Link and Graph-Imbedding Models for Cyclic Weaving on Surfaces

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Abstract

We extend graph rotation systems into a solid mathematical model for the development of an interactive-graphics cyclic-weaving system. It involves a systematic exploration and characterization of dynamic surgery operations on graph rotation systems, such as edge-insertion, edge-deletion, and edge-twisting. This talk explains the underlying mathematics and some high-level aspects of the programming system for the interactive-graphics system.

* Presenter of this talk. An implementation-focused version will be presented at SIGGRAPH 2009 in New Orleans.
1 Introduction

Several recent papers [2, 4, 5, 6] on graphics by my co-authors Ergun Akleman and Jianer Chen use classical topological graph theory [12, 15, 17], especially graph rotation systems, as a solid mathematical basis for 3D-mesh modeling and sculpturing systems.

![Figure 1: Weaves on 3D-meshes.](image)

Some advantages of this formal mathematical model:

(A1) universal: its techniques can be adopted by any existing modeling software system;

(A2) robust: it never generates invalid non-manifold structures;

(A3) powerful: it can perform all necessary topological surgery operations;

(A4) the primary operations are simple and intuitive;

(A5) many secondary operations at user-level can be readily implemented using the built-in primary operations.
2 Links, surfaces, and cyclic weaving

Definition. A link $\sigma : \cup C \rightarrow \mathbb{R}^3$ is a homeomorphism from a set $C = \{c_1, \ldots, c_k\}$ of disjoint circles into $\mathbb{R}^3$.

Definition. A projection of a link onto a surface $S \subset \mathbb{R}^3$ is an immersion $\sigma : \cup C \rightarrow S$, with finitely many singular points in $\cup C$, such that the preimages of each crossing point $y \in S$ are ordered, and such that each preimage has at most two points.

![Diagram of an alternating link projection.](image)

Figure 2: An alternating link projection.

Some features of such an immersion:

- local homeorphism: $(\forall x \in \cup C) \ x$ has a nbhd that is mapped homeomorphically to $S$;
- every intersection is a true crossing (no tangencies);
- the images of two circles may intersect;
- the image of a circle may self-intersect;

A knot is a link with only one component.
**Definition.** A link projection is *alternating* if on a traversal of each of its components, the over-crossings and under-crossings alternate, as on the left of Figure 3. An *alternating link* is a link that has an alternating projection.

![Figure 3: Two projections of the Whitehead link.](image)

![Figure 4: The Borromean link.](image)

![Figure 5: The trefoil knot.](image)
Surfaces

A closed surface in 3-space separates 3-space into two parts, by a 3-dimensional analogue of the Jordan curve theorem. The part that goes to infinity is called the outside and the other part is called the inside.

Definition. Restoration of a link $L$ from a projection onto a surface is the result of pulling each crossing apart: a small over-crossing segment is pulled outside the surface and a small undercrossing segment is pushed inside the surface.

Figure 6: An unknotted torus in $\mathbb{R}^3$. 

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Knots, links, and surfaces

Theorem 2.1 (Seifert, 1934) Every knot or link in space is the boundary of a connected surface.

Proof. [16] or [1]. □

Figure 7: Seifert surface for a trefoil knot linked to an unknot.

A torus knot is a knot that lies on an unknotted torus in 3-space.

Figure 8: A torus knot.

The theory of alternating projections onto knotted higher-genus surfaces has apparently not yet been developed by topologists.
Proposition 2.2  For any link $L$ in $\mathbb{R}^3$ with $n$ components, there is a closed orientable surface $S$ of genus $n$ in $\mathbb{R}^3$ on which $L$ is imbedded.

Proof.  Thicken each component $C_j$ into a solid torus, so that $S_j$ lies on the surface of that solid torus, and so that the solid tori are mutually disjoint. Nest discard the interiors of the solid tori, so that each component of the link lies on a torus. Then connect the $n$ tori with $n-1$ tubes, to obtain a copy $S$ of the surface $S_n$ of genus $n$. $\Box$

Corollary 2.3  Every link $L$ in 3-space has an alternating projection onto some closed surface in $\mathbb{R}^3$.

Proof.  By Proposition 2.2, there is a closed orientable surface $S$ in $\mathbb{R}^3$ such that $L$ is imbedded on $S$. An imbedding is an alternating projection with zero crossings. $\Box$

Cyclic plain-weaving

Definition.  A cyclic plain-weaving is an alternating projection of a link onto a surface in $\mathbb{R}^3$.

