HW #1

ELEN E4710 - Intro to Network Engineering — Due 9/20/2004
Fall 2004 — Prof. Rubenstein

Homework must be turned in at the beginning of class on the due date indicated above. CVN students have one additional day. Late assignments will not be accepted.

1. A fair n-sided die (each side is equally likely to come up) with sides numbered from 1 to n is rolled until the value 5 appears on top. How many rolls are expected?

   Ans:)

   Ans: Define X to be the number of rolls until the value 5 appears on top.
   $E[X] = \sum_{x=1}^{\infty} xP\{X = x\} = \sum_{x=1}^{\infty} x(1-p)^{x-1}p = p + 2p(1-p) + 3p(1-p)^2 + ..$
   Let $f(x) = (1-p)E[X] = p(1-p) + 2p(1-p)^2 + 3p(1-p)^3 + ...$
   So $E[X] - f(x) = pE[X] = p + p(1-p) + p(1-p)^2 + ... = p\frac{1}{1-(1-p)} = 1$
   $E[X] = 1/p = n$

2. A fair coin is tossed 10 times. What is the probability that either the first five tosses are heads or the last 5 tosses are tails?

   Ans: Define A to be the Indictor Random Variable such that $A = 1$ if five heads happen in the first fives tosses and $A = 0$ otherwise. Define B to be the Indictor Random Variable such that $B = 1$ if five tails happen in the last fives tosses and $B = 0$ otherwise.

   $P\{A = 1 \cup B = 1\} = P\{A = 1\} + P\{B = 1\} - P\{A = 1 \cap B = 1\} = 2^5/2^{10} + 2^5/2^{10} - 1/2^{10} = 65/1024.$ ■

3. You lose at craps if on your first roll, you roll 2,3 or 12. You win if on your first roll, you roll 7 or 11. You win if you roll anything else (4-6, 8-10) on your first roll and roll this value again before rolling a 7. Otherwise you lose.

   (a) What is the probability that you will win? If you do this right, the odds should be in the house's favor (i.e., less than .5), but by very little.

   Ans: Define N_i to be the outcome of your ith roll. Define L to be the Indicator Random Variable such that $L = 1$ if you lose and $L = 0$ if you win.
   The probability desitity for N_i is:
   $P\{N_i = 2\} = P\{N_i = 12\} = 1/36$
   $P\{N_i = 3\} = P\{N_i = 11\} = 2/36$
   $P\{N_i = 4\} = P\{N_i = 10\} = 3/36$
   $P\{N_i = 5\} = P\{N_i = 9\} = 4/36$
   $P\{N_i = 6\} = P\{N_i = 8\} = 5/36$
   $P\{N_i = 7\} = 6/36$

   $P\{L = 1\} = \sum_{i=2}^{12} P\{L = 1 | N_1 = i\}P\{N_1 = i\}$
   $= \sum_{i=2,3,12} P\{N_1 = i\} + \sum_{i=4,5,6,8,9,10} P\{L = 1 | N_1 = i\}P\{N_1 = i\}$
   So the remaining unknowns are $P\{L = 1 | N_1 = i\}$ for $i = 4, 5, 6, 8, 9, 10$.
   $P\{L = 1 | N_1 = i\} = P\{$ you roll a 7 before i $\}$ You can consider the sample space becomes the set {7,i}.
   $P\{L = 1 | N_1 = i\} = P\{N_i = 7\}/(P\{N_i = 7\} + P\{N_i = i\})$.
   Finally, the probability you win is $1 - P\{L = 1\} = 244/495$. ■

(b) If you gain a dollar for each win and lose a dollar for each loss, how much money will you have left on average after playing 100 games?

Ans: Define X_i to be the money you gain for game i.
$E[X_i] = 1(244/495) + (-1)(1 - 244/495) = -7/495$
$E[\sum_{i=1}^{100} X_i] = \sum_{i=1}^{100} E[X_i] = 100E[X_i] = -700/495 = -140/99.$

■

4. Consider a room containing n people in which each person is equally likely to be born on any day of the year (i.e., assume no leap years and $P(G_i = k) = 1/365$, where G_i is the birth date of the ith person and k is a number between 1 and 365 indicating that person's birth date).

(a) What is the probability that no two people have the same birthday?

Ans: The sample space contains 365^n outcomes.
We can choose n different dates from a year. These n dates form $n!$ permutations of the outcome that no two people have the same birthday.
$P\{$ no two people have the same birthday $\} = \binom{365}{n} n!/365^n$

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(b) What is the probability that k people were born on January 24 where $k < n$?

