CS E6204 Lectures 9b and 10
Alexander-Conway and Jones Polynomials*

Abstract

Before the 1920's, there were a few scattered papers concerning knots, notably by K. F Gauss and by Max Dehn. Knot theory emerged as a distinct branch of topology under the influence of J. W. Alexander. Emil Artin, Kurt Reidemeister, and Herbert Seifert were other important pioneers. Knot theory came into maturity in the 1940's and 1950's under the influence of Ralph Fox.

Calculating knot polynomials is a standard way to decide whether two knots are equivalent. We now turn to knot polynomials, as presented by Kunio Murasugi, a distinguished knot theorist, with supplements on the classical Alexander matrix and knot colorings from *Knot Theory*, by Charles Livingston.

* Extracted from Chapters 6 and 11 of *Knot Theory and Its Applications*, by Kunio Murasugi (Univ. of Toronto).

Selections from Murasugi

§6.2 The Alexander-Conway polynomial

§6.3 Basic properties of the A-C polynomial

§11.1 The Jones polynomial

§11.2 Basic properties of the Jones polynomial

1 The Alexander-Conway polynomial

Calculating Alexander polynomials used to be a tedious chore, since it involved the evaluation of determinants. Conway made it easy by showing that they could be defined by a skein relation.

The ***Alexander-Conway polynomial*** $\nabla_K(z)$ for a knot K is a Laurent polynomial in z, which means it may have terms in which z has a negative exponent.

Axiom 1: If K is the trivial knot, then $\nabla_K(z) = 1$.

Axiom 2: For the skein diagrams D_+, D_-, and D_0 in Fig 1.1, the following skein relation holds:

$$\nabla_{D_+} \;=\; \nabla_{D_-}(z) \,+\, z\nabla_{D_0}(z) \tag{1.1}$$

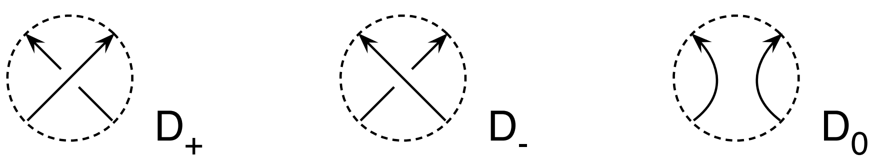


Figure 1.1: Diagrams for the Conway skein relation.

Remark In what follows, we take $\nabla_K(z)$ to be invariant under Reidemeister moves.

Proposition 1.1 *Let O_μ be the trivial link with μ components. Then*

$$\nabla_{O_\mu}(z) \;=\; 0 \qquad \textit{for } \mu \geq 2 \tag{1.2}$$

Proof In the Conway skein

$$\nabla_{D_+}(z) \;=\; \nabla_{D_-}(z) + z\nabla_{D_0}(z)$$

for the link of Figure 1.2, we have

$$D_+ = D_{O_{\mu-1}} \qquad D_- = D_{O_{\mu-1}} \qquad D_0(z) = D_{O_\mu}$$

Thus, $\nabla_{D_+}(z) = \nabla_{D_-}(z)$, and

$$z\nabla_{D_0}(z) \;=\; \nabla_{D_+}(z) - \nabla_{D_-}(z) \;=\; 0$$

from which it follows that

$$\nabla_{O_\mu}(z) \;=\; \nabla_{D_0}(z) \;=\; 0$$

$\diamondsuit$

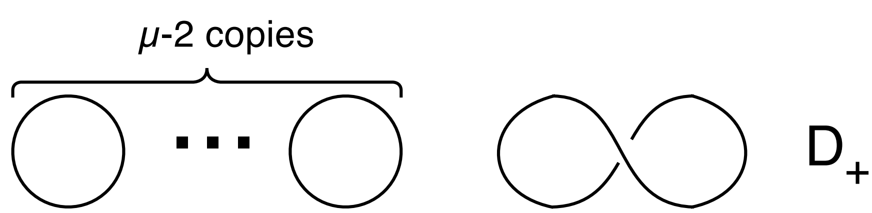


Figure 1.2: D_+ in a skein for O_μ.

Example: Right Trefoil Knot

We use a ***skein tree*** with the knot itself at the root and instances of O_μ at its leaves.

