.
We have T_1, T_2 and T_3 identically and independently distributed.
$P\{$the third bus takes more than an hour to arrive $\} = P\{T_1 + T_2 + T_3 > 1\} = \int_0^\infty P\{T_2 + T_3 > 1 - t_1\} dens\{T_1 = t_1\} dt_1 = \int_1^\infty dens\{T_1 = t_1\} dt_1 + \int_0^1 P\{T_2 + T_3 > 1 - t_1\} dens\{T_1 = t_1\} dt_1$

Since, $P\{T_2+T_3 > t\} = \int_0^\infty P\{T_3 > t - t_2\} dens\{T_2 = t_2\} dt_2 = \int_t^\infty dens\{T_2 = t_2\} dt_2 + \int_0^t P\{T_3 > t - t_2\} dens\{T_2 = t_2\} dt_2 = e^{-\lambda t} + \int_0^t e^{-\lambda(t-t_2)}\ \lambda e^{-\lambda t_2} dt_2 = e^{-\lambda t} + \int_0^t e^{-\lambda t}\ \lambda dt_2 = e^{-\lambda t} + (t - 0)e^{-\lambda t}\ \lambda = (1 + \lambda t)e^{-\lambda t}$

$P\{T_1 + T_2 + T_3 > 1\} = \int_1^\infty dens\{T_1 = t_1\} dt_1 + \int_0^1 P\{T_2 + T_3 > 1 - t_1\} dens\{T_1 = t_1\} dt_1 = e^{-\lambda} + \int_0^1 (1 + \lambda(1 - t_1))e^{-\lambda(1-t_1)} dens\{T_1 = t_1\} dt_1 = e^{-\lambda} + \int_0^1 (1 + \lambda(1 - t_1))e^{-\lambda(1-t_1)} \lambda e^{-\lambda t_1} dt_1 = e^{-\lambda} + \int_0^1 (\lambda + \lambda^2 - \lambda^2 t_1)e^{-\lambda} dt_1 = (1 + \lambda + \lambda^2)e^{-\lambda} - \lambda^2 e^{-\lambda} \int_0^1 t_1 dt_1 = (1 + \lambda + \lambda^2/2)e^{-\lambda}$ ∎

5. Suppose that the arrival times of three buses numbered 1,2,3 are exponentially distributed with rates λ_1, λ_2, and λ_3 (here, the any bus can come first, i.e., the clocks for all buses start at time 0).

   (a) For the case where $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, What is the probability that it takes more than an hour for all three buses to arrive?

   Ans) Define T_i to be the arrival time (hours) of the ith bus.
   $P\{$ it takes more than an hour for all buses to arrive$\} = 1 - P\{$all buses takes $< 1\} = 1 - (1 - e^{-\lambda})^3$ ∎

   (b) What is the probability that buses arrive in the order 1,2,3?

   Ans) $P\{T_1 < T_2, T_2 < T_3\} = P\{T_1 < T_2, T_3\} P\{T_2 < T_3\} = \int_0^\infty dens\{T_1 = t\} P\{T_2 > t\} P\{T_3 > t\} dt \int_0^\infty dens\{T_2 = t\} P\{T_3 > t\} dt = \int_0^\infty \lambda_1 e^{-\lambda_1 t} e^{-\lambda_2 t} e^{-\lambda_3 t} dt \int_0^\infty \lambda_2 e^{-\lambda_2 t} e^{-\lambda_3 t} dt = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \frac{\lambda_2}{\lambda_2+\lambda_3}$.
   The intuitive idea is that bus 1 wins the competition by the chance of $\lambda_1/(\lambda_1 + \lambda_2 + \lambda_3)$. After that, by the memoryless property, bus 2 and bus 3 begin a new competition where bus 2 wins with probability $\lambda_2/(\lambda_2 + \lambda_3)$. ∎

   (c) What is the probability that bus 2 arrives more than time t after bus 1, given that bus 2 arrives after bus 1?

   Ans) By the memoryless property, $P\{T_2 > T_1 + t | T_2 > T_1\} = P\{T_2 > t\} = e^{-\lambda_2 t}$. ∎

   (d) What is the probability that bus 2 takes more than time t to arrive given that bus 1 comes after bus 2?