Ans: Define N_i to be the number of people who have the birthday on the ith (i=1,2,..,365) day.
$P\{N_{24} = k\} = P\{N_i = k\} = \binom{n}{k} p^k (1-p)^{n-k}$, where $p = 1/365$, regardless of the value of i.
Because one person has the birthday on Jan 24th with probability p. And we choose exactly k people from the group of n people.

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(c) How many pairs of people can we expect to have the same birthday? Give a closed-form solution without summations.

Ans: Define $I_{i,j}$ to be the Indicator Random Variable such that $I_{i,j} = 1$ if the ith person and the jth person have the same brithday and $I_{i,j} = 0$ otherwise.
We would like to know $E[\sum_{i<j} I_{i,j}]$.
$P\{I_{i,j} = 1\} = P\{i \text{ and } j \text{ have the same birthday}\} = p = 1/365.$
$E[\sum_{i<j} I_{i,j}] = \sum_{i<j} E[I_{i,j}] = \binom{n}{2} E[I_{i,j}] = \frac{n(n-1)}{2}\frac{1}{365}$

■

5. Let X, Y, and Z be random variables such that X and Y are independent, X and Z are independent, and Y and Z are independent (i.e., $P(X,Y) = P(X)P(Y), P(X,Z) = P(X)P(Z), P(Y,Z) = P(Y)P(Z)$. Show that this does not imply that X, Y and Z are independent (i.e., that it is not always true that $P(X,Y,Z) = P(X)P(Y)P(Z)$). Think about rolling 2 dice...

Ans: There may be alternative examples. Let's consider a simple example where two fair coins are tossed.
We define X to be the Indicator Random Variable of the first coin toss such that $X = 1$ if the first coin appears a Head and $X = 0$ if it appears a Tail. Likewise, we define Y to be the Indicator Random Variable associated with the outcome of the second coin toss. We also define Z to be the random variable such that $Z = 1$ if $X = Y$ and $Z = 0$ otherwise. Or we can write $Z = (X + Y)$ mod 2.

$P\{Z = 0\} = P\{X = 0\}P\{Y = 0\} + P\{X = 1\}P\{Y = 1\} = 0.5$ and
$P\{Z = 1\} = P\{X = 0\}P\{Y = 1\} + P\{X = 1\}P\{Y = 0\} = 0.5.$

$P\{Z=0, X=0\} = P\{Y=0, X=0\} = P\{Y=0\}P\{X=0\} = 1/4 = P\{Z=0\}\{X=0\}$ similarly, we can find $P\{Z=z, X=x\} = P\{Z=z\}P\{X=x\}$ for all (x, z). So we know Z is independent of X and Y. But, although $P\{Z=0\}P\{X=0\}P\{Y=1\} = 1/8$, the event $\{Z=0, X=0, Y=1\}$ never happens. ∎

6. With probability p_1, TA Ma has a message to give to the class. With probability p_2, Professor Rubenstein will remember to deliver the message. With probability p_3, Mary, a student in the class arrives on time to hear any announcements Professor Rubenstein makes. Assuming these three events are independent (such that the probability that Mary hears an announcement is $p_1p_2p_3$), what is the probability that TA Ma gave an announcement to Professor Rubenstein, given that Mary did not hear an announcement?

Ans: We define the following events:
A: TA Ma has a message. $P\{A\} = p_1$
B: Prof. Rubenstein delivers messages. $P\{B\} = p_2$
C: Mary hears any announcement Prof. Rubenstein makes. $P\{C\} = p_3$
D: Mary hears TA Ma's message from Prof. Rubenstein. $P\{D\} = P\{A \cap B \cap C\} = p_1p_2p_3$

Then we'd like to know $P\{A|D^c\}$.
$P\{A|D^c\} = \frac{P\{AD^c\}}{P\{D^c\}} = \frac{P\{AD^c\}}{1-P\{D\}}$
To know $P\{AD^c\}$, we have two mutually exclusive cases:
(1) TA Ma has a message and Prof. Rubenstein will not remember to deliver it. This happens with probability $p_1(1-p_2)$.
(2) TA Ma has a message and Prof. Rubenstein also forwards that message. But Mary doesn't get the message. This happens with probability $p_1p_2(1-p_3)$.
$P\{AD^c\} = p_1(1-p_2) + p_1p_2(1-p_3)$
Therefore, $P\{A|D^c\} = \frac{P\{AD^c\}}{1-P\{D\}} = \frac{p_1(1-p_2)+p_1p_2(1-p_3)}{1-p_1p_2p_3}$

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