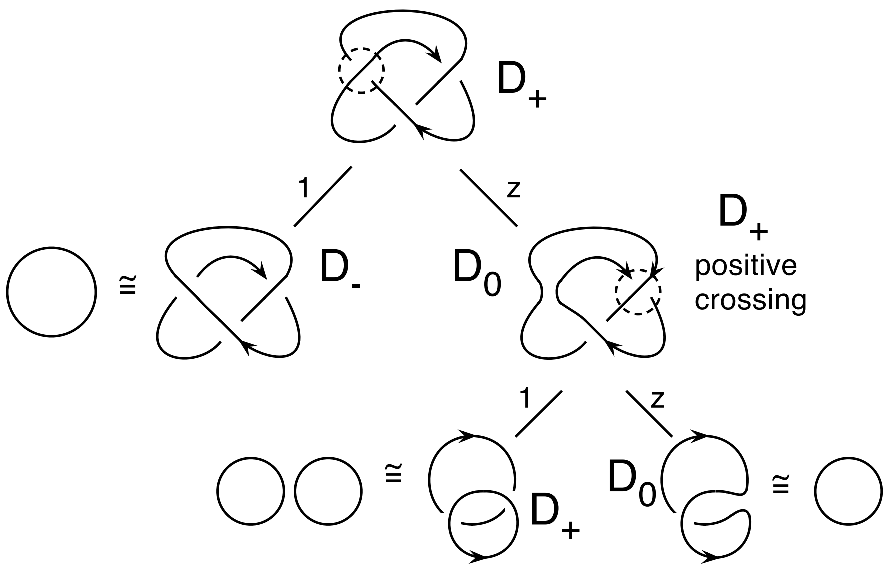


Figure 1.3: Skein tree for the right trefoil knot.

$$\begin{aligned}
\nabla_{3_1} &= 1 \cdot \nabla_{O_1} + z \cdot 1 \cdot \nabla_{O_2} + z^2 \cdot \nabla_{O_1} \\
&= 1 \cdot 1 + z \cdot 1 \cdot 0 + z^2 \cdot 1 \\
&= 1 + z^2 \qquad (1.3)
\end{aligned}$$

The (classical) ***Alexander polynomial*** $\Delta_K(t)$ is obtained by substituting

$$z = t^{1/2} - t^{-1/2}$$

into the Alexander-Conway polynomial. Thus,

$$\begin{aligned}
\Delta_{3_1} &= 1 + (t^{1/2} - t^{-1/2})^2 \\
&= 1 + t - 2 + t^{-1} = t^{-1} - 1 + t \qquad (1.4)
\end{aligned}$$

Example: Figure8 Knot

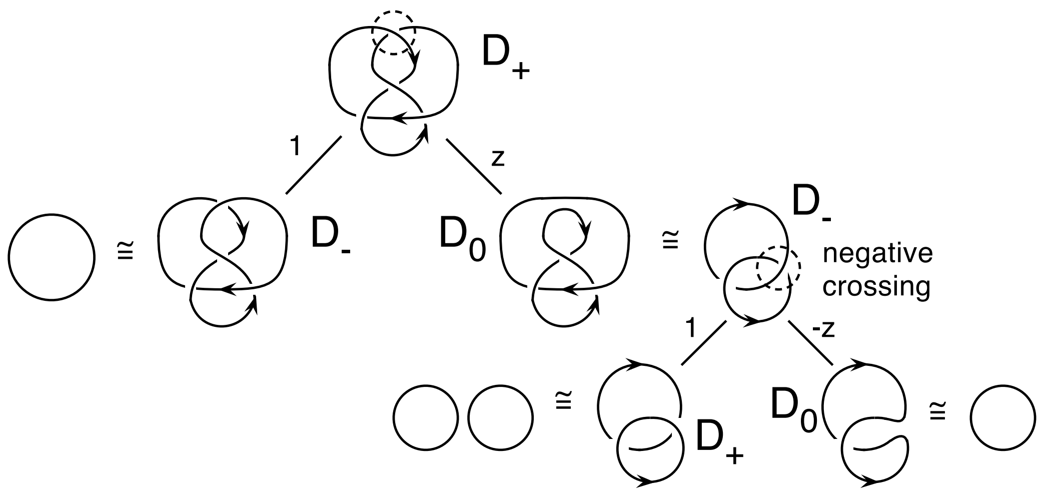


Figure 1.4: Skein tree for the figure-eight knot.

