

**CSEE 6861 CAD of Digital Systems**  
**Handout: Lecture #3**  
**2/4/16**

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**ESPRESSO Algorithm:**  
**The “EXPAND” Step, continued**

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## “EXPAND” Step: Expansion Direction (REVIEW)

### Basic idea:

- Once **cube c** in **cover F** is selected for expansion, expand it in “good directions”

### 3 Heuristic Expansion Steps (in sequence): to expand a **cube c**

- Expand **c** to completely contain (i.e. “swallow up”) as many other cubes of **cover F** as possible  
 \_\_\_\_\_ - delete these completely-contained cubes immediately!
- Once #1 done -- continue to expand **c** to partially overlap as many other cubes of **cover F** as possible  
 - the goal is to “induce redundancies”, which then get deleted in the IRRED step
- Once #2 done -- continue to expand **c** into a “maximal size” prime implicant

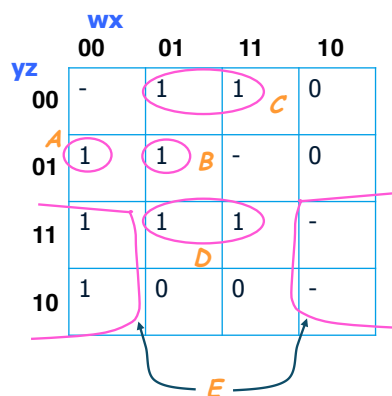
⇒ **New Focus: Step #2 and #3**

#3

## “EXPAND” Step: Expansion Direction

Example #1: illustrates Step #1 + Step #2

Suppose cube A has been picked for expansion

