CS E6204 Lectures 8b and 9a
Knots and Graphs*

Abstract

Heffter's invention of rotation systems made it possible to investigate imbeddings of graphs from a predominantly combinatorial perspective. Analogously, the Seifert matrix and the Alexander matrix made knot theory amenable to algebraic methods. Subsequently, Conway's rederivation of the Alexander polynomial with skeins and the discovery of the Jones polynomial, by Jones, and its subsequent combinatorialization by Kauffman and others has greatly facilitated the calculation of knot and link invariants.

* Extracted from Chapters 16,17 of *Algebraic Graph Theory*, by Chris Godsil (U of Waterloo) and Gordon Royle (U of Western Australia).

Chapter 16 of Godsil and Royle

1. Knots and their projections
2. Reidemeister moves on knots
3. Signed plane graphs
4. Reidemeister moves on graphs
5. Reidemeister invariants
6. The Kauffman bracket polynomial
7. The Jones polynomial
8. Connectivity

1 Knots and Their Projections

A ***knot*** is a piecewise-linear (abbr. PL) mapping of the circle S^1 into S^3. We often refer to the image of a knot as the knot. In dimension 3, the PL category is equivalent to the differentiable category, so **we may think of knots as smooth closed curves**.

A ***link*** is a collection of pairwise disjoint knots. For simplicity of exposition, we may sometimes say "knot" when our meaning is either a knot or a link.

A ***normal projection*** of a link (aka ***shadow***) is a 4-regular graph imbedded in the plane.

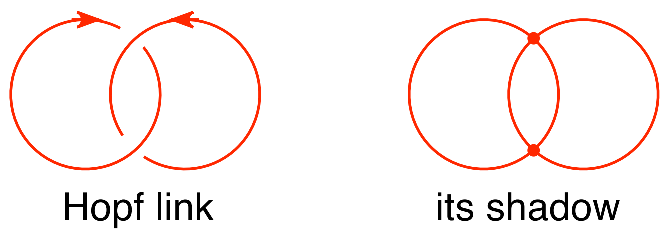


Figure 1.1: The Hopf link and its shadow.

A usual way to represent a link is a ***link diagram*** in which a shadow of the link is augmented so as to indicate at each crossing which strand of the link is "closer to the source of light" (the ***overcrossing***).

The usual augmentation is to install an opening in the strand containing the undercrossing, as illustrated in this figure of a trefoil knot.

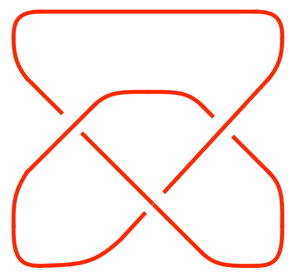


Figure 1.2: Knot diagram = augmented shadow.

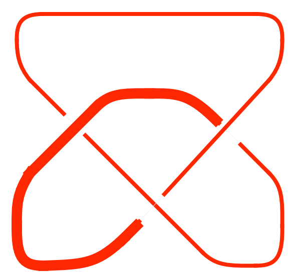


Figure 1.3: An arc of a knot.

We observe that when making a complete traversal of all the components of a link, each crossing occurs once as an overcrossing and once as an undercrossing. Since the number of arcs of a knot equals the number of undercrossings (except for the unknot), it follows that the number of arcs equals the number of crossings.

The links L and L' are equivalent if there is a homeomorphism

$$\phi : S^3 \times [0,1] \to S^3 \times [0,1]$$

such that

- $\phi|_{S^3 \times \{0\}}$ is the identity mapping;
- $\phi|_{S^3 \times \{1\}}$ maps $L \times \{1\}$ to $L' \times \{1\}$.

Such a mapping ϕ is called an ***ambient isotopy*** from the link L to the link L'. One imagines the link L being deformed little by little into the link L'. Imagine a continuum of links, with initial link L and final link L'.

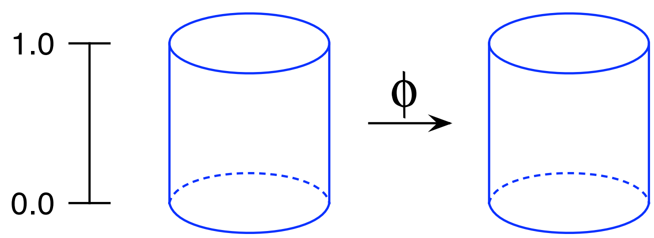


Figure 1.4: Conceptualization of an isotopy.

Alternatively, two links are ***equivalent*** if there is an orientation-preserving homeomorphism of pairs

$$h : (S^3,\, L) \to (S^3,\, L')$$

Any knot that is equivalent to a circle in the xy-plane is called an ***unknot***.

A link that is equivalent to its mirror image is ***amphichiral***. Famously, the right trefoil and the left trefoil are not equivalent.

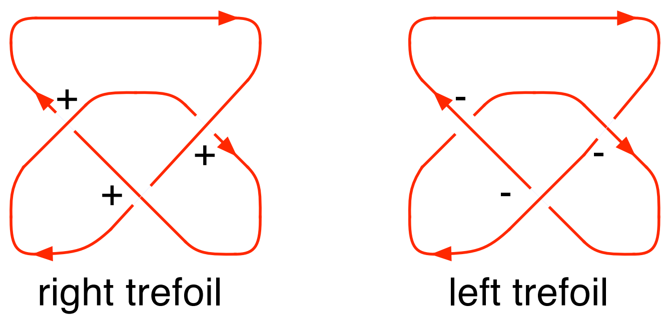


Figure 1.5: Right and left trefoil knots.