$$\begin{aligned}\nabla_{4_1} &= 1 \cdot \nabla_{O_1} + z \cdot 1 \cdot \nabla_{O_2} - z^2 \cdot \nabla_{O_1} \\ &= 1 \cdot 1 + z \cdot 1 \cdot 0 - z^2 \cdot 1 \\ &= 1 - z^2 \qquad (1.5)\end{aligned}$$

Thus,

$$\begin{aligned}\Delta_{4_1} &= 1 - (t^{1/2} - t^{-1/2})^2 \\ &= 1 - t + 2 - t^{-1} = -t^{-1} + 3 - t \qquad (1.6)\end{aligned}$$

Example: Hopf Links

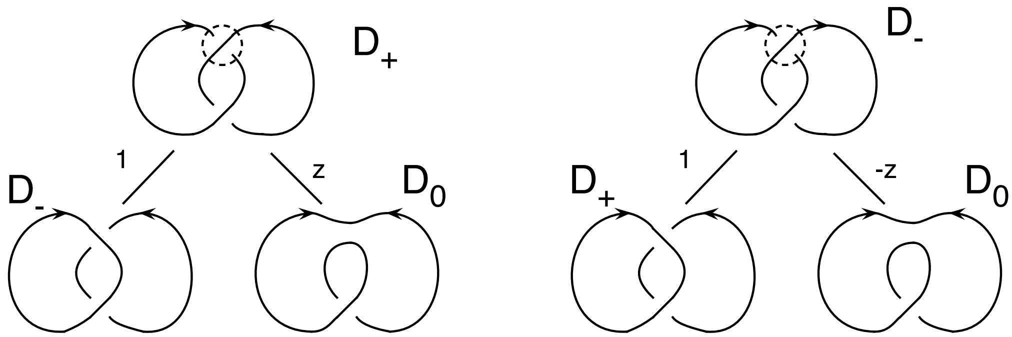


Figure 1.5: Skein trees for the Hopf links.

$$\begin{aligned}\nabla_{H_+} &= 1 \cdot \nabla_{O_2} + z \cdot 1 \cdot \nabla_{O_1} \\ &= 0 + z = z \end{aligned} \tag{1.7}$$

and, thus,

$$\Delta_{H_+} = t^{1/2} - t^{-1/2} \tag{1.8}$$

$$\begin{aligned}\nabla_{H_-} &= 1 \cdot \nabla_{O_2} - z \cdot 1 \cdot \nabla_{O_1} \\ &= 0 - z = -z \end{aligned} \tag{1.9}$$

and, thus,

$$\Delta_{H_-} = t^{-1/2} - t^{1/2} \tag{1.10}$$

Example: Whitehead Link

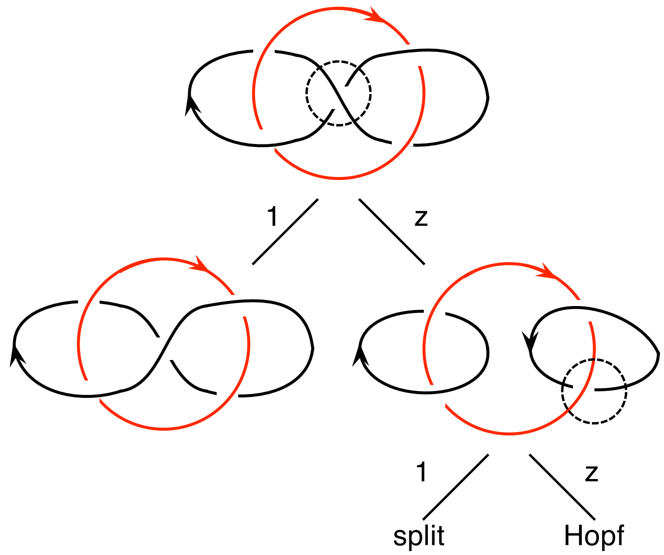


Figure 1.6: Skein tree for the Whitehead link.

$$\begin{aligned}\nabla_{5_1^2} &= 1 \cdot \nabla_{O_2} + z^2 \cdot \nabla_{H_+} \\ &= 0 + z^2 \cdot z = z^3 \qquad (1.11)\end{aligned}$$

Thus,

$$\begin{aligned}\Delta_{5_1^2} &= (t^{1/2} - t^{-1/2})^3 \\ &= -t^{-3/2} + 3t^{-1/2} - 3t^{1/2} + t^{3/2} \qquad (1.12)\end{aligned}$$

2 The classical Alexander polynomial

Let K be an oriented knot (or link) with n crossings. Label the crossings 1, 2, $\ldots$, n, and label the n arcs y_1, y_2, $\ldots$, y_n, as in Figure 2.1. We observe that all three crossings are positive.