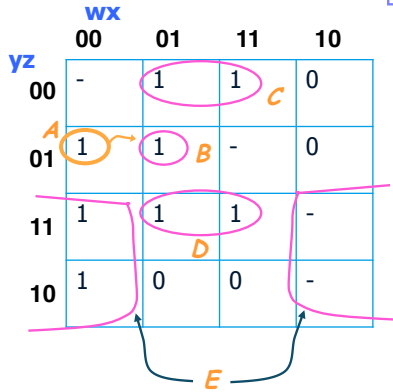


#4

## “EXPAND” Step: Expansion Direction

Example #1 (cont.): illustrates Step #1 + Step #2

**STEP #1:** expand cube A to fully contain as many other cubes as possible

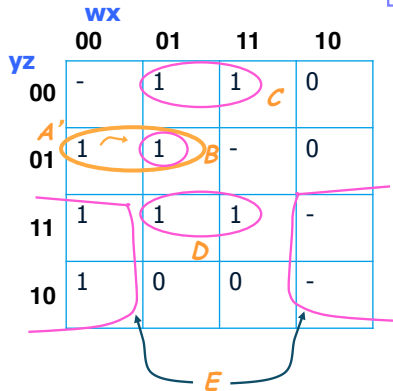


#5

## “EXPAND” Step: Expansion Direction

Example #1 (cont.): illustrates Step #1 + Step #2

**STEP #1:** expand cube A to fully contain as many other cubes as possible

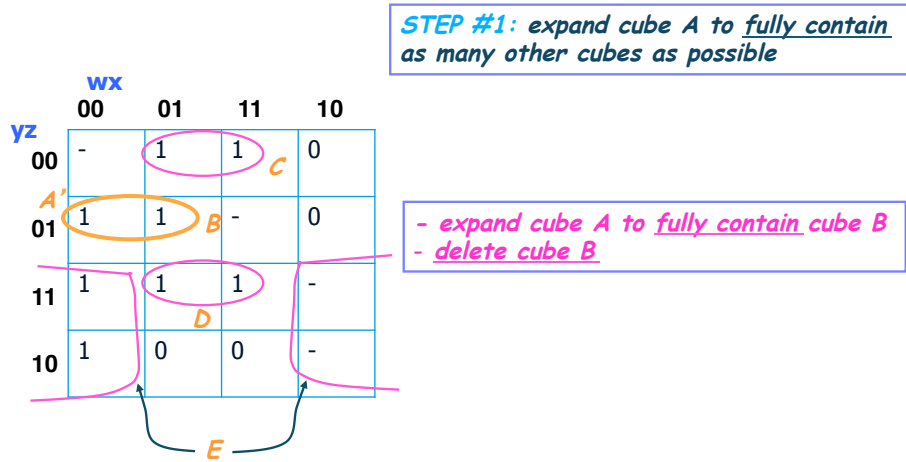


- expand cube A to fully contain cube B

#6

## “EXPAND” Step: Expansion Direction

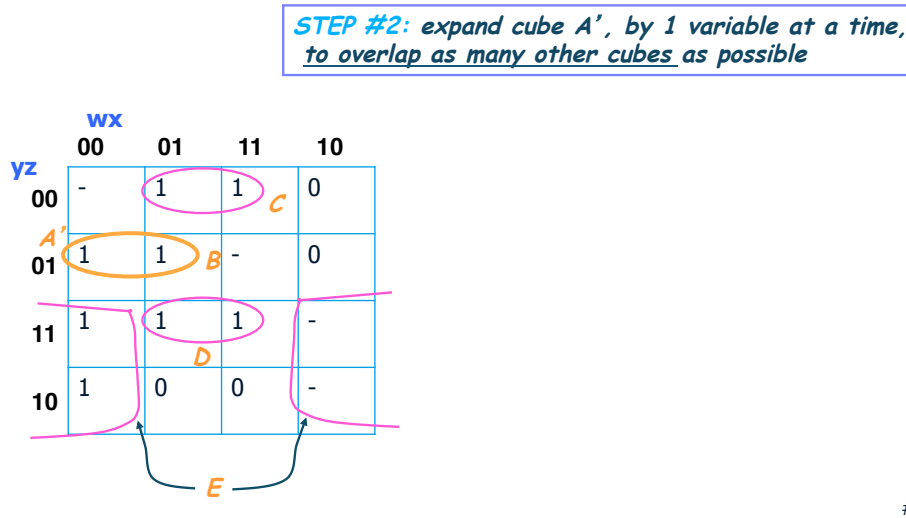
Example #1 (cont.): illustrates Step #1 + Step #2



#7

## “EXPAND” Step: Expansion Direction

Example #1 (cont.): illustrates Step #1 + Step #2

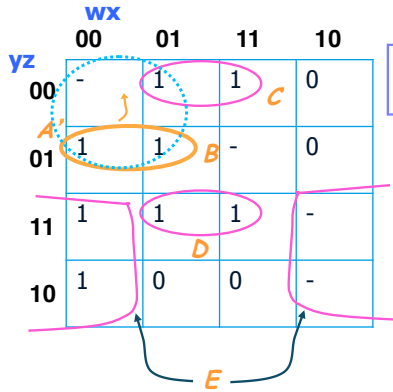


#8

## “EXPAND” Step: Expansion Direction

Example #1 (cont.): illustrates Step #1 + Step #2

**STEP #2:** expand cube  $A'$ , by 1 variable at a time, to overlap as many other cubes as possible



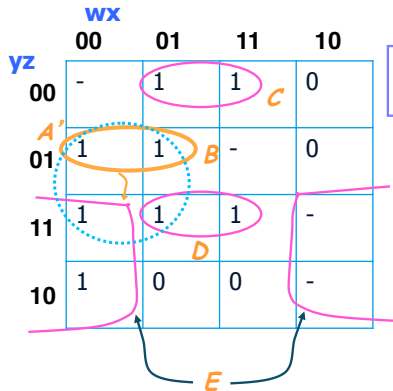
**Option 1:** expand cube  $A'$  in “z” dimension”: overlaps 1 additional cube (C)