Remark  A cyclic weaving is like a cyclic plain weaving, except that the projection of the underlying link need not be alternating, and crossings on the surface $S$ may have pre-images in the link $L$ with more than two points. The thickness of a weaving is the maximum number of points in a preimage.
3 Graph imbeddings and rotation systems

This presentation of topological graph theory is consistent with more detailed discussions of these issues to be found in [12].

Topological graphs may have multi-edges and self-loops.

An edge ALWAYS has two edge-ends, which are small neighborhoods of the limit points 0 and 1 of a parametrization of the edge, even when there is only one endpoint.

Each edge $e$ induces two oriented edges, each running from one edge-end of edge $e$ to the other edge-end of $e$. 
Surfaces and imbeddings

- **surface**: a closed, compact 2-dimensional manifold;
- **imbedding**: a homeomorphism \( G \rightarrow S \) of a graph \( G \) onto a topological subspace of the surface \( S \);
- **cellular**: every connected component of \( S - G \) is homeomorphic to an open disk.
- **rotation** at a vertex \( v \) of \( G \): a cyclic ordering of the oriented edges originating at \( v \);
- **(pure) rotation system** of graph \( G \): a set of \( n \) rotations, one for each vertex of \( G \).

Figure 11: Two inequivalent rotation systems for \( K_4 \).
Two \textit{equivalent orientable imbeddings} of a graph $G$ have the same rotation at every vertex of $G$.

\textbf{Example 3.1} Imbeddings of the complete graph $K_4$.

- 2 in $S_0$ with four 3-gons, like top drawing
- 8 in $S_1$ with 3-gon and 9-gon, like bottom drawing
- 6 in $S_1$ with a 4-gon and an 8-gon

Thus, the genus distribution of $K_4$ is

\[ g_0(K_4) = 2 \quad g_1(K_4) = 14 \]

\textbf{Example 3.2} Two more genus distributions.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig12}
\caption{Two non-isomorphic graphs.}
\end{figure}

\begin{enumerate}
\item \textbf{g-dist} = (2, 38, 24)
\item \textbf{g-dist} = (0, 40, 24)
\end{enumerate}

\textbf{Two basic artifacts of context}

\textbf{Proposition 3.3} For any graph $G$,

\[ \sum_{i \geq 0} g_i(G) = \prod_{v \in V(G)} ((\text{deg}(v) - 1)!) \]

\textbf{Theorem 3.4} The minimum-genus problem is NP-complete.
Face-tracing

The *face-tracing algorithm* [9, 12] constructs the fb-walks. Matching the perimeter of each $s$-sided polygon to each fb-walk of length $s$ reconstructs the surface $S$.

A **face corner** is a triple $(v, e, e')$ comprising a vertex $v$ and two oriented edges $e$ and $e'$, both oriented out of $v$, where the $v$-edge-end of $e'$ immediately follows the $v$-edge-end of $e'$ in the rotation at $v$. If neither $e$ nor $e'$ is a self-loop, we say that $e'$ is 0-next to $e$ at $v$ and that $e$ is 1-next to $e'$ at $v$. For a self-loop, we must say which orientation is 0-next or 1-next.

**Subroutine FaceTrace**($t_0, (u_0, w_0)$)

Input: $(u_0, w_0)$ is an oriented edge, and $t_0 \in \{0, 1\}$ is its “trace type”.

Output: the sequence of oriented edges in the fb-walk containing $(u_0, w_0)$.

1. trace and print $(u_0, w_0)$;
2. $t = t_0 + \text{type}([u_0, w_0]) \mod 2$;
3. $(u, w) = \text{the } t\text{-next to } (w_0, u_0) \text{ at } w_0$; \(u = w_0\)
4. while $(u, w) \neq (u_0, w_0)$ or $(t \neq t_0)$ do
   trace and print $(u, w)$;
   $t = t + \text{type}([u, w]) \mod 2$;
   $(w', u') = (w, u)$;
   $(u, w) = \text{the } t\text{-next to } (w', u') \text{ at } w'$. \(u = w'\)

**Algorithm FbWalks**($\rho(G)$)

Input: $\rho(G)$ is a general graph rotation system.