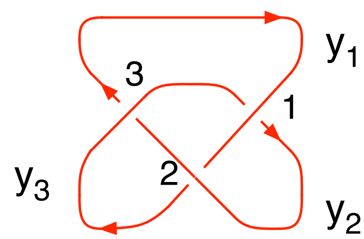


Figure 2.1: Trefoil knot with crossings and arcs labeled.

Construct an $n \times n$ matrix M, such that row r corresponds to the crossing labeled r and column s corresponds to the arc labeled y_s. Suppose that at crossing r the overpassing arc is labeled y_i, that arc y_j ends at crossing r, and that arc y_k begins at crossing r. Suppose also that i, j, and k are mutally distinct. Suppose also that crossing r is positive. Then

$$M(r,i) = 1-t \qquad M(r,j) = -1 \qquad M(r,k) = t$$

and $A_K(r,s) = 0$, otherwise. Thus, when K is the trefoil,

$$M \;=\; \begin{pmatrix} 1-t & t & -1 \\ -1 & 1-t & t \\ t & -1 & 1-t \end{pmatrix}$$

When crossing r is negative, then

$$M(r,i) = 1-t \qquad M(r,j) = t \qquad M(r,k) = -1$$

In the exceptional case where the three arcs are not distinct, the sum of the entries described above goes into the appropriate column.

The ***Alexander matrix*** A_K is defined to be the matrix obtained from matrix M by deleting row n and column n. Thus, the Alexander matrix for the trefoil knot is

$$A_K = \begin{pmatrix} 1-t & t \\ -1 & 1-t \end{pmatrix}$$

We observe that the Alexander matrix is unique, at best, only up to a permutation of the rows and columns. (It is, in fact, not quite unique even in that sense.)

The ***(classical) Alexander polynomial*** $\Delta_K(t)$ of a knot K is the determinant of its Alexander matrix.

For the trefoil knot, we obtain

$$\Delta_{3_1}(t) = (1-t)^2 + t = 1 - t + t^2$$

We recall that we previously calculated that

$$\Delta_{3_1}(t) = t^{-1} - 1 + t$$

We regard two Alexander polynomials as equivalent if they differ only by multiplication by $\pm t^k$.

Example 2.1 (Figure-8 Knot)

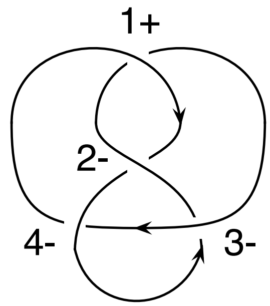


Figure 2.2: Figure-8 knot with crossings and arcs labeled.

$$A_K \;=\; \begin{pmatrix} 1-t & t & -1 \\ t & 1-t & 0 \\ 0 & t & 1-t \end{pmatrix}$$

$$\begin{aligned} \Delta_{4_1}(t) &= (1-t)\cdot(1-t)^3 \;-\; t\cdot t(1-t) \;+\; (-1)\cdot t^2 \\ &= (1-3t+3t^2-t^3) \;-\; (t^2-t^3) \;-\; t^2 \\ &= 1-3t+t^2 \end{aligned}$$

We previously calculated

$$\Delta_{4_1} \;=\; -\,t^{-1}+3-t$$

which is equivalent, under the rules of equivalency given here.