#9

## “EXPAND” Step: Expansion Direction

Example #1 (cont.): illustrates Step #1 + Step #2

**STEP #2:** expand cube  $A'$ , by 1 variable at a time, to overlap as many other cubes as possible



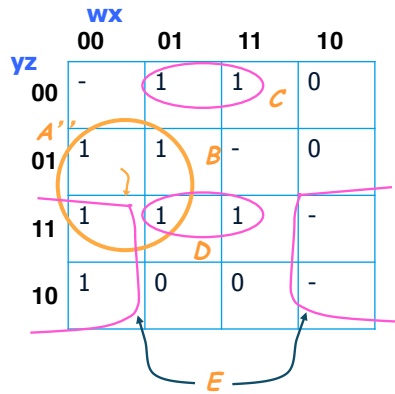
**Option 2:** expand cube  $A'$  in “y” dimension”: overlaps 2 additional cubes (D, E)

#10

## “EXPAND” Step: Expansion Direction

Example #1 (cont.): illustrates Step #1 + Step #2

STEP #2: expand cube A', by 1 variable at a time, to overlap as many other cubes as possible



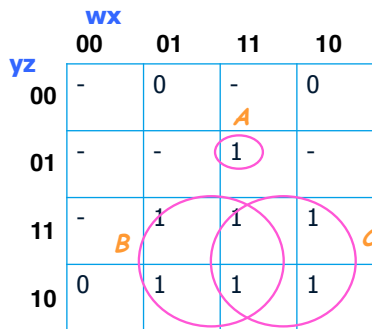
Final choice = Option 2: expand cube A' in "y dimension"

#11

## “EXPAND” Step: Expansion Direction

Example #2: illustrates Step #2 + Step #3

Suppose cube A has been picked for expansion



#12

## “EXPAND” Step: Expansion Direction

Example #2 (cont.): illustrates Step #2 + Step #3

*STEP #2: expand cube A, by 1 variable at a time, to overlap as many other cubes as possible*

	wx			
yz	00	01	11	10
00	-	0	-	0
01	-	-	1	-
11	-	1	1	1
10	0	1	1	1

Diagram showing three overlapping cubes: A (orange dashed circle around the 1 at wx=11, yz=01), B (pink circle around wx=01, yz=11 and wx=11, yz=11), and C (pink circle around wx=11, yz=11 and wx=10, yz=11).

*Final choice =  
expand cube A in “y dimension”  
⇒ overlaps cubes B & C*

#13

## “EXPAND” Step: Expansion Direction

Example #2 (cont.): illustrates Step #2 + Step #3

*STEP #2: expand cube A, by 1 variable at a time, to overlap as many other cubes as possible*

	wx			
yz	00	01	11	10
00	-	0	-	0
01	-	-	1	-
11	-	1	1	1
10	0	1	1	1

Diagram showing three overlapping cubes: A' (orange dashed circle around the 1 at wx=11, yz=01), B (pink circle around wx=01, yz=11 and wx=11, yz=11), and C (pink circle around wx=11, yz=11 and wx=10, yz=11).

*Final choice =  
expand cube A in “y dimension”*

#14

## “EXPAND” Step: Expansion Direction

Example #2 (cont.): illustrates Step #2 + Step #3

**STEP #3:** expand cube  $A'$  into a maximal size prime implicant

		wx			
		00	01	11	10
yz	00	-	0	-	0
	01	-	-	1	-
	11	-	1	1	1
	10	0	1	1	1

Diagram illustrating Step #3: Expansion Direction. The Karnaugh map shows three prime implicants:  $A'$  (orange oval),  $B$  (pink oval), and  $C$  (pink oval).  $A'$  is a cube of size 2,  $B$  is a cube of size 4, and  $C$  is a cube of size 4.