Output: the collection of all fb-walks in $\rho(G)$.

while there is an untraced face corner $(u, e, e')$ in $\rho(G)$ do
   suppose that $e' = (u, w)$; call FaceTrace($0, (u, w)$).
4 Rotation systems and surgery

Surgery operations on pure graph rotation systems (thus, on graph imbeddings) have been extensively studied [7, 12]. They are relatively easy to understand.

Pure edge-insertion and edge-deletion surgery

Edge-Insert-0

(a) If both ends of a new edge $e$ are inserted into corners of the same face, then $e$ splits that face into two faces, and the two oriented edges corresponding to $e$ belong to the different fb-walks in the new imbedding.

(b) If the two ends of $e$ are inserted into corners of two different faces, then $e$ merges those two faces into a single face, and the two oriented edges corresponding to $e$ belong to the fb-walk of that single face in the new imbedding.

Figure 13: Adding an edge to an orientable imbedding.
The operation of edge deletion on a pure rotation system “reverses” edge insertion.

**Edge-Delete-0**

If the two oriented edges corresponding to an edge $e$ appear in the boundary walks of two different faces, then deleting the edge $e$ merges the two faces into a single face.

If the two oriented edges corresponding to an edge $e$ belong to the boundary walk of a single face, then deleting the edge $e$ splits that face into two faces.

![Figure 14: Edge-delete from orientable imbedding (R to L).](image)

*** REMARK ***

The fb-walks for an imbedding of a graph on an orientable surface represent a trivial weave, i.e., a link whose components are completely unlinked and individually unknotted.

We now turn our focus to non-trivial weaves.
5 General rotation systems

A general rotation system of a graph \( G = (V, E) \) consists of a pure rotation system of \( G \) plus a function \( t : E \to \{0, 1\} \) that assigns to each edge of \( G \) an edge-type.

This augmentation of graph rotation systems is sufficient to represent imbeddings on non-orientable surfaces. For this, we regard type-0 edges as flat and type-1 edges as twisted.

The following figure represents band-decompositions for imbeddings \( K_4 \to S_0 \) and \( K_4 \to N_1 \).

![Band-decompositions for \( K_4 \to S_0 \) and \( K_4 \to N_1 \).](image)

**Figure 15:** Band-decompositions for \( K_4 \to S_0 \) and \( K_4 \to N_1 \).

**Proposition 5.1** A twist that joins two fb-walks decreases the Euler characteristic by 1. \( \square \)
**Proposition 5.2** The imbedding surface specified by a general rotation system is orientable if and only if the following condition holds for every pair of vertices $u$ and $v$:

The parity of the number of twisted edges is the same along every path between $u$ and $v$. □

The *induced weaving* of a general rotation system is the projection of the boundary of its band-decomposition onto the surface specified by its underlying pure rotation system.

**Example 5.3** The second band-decomposition also gives us a non-trivial weaving on $S_0$ of a link with three components, which are the fb-walks for the imbedding $K_4 \to N_1$ projected onto $S_0$.

![Band-decomps for $K_4 \to S_0$ and $K_4 \to N_1$ again.](image_url)

Figure 16: Band-decomps for $K_4 \to S_0$ and $K_4 \to N_1$ again.
6 On edge-twisting surgery

Twisting an edge in a general rotation system means changing its type, either from 0 to 1, or from 1 to 0.

Remark When traversing a twisted edge during face-tracing, the cyclic direction at the terminating vertex (at which one selects the next oriented edge) is taken to be opposite from the direction at the originating vertex.

Theorem 6.1 Twisting an edge $e$ in a general rotation system $\rho(G)$ satisfies the following rules:

(A) Suppose that the two trace-pairs induced by $e$ belong to the boundaries of two different faces in the imbedding. Then twisting $e$ merges the two faces into a single face;

(B) Suppose that the two trace-pairs induced by $e$ belong to the boundary of the same face $F$ in the imbedding.

(B1) If the two trace-pairs induced by $e$ use the same oriented edge, then twisting $e$ splits the face $F$ into two faces;

(B2) If the two trace-pairs induced by $e$ use different oriented edges, then twisting $e$ converts the face $F$ into a new single face.