3 Properties of the A-C polynomial

Proposition 3.1 *For any link K, we have*

$$\nabla_L(0) \;=\; \nabla_{O_\mu}(0) \;=\; (\mu = 1)$$

Proof The skein relation

$$\nabla_{D_+}(z) \;=\; \nabla_{D_-}(z) \,+\, z\nabla_{D_0}(z)$$

implies that

$$\nabla_{D_+}(0) \;=\; \nabla_{D_-}(0)$$

Accordingly, the value of $\nabla_L(0)$ is unchanged when the crossing-type is changed. Changing sufficiently many crossings-types turns any knot into a split union of unknots. Thus,

$$\nabla_L(0) \;=\; \nabla_{O_\mu}(0) \qquad \diamondsuit$$

Corollary 3.2 *For any link L, we have $\Delta_L(1) \;=\; (\mu(L) = 1)$.*

Proof We have defined the Alexander polynomial $\Delta_L(t)$ by the equation

$$\Delta_L(t) \;=\; \nabla_L(t^{1/2} - t^{-1/2})$$

Thus, by Proposition 3.1, we have

$$\Delta_L(1) \;=\; \nabla_L(0) \;=\; (\mu(L) = 1) \qquad \diamondsuit$$

4 The Jones polynomial

The ***Jones polynomial*** $V_K(t)$ for a knot K is a Laurent polynomial in $\sqrt{t}$, which means it may have terms in which $\sqrt{t}$ has a negative exponent. We write t for $\sqrt{t}^{\,2}$.

Axiom 1: If K is the trivial knot, then $V_K(t) = 1$.

Axiom 2: For the skein diagrams D_+, D_-, and D_0 in Fig 4.1, the following skein relation holds:

$$\frac{1}{t}\,V_{D_+}(t) \;-\; t\,V_{D_-}(t) \;=\; \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{D_0}(t) \qquad (4.1)$$

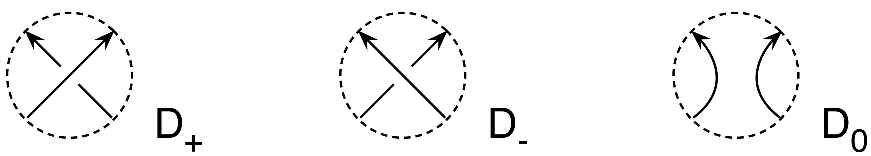


Figure 4.1: Diagrams for the Jones skein relation.

We recall these two relations:

$$V_{D_+}(t) \;=\; t^2 V_{D_-}(t) \,+\, tzV_{D_0}(t) \qquad (4.2)$$

$$V_{D_-}(t) \;=\; t^{-2} V_{D_+}(t) \,-\, t^{-1}zV_{D_0}(t) \qquad (4.3)$$

where

$$z \;=\; \left(\sqrt{t} \,-\, \frac{1}{\sqrt{t}}\right) \qquad (4.4)$$

We have previously proved the following:

Proposition 4.1 *Let O_μ be a trivial link with μ components. Then*

$$V_{O_\mu}(t) \;=\; (-1)^{\mu-1}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{\mu-1} \qquad (4.5)$$

A ***skein-tree diagram*** reduces the calculation of $V_K(t)$ to applications of Prop 4.1 at the nodes of a tree.

Example 4.1 (The Figure-Eight Knot 4_1)

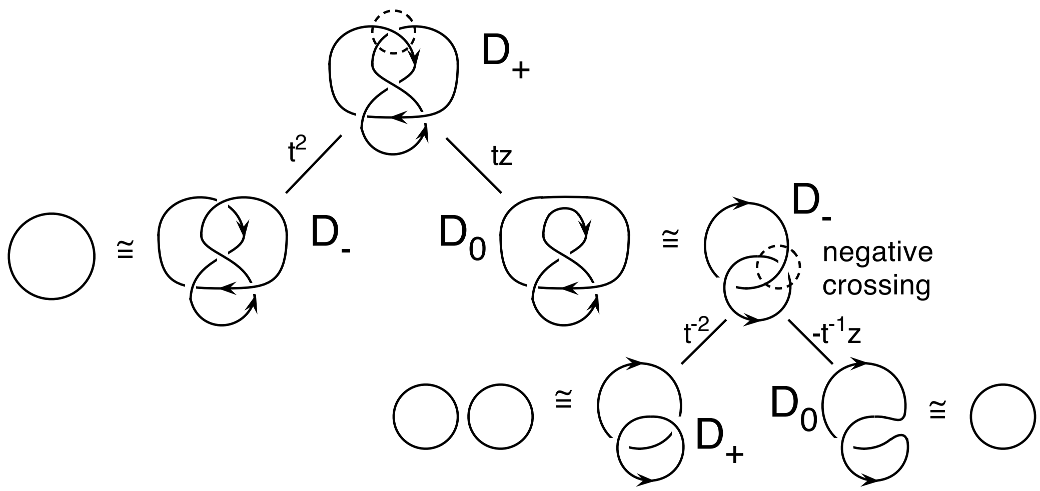


Figure 4.2: Skein diagram for the figure-eight knot.