#15

## “EXPAND” Step: Expansion Direction

Example #2 (cont.): illustrates Step #2 + Step #3

**STEP #3:** expand cube  $A'$  into a maximal size prime implicant

		wx			
		00	01	11	10
yz	00	-	0	-	0
	01	-	-	1	-
	11	-	1	1	1
	10	0	1	1	1

Diagram illustrating Step #3: Expansion Direction. The Karnaugh map shows three prime implicants:  $A'$  (orange oval),  $B$  (pink oval), and  $C$  (pink oval).  $A'$  is a cube of size 2,  $B$  is a cube of size 4, and  $C$  is a cube of size 4. Arrows indicate the expansion of  $A'$  in the 'z' dimension.

*Option 1: expand cube  $A'$  in “z” dimension*  
*-  $A'$  becomes wx*

#16



## “EXPAND” Step: Expansion Direction

Example #2 (cont.): illustrates Step #2 + Step #3

**STEP #3:** expand cube  $A'$  into a maximal size prime implicant

	wx			
yz	00	01	11	10
00	-	0	-	0
01	-	-	1	-
11	-	1	1	1
10	0	1	1	1

Diagram showing prime implicants:  $A'$  (orange oval),  $B$  (pink oval), and  $C$  (pink oval).  $A'$  is a cube covering cells (01,11) and (11,11).  $B$  is a cube covering cells (11,11) and (10,11).  $C$  is a cube covering cells (11,11) and (10,10).

**Option 2:** expand cube  $A'$  in  $w'/x'$  dimensions  
-  $A'$  becomes  $z$

#17

## “EXPAND” Step: Expansion Direction

Example #2 (cont.): illustrates Step #2 + Step #3

**STEP #3:** expand cube  $A'$  into a maximal size prime implicant

	wx			
yz	00	01	11	10
00	-	0	-	0
01	-	-	1	-
11	-	1	1	1
10	0	1	1	1

Diagram showing prime implicants:  $A''$  (orange oval),  $B$  (pink oval), and  $C$  (pink oval).  $A''$  is a larger cube covering cells (01,11), (11,11), (11,10), and (10,10).  $B$  is a cube covering cells (11,11) and (10,11).  $C$  is a cube covering cells (11,11) and (10,10).

**Final choice = Option 2**  
- 'larger' prime ( $A'' = z$ ) has fewer literals

#18

## ESPRESSO Algorithm: The “IRREDUNDANT” Step

### The “IRREDUNDANT” Step

Example Cover: after “expand” = prime cover

Given: cover  $F$  (after EXPAND)

	wx			
yz	00 $P1$	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	0	1	1
10	-	-	1	0

Diagram illustrating the prime cover of a function  $F$  after expansion. The Karnaugh map shows six prime implicants circled in pink:  $P1$  (top-left),  $P2$  (middle-left),  $P3$  (top-middle),  $P4$  (middle-right),  $P5$  (bottom-right), and  $P6$  (bottom-middle).

Goal: make  $F$  irredundant = delete max # of implicants while still maintaining a valid cover

#20

## The “IRREDUNDANT” Step

Approach #1 (SUBOPTIMAL!): greedily remove one redundant cube at a time

	wx			
yz	00 P1	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	0	1	1
10	-	-	1	0

Diagram showing prime implicants P1, P2, P3, P4, P5, and P6. P3 is highlighted in orange.

before:

	wx			
yz	00 P1	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	0	1	1
10	-	-	1	0

Diagram showing prime implicants P1, P2, P4, P5, and P6. P3 is removed.

after deleting P3:  
irredundant suboptimal cover

#21

## The “IRREDUNDANT” Step

Approach #2 (OPTIMAL!): form & solve an exact optimization problem

This is approach used in ESPRESSO...:

	wx			
yz	00 P1	01	11	10
00	1	1	0	0
01	0	1	1	0
11	0	0	1	1
10	-	-	1	0

Diagram showing prime implicants P1, P2, P3, P4, P5, and P6. P3 is highlighted in orange.

before:

Goal: set up a simplified PI table,  
and solve it exactly, to determine  
min # of cubes to keep  
(delete all others)

#22

## The "IRREDUNDANT" Step

Simplified PI Table:

		prime implicants					
		P1	P2	P3	P4	P5	P6
ON-set minterms	0000	X					
	0100	X	X				
	0101		X	X			
	1101			X	X		
	1111				X	X	
	1011					X	
	1110						X
	1110						X

#23

## The "IRREDUNDANT" Step

Simplified PI Table:

Approach: solve PI table exactly

		prime implicants					
		P1	P2	P3	P4	P5	P6
ON-set minterms	0000	X					
	0100	X	X				
	0101		X	X			
	1101			X	X		
	1111				X	X	
	1011					X	
	1110						X
	1110						X

**Optimal Solution:**  
 - select {P1, P2, P5, P6}  
 - discard rest....

#24

## The “IRREDUNDANT” Step

### An Issue:

- How can afford to compute an **exact solution to prime implicant table** (like the expensive Quine-McCluskey method!) in the inner loop of a fast heuristic algorithm (espresso)?

### Justification:

- This is **not the same** as the QM method!

**Key Observation:** columns in this table are **only the primes in current cover**,  
-- **not all the primes of the function!**

Hence, there is no prime generation step, and often a (much) smaller table!

#25

## The “IRREDUNDANT” Step

Approach #2 (OPTIMAL): final solution

		wx			
		00 P1	01	11	10
yz	00	1	1	0	0
	01	0	1	1	0
	11	0	0	1	1
	10	-	-	1	0

before:

		wx			
		00 P1	01	11	10
yz	00	1	1	0	0
	01	0	1	1	0
	11	0	0	1	1
	10	-	-	1	0

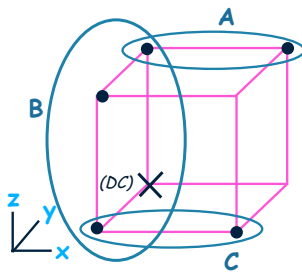
after:  
irredundant optimal cover

#26

## ESPRESSO Algorithm: The “REDUCE” Step

### The “REDUCE” Step

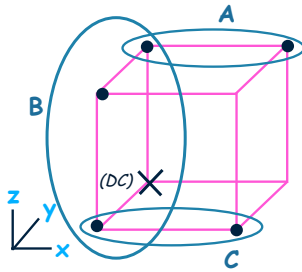
Example Cover: after “expand” and “irredundant” = prime irredundant cover



#28

## The "REDUCE" Step

Example Cover: after "expand" and "irredundant" = prime irredundant cover



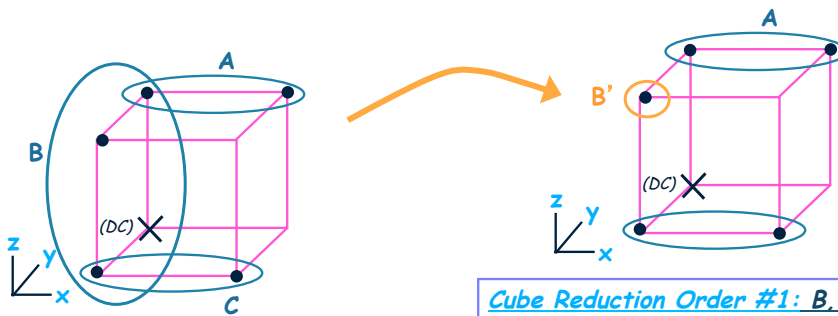
Given: cover F (after EXPAND/IRRED)

Goal: maximally reduce all cubes of cover F, in some order, while still maintaining a valid cover

#29

## The "REDUCE" Step

Key Observation: reduce operation is **order-dependent**



Cube Reduction Order #1: B, A, C  
- only B can be reduced

#30

# The "REDUCE" Step

**Key Observation:** reduce operation is *order-dependent*