   Ans) $P\{T_2 > t | T_1 > T_2\} = P\{T_2 > t, T_1 > T_2\}/P\{T_1 > T_2\} = \int_t^\infty dens\{T_2 = t\} P\{T_1 > t\} dt / (\frac{\lambda_2}{\lambda_1+\lambda_2}) = \int_t^\infty \lambda_2 e^{-\lambda_2 t} e^{-\lambda_1 t} dt / (\frac{\lambda_2}{\lambda_1+\lambda_2}) = \lambda_2 [0 - (-\frac{1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t}] / (\frac{\lambda_2}{\lambda_1+\lambda_2}) = e^{-(\lambda_1+\lambda_2)t}$ ∎

6. Consider a 2-D parity check code.

   (a) Prove that any combination of 1, 2, or 3 bit errors is detectable.

   Ans) If a one-bit error occurs, there exist one row and one column where parity check errors occur.
   If a two-bit error occurs, there are three cases:

   - if two error bits occur in a same row, then there exist two columns where parity check errors occur.
   - if two error bits occur in a same column, then there exist two rows where parity check errors occur.

- otherwise, there exist two rows and two columns where parity check errors occur.

If a three-bit error occurs, by the same reason, there exists at least one row or one column where a parity check error occurs. ∎

(b) For a 16-bit data word (i.e., code bits are extra bits), show a 4 bit error combination that cannot be detected, and show one that can.

Ans) An example for a non-detectable 4 bit error:

1	0	1	1
0	R	R	0
1	R	R	1
1	0	1	0

An example for a detectable 4 bit error:

R	0	1	1
0	R	1	0
1	0	R	1
1	0	1	R

∎

(c) For a data word with k^2 bits, given that bit flips are a Bernoulli process (independent) with probability p, compute the probability that 4 bit-errors occur and that the 2-D parity check fails to detect that the word is corrupted.

Ans) The pattern of a non-detectable 4 bit error is that the four errors form a rectangle. So each of these non-detectable 4 bit error can be described by the entry (i_1, i_2, j_1, j_2) uniquely, where i_1, i_2 are the row indexes and j_1, j_2 are the column indexes. Therefore, the number of different non-detectable 4-bit errors is $\binom{k}{2}\binom{k}{2}$.

$P\{$a 4 bit error occurs and non-detectable $\} = P\{$ non-detectable | a 4 bit error occurs $\}P\{$ a 4 bit error occurs $\} = \frac{\binom{k}{2}\binom{k}{2}}{\binom{k^2}{4}} * \binom{k^2}{4} p^4 (1-p)^{k^2-4} = \frac{1}{4}k^2(k-1)^2 p^4 (1-p)^{k^2-4}$. ∎

$$\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array}$$

(d) Fix the bit error in the above codeword.

Ans)

$$\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1^* & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array}$$

Because the fourth row and the third column have the parity check errors. ∎

7. Consider the (7,4) Linear code whose code bits are generated as follows:

- $c_1 = b_1 \oplus b_3 \oplus b_4$

- $c_2 = b_1 \oplus b_2 \oplus b_4$
- $c_3 = b_2 \oplus b_3 \oplus b_4$

Suppose you receive the codeword 1110111 which was generated using the above linear code. What codeword was most likely transmitted?

Ans)

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$H(1110111)^T = (111)^T$. The error pattern is matched at the fourth column of H. So codeword 1111111 was most likely transmitted. ∎

8. The above codes work best when p, the probability of a bit being flipped, is very small. What if p were very large (e.g., $1 - \epsilon$ for some very small ϵ) and Bernoulli. How would you modify the coding technique to get guarantees that were as good as if p were very small (i.e., $p = \epsilon$)?

   Ans) We flip each bit of the original codeword to construct the new codeword. The probability distribution of the receiving codeword is exactly the same, as if we have a very small p and transmit the original codeword. ∎

9. Suppose we switch to the linear code:

   - $c_1 = b_1 \oplus b_3$
   - $c_2 = b_2 \oplus b_4$
   - $c_3 = b_2 \oplus b_3$

   (a) Can this code always detect single-bit errors? Explain why or why not.

   Ans) This code can always detect single-bit errors.

   $$\mathbf{H} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   Because each single-bit error can be detected by an error pattern, which is a non-zero vector(column) in matrix H. ∎

   (b) List the sets of single-bit errors that should be detected, but not repaired (because 2 possible repairs are equally likely).

   Ans) The sets are $\{1, 5\}$ and $\{4, 6\}$.

   Notice that if one of the 1st or 5th (4th or 6th) bit is flipped, we will get the same error pattern from matrix H. Therefore, we cannot repair this error since we don't know which bit is actually flipped. ∎