$$\begin{aligned} V(4_1)(t) &= t^2V(O_1)(t) + t^{-1}zV(O_2)(t) - z^2V(O_1)(t) \\ &= t^2 - t^{-1}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right) - \left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^2 \\ &= t^{-2} - t^{-1} + 1 - t + t^2 \end{aligned}$$

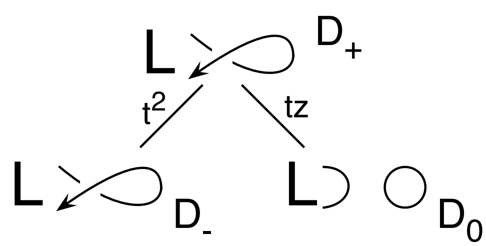
5 Basic properties of the Jones polynomial

NOTATION We denote by $L \sqcup L'$ a split union of links L and L'.

Proposition 5.1 *Let L be any link. Then*

$$V_{L\sqcup O_1}(t) \;=\; -\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)V_L(t) \tag{5.1}$$

Proof Consider the following skein tree:



We calculate

$$\begin{aligned} V(L)(t) &= t^2V(L)(t) \,+\, tzV(L\sqcup O)(t) \\ \therefore V(L\sqcup O)(t) &= \frac{1-t^2}{tz}V(L)(t) \;=\; \frac{t^{-1}-t}{\sqrt{t}-\sqrt{t}^{-1}} \\ &= (-1)\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)V(L)(t) \end{aligned}$$

Proposition 5.2 *Let O_μ be a trivial link with μ components. Then*

$$V_{L\sqcup O_\mu}(t) \;=\; (-1)^\mu\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^\mu V_L(t) \tag{5.2}$$

Proof Induction on Proposition 5.1. $\diamondsuit$

NOTATION The connected sum of two knots is denoted $K_1 \# K_2$.

Theorem 5.3 *Let K_1 and K_2 be knots. Then*

$$V_{K_1 \# K_2}(t) \;=\; V_{K_1}(t) V_{K_2}(t) \tag{5.3}$$

Proof Suppose we temporarily regard K_2 as a point on K_1 and we expand the skein tree for K_1 as follows:

$$V_{K_1}(t) \;=\; f_1(t) V_{O_1}(t) + \cdots + f_m(t) V_{O_m}(t)$$

By replacing the special point by a copy of K_2, we obtain

$$\begin{aligned} V_{K_1 \# K_2}(t) \;=\; & f_1(t) V_{K_2}(t) + f_2(t) V_{K_2 \sqcup O_1}(t) + \cdots \\ & + f_{m-1}(t) V_{K_2 \sqcup O_{m-2}}(t) + f_m(t) V_{K_2 \sqcup O_{m-1}}(t) \end{aligned}$$

By Proposition 5.2, we have

$$V_{K_2 \sqcup O_\mu}(t) \;=\; (-1)^\mu \left(\sqrt{t} + \frac{1}{\sqrt{t}} \right)^\mu V_{K_2}(t)$$

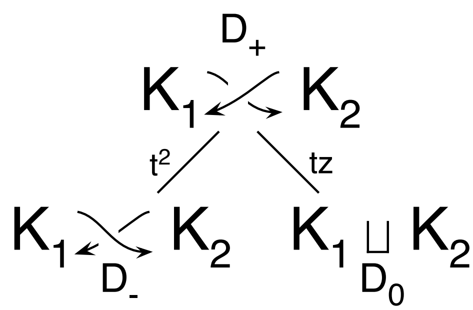
Combining these two equations yields

$$\begin{aligned} V_{K_1 \# K_2}(t) \;&=\; f_1(t) V_{K_2}(t) + f_2(t) V_{O_1} V_{K_2}(t) + \cdots \\ & \qquad + f_{m-1}(t) V_{O_{m-2}} V_{K_2}(t) + f_m(t) V_{O_{m-1}} V_{K_2}(t) \\ &=\; \big(f_1(t) + f_2(t) V_{O_1} + \cdots \\ & \qquad + f_{m-1}(t) V_{O_{m-2}} + f_m(t) V_{O_{m-1}} \big) V_{K_2}(t) \\ &=\; V_{K_1}(t) V_{K_2}(t) \qquad \diamondsuit \end{aligned}$$