The results of twisting an edge both of whose induced trace-pairs belong to the boundary walk of the same face, in particular case (B2) of Theorem 6.1, seem to have been absent from the existing literature in topological graph theory. It takes several pages of technical detail to close this gap.
7 General edge-inserts and edge-deletes

One might expect that most results for pure graph rotation systems would extend naturally to general graph rotation systems. However, there seem to be some subtle issues that are quite different, which, to our knowledge, have not been thoroughly studied in the literature.

Example 7.1 Figure 17(1) corresponds to a 1-face imbedding of the bouquet $B_1$ (one vertex with one self-loop) on the projective plane. In particular, face corners $c_1$ and $c_2$ in Figure 17(1) belong to the same face. Now suppose that we insert a new type-0 edge $e_2$ between these two face corners, as depicted in Figure 17(2).

![Figure 17: Inserting an edge into a general rotation system](image)

The rules in §2 for pure graph rotation systems say that an edge insertion (necessarily type-0 for pure rotation systems) between two corners of the same face would split that face into two faces. However, by applying the general face-tracing algorithm to Figure 17(2), we find out that the resulting rotation system corresponds to a 1-face imbedding of the bouquet $B_2$ (on the Klein bottle)!
Rules for edge-insertion surgery

Theorem 7.2 Suppose that we insert the ends of a type-0 edge e into two face corners c₁ and c₂ in a general rotation system ρ(G). Then the following rules hold:

(A) Suppose that corners c₁ and c₂ belong to two different faces. Then inserting edge e between c₁ and c₂ merges the two faces into a single face.

(B) Suppose that corners c₁ and c₂ belong to the same face. Then

(B1) if c₁ and c₂ have the same corner-type, then inserting edge e between c₁ and c₂ splits the face into two faces;

(B2) if c₁ and c₂ have different corner-types, then inserting e between c₁ and c₂ results in a new face.
Rules for edge-deletion surgery

Now we turn to edge-deletion on general rotation systems. Since deleting a type-1 edge $e$ can be implemented by first twisting $e$ then deleting the twisted $e$ that is of type-0, it is sufficient to focus on deleting a type-0 edge.

**Theorem 7.3** Deleting a type-0 edge $e$ from a general graph rotation system $\rho(G)$ satisfies the following rules:

(A) Suppose that the two trace-pairs induced by $e$ belong to the boundary walks of two different faces of the imbedding. Then deleting edge $e$ merges the two faces into a single face.

(B) Suppose that the two trace-pairs induced by $e$ both belong to the boundary walk of the same face $F$ in the imbedding.

(B1) If the two trace-pairs induced by edge $e$ use different oriented edges, then deleting edge $e$ splits the boundary walk of the face $F$ into two closed walks, each the boundary of a new face of the resulting imbedding.

(B2) If the two trace-pairs induced by edge $e = [u, w]$ use the same oriented edge, then deleting edge $e$ changes the boundary walk of face $F$ into the boundary walk of a single new face.
8 Extended graph rotation systems

Topologically, tracing a twisted edge “reverses” the local orientation of the rotation system. Accordingly, retwisting an edge is equivalent to untwisting. Here are the differences in our model for cyclic weaving:

- We record which segment goes over and which segment goes under at the crossing point.
- We record by how many turns an edge is twisted.

![Diagram showing different twist states](image)

Figure 18: (1) an untwisted edge. (2) a clockwise twisted edge. (3) a counterclockwise twisted edge. (4) a double clockwise twisted edge. (5) a double counterclockwise twisted edge.

An extended rotation system for a graph $G$ is obtained from a pure rotation system by assigning to every edge $e$, a number $k(e)$ of twists, with $k(e) \in \mathbb{Z}$. 
9  Cyclic plain-weaving on surfaces

Theorem 9.1 Let $\rho_0(G)$ be a pure rotation system for an imbedding $\pi_0 : G \to S$ of a graph on an orientable surface. Let $A$ be an arbitrary subset of edges of $G$. If we twist all edges in $A$ positively, or if we twist all edges in $A$ negatively, then the resulting extended rotation system induces a cyclic plain weaving on $S$.

Figure 19: Close-up view of very small ERS-weaves.

Figure 20: Some easily implemented ERS weaves.
**Theorem 9.2** Every cyclic plain-weaving on the sphere can be specified by an ERS.

Figure 21: Constructing the graph and the ERS for a link.

**Theorem 9.3** Every CELLULAR cyclic plain-weaving on an orientable surface can be specified by an ERS.
References


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