Theorem 5.4 *Let K_1 and K_2 be knots. Then*

$$V_{K_1 \sqcup K_2}(t) \;=\; -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) V_{K_1}(t)\, V_{K_2}(t) \qquad (5.4)$$

Proof From this skein-tree



we calculate

$$\begin{aligned} V_{K_1 \# K_2}(t) &= t^2\, V_{K_1 \# K_2}(t) \,+\, tz\, V_{K_1 \sqcup K_2}(t) \\ \therefore V_{K_1 \sqcup K_2}(t) &= \frac{1-t^2}{tz} V_{K_1 \# K_2}(t) \;=\; \frac{t^{-1}-t}{\sqrt{t}-\sqrt{t}^{-1}} \\ &= (-1)\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) V_{K_1}(t)\, V_{K_2}(t) \qquad \diamondsuit \end{aligned}$$

Theorem 5.5 *Let $-K$ be the knot obtained by reversing the orientation on knot K. Then*

$$V_{-K}(t) \;=\; V_K(t)$$

Proof The knots $-K$ and K have the same skein tree, because each crossing retains its sign. $\diamondsuit$

Theorem 5.6 *Let K^* be the mirror image of the knot K. Then*

$$V_{K*}(t) \;=\; V_K(t^{-1})$$

Proof Suppose we regard z as a function

$$z(t) \;=\; t^{1/2} - t^{-1/2}$$

Then

$$z(t^{-1}) \;=\; t^{-1/2} - t^{1/2} \;=\; -\,z(t)$$

Accordingly, the skein relations

$$V_{D_+}(t) \;=\; t^2 V_{D_-}(t) \,+\, tz V_{D_0}(t) \tag{5.5}$$

$$V_{D_-}(t) \;=\; t^{-2} V_{D_+}(t) \,-\, t^{-1} z V_{D_0}(t) \tag{5.6}$$

could be rewritten as

$$V_{D_+}(t) \;=\; t^2 V_{D_-}(t) \,+\, tz(t) V_{D_0}(t) \tag{5.7}$$

$$V_{D_-}(t) \;=\; t^{-2} V_{D_+}(t) \,+\, t^{-1} z(t^{-1}) V_{D_0}(t) \tag{5.8}$$

In transforming the skein tree for K into the skein tree for K^*, each application of Equation (5.5) is replaced by an application of Equation (5.6), and vice versa. $\diamondsuit$

A knot is ***amphichiral*** if it is equivalent to its mirror image.

Corollary 5.7 *If K is amphichiral, then its Jones polynomial is symmetric.*

Proof This follows immediately from Theorem 5.6. $\diamondsuit$

Example 5.1 The trefoil knot is not amphichiral, since its Jones polynomial is

$$V_{3_1}(t) \;=\; t + t^3 - t^4$$

Proposition 5.8 *Let L be an oriented link with μ components. Then*

$$V_L(1) \;=\; (-2)^{\mu-1} \tag{5.9}$$

Proof For $t = 1$, the skein equation (see Eq (4.2))

$$V_{D_+}(t) \;=\; t^2 V_{D_-}(t) \;+\; zt V_{D_0}(t)$$

becomes (since $z(1) = 0$)

$$V_{D_+}(1) \;=\; V_{D_-}(1)$$

It follows that changing overcrossings to undercrossings or vice versa has no effect on the value of $V_L(1)$. Of course, one could select a collection of crossing-type reversals that reduces L to the trivial link with μ components. Thus, by Proposition 4.1, we have

$$\begin{aligned} V_L(1) \;&=\; V_{O_\mu}(1) \\ &=\; (-1)^{\mu-1}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{\mu-1}\Bigg|_t = 1 \\ &=\; (-1)^{\mu-1} 2^{\mu-1} \;=\; (-2)^{\mu-1} \end{aligned}$$

$\diamondsuit$