In a projection of the right trefoil knot, the angle of motion from the forward direction on the overcrossing strand ("positive x direction) to the forward direction on the undercrossing strand ("positive y direction") is counterclockwise. (Your right thumb points in the positive z direction.) In a projection of the left trefoil knot, it is clockwise.

Remark It does not matter which way you orient the two trefoil knots. These rules of thumb give consistent results.

2 Reidemeister Moves

The three Reidemeister moves can clearly be implemented by isotopies. Accordingly, they preserve knot type.

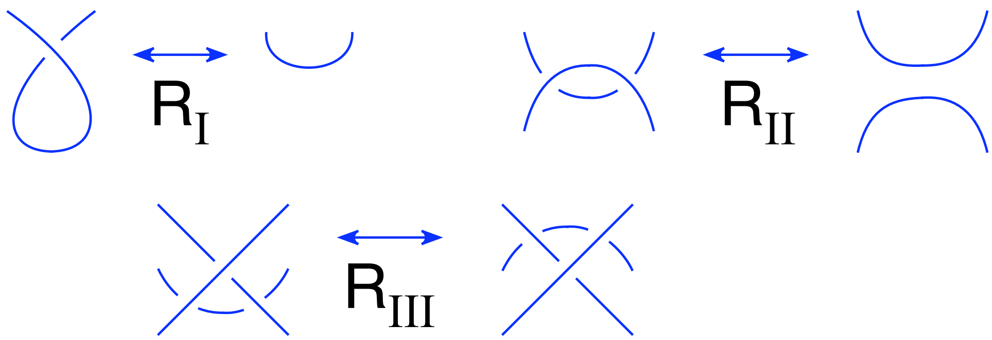


Figure 2.1: The three Reidemeister moves.

The following theorem is attributed to Reidemeister [Rei32]. It is a major step in the combinatorialization of links. The proof (omitted) is topological.

Theorem 2.1 *Two link diagrams determine the same link if and only if one can be obtained from the other by a sequence of Reidemeister moves and planar isotopies.*

The fundamental problem of know theory is to decide whether two knots are equivalent. Since there are infinitely possible sequences of Reidemeister moves, one cannot simply try them all. Wolfgang Haken [**?**] produced an algorithm to make this decision, but it is too complicated to be practical.

A property of links, such as a number, a polynomial, or a matrix, whose value is unchanged by any of the Reidemeister moves is called a ***link invariant***. Thus, if a link invariant has different values on two different diagrams, then the two diagrams represent different links.

Example 2.1 We define a ***proper 3-coloring of a link*** to be an onto assignment of three colors to the arcs so that at each crossing

- either all three colors occur, or
- only one color occurs

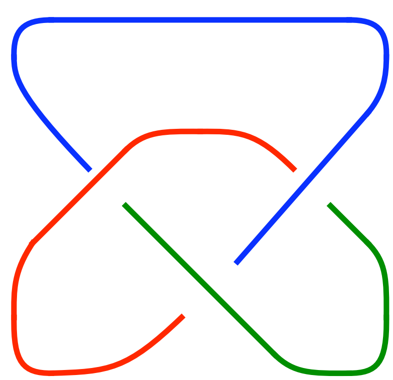


Figure 2.2: A 3-coloring of the trefoil knot.

Example 2.2 We now prove that the diagram below of the Whitehead link has no 3-coloring.

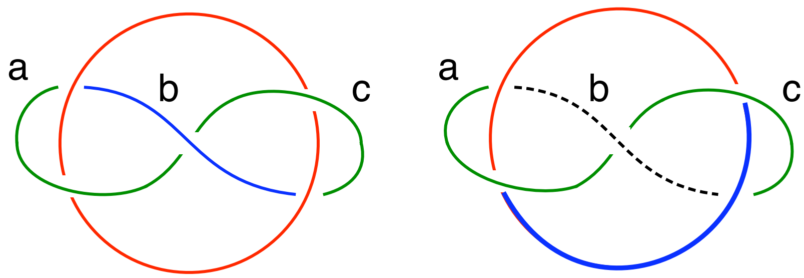
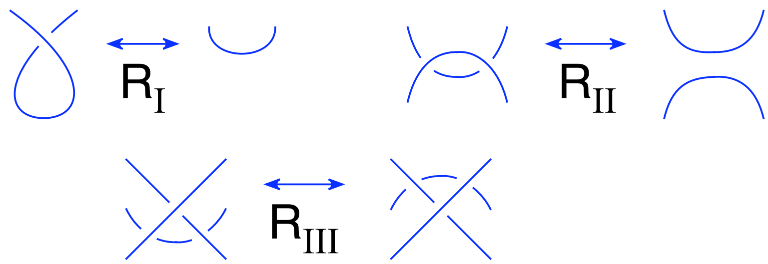


Figure 2.3: Attempts at 3-coloring the Whitehead link.

On the left, we suppose that the upper and lower semi-circles are assigned the same color, say, red. If arc b is red, then arcs a and c would have to be red also. If arc b is blue, as shown, then arcs a and c both have to be a third colr, say, green. However, this creates three crossing with two colors.

On the right, we suppose that the upper and lower semi-circles are assigned red and blue, respectively. This forces arcs a and c to be green, the third color. However, then there is no satisfactory color for arc b.

Theorem 2.2 *3-colorability is invariant under all three Reidemeister moves.*



Proof For R_I, the same color must be assigned to both arcs on the left. Use it again for the arc on the right.

For R_{II}, if both arcs on the right have the same color, then use that color for all four arcs on the left. If those two arcs are colored differently, then assign the third color to the short middle arc on the left.

For R_{III}, if the big X uses only one color, then the three curved arcs all have that same color. If the big X uses three colors, then there are several cases to be considered: in each of them, the 3-coloring property can be preserved by changing the color of only the middle curved arc. $\diamondsuit$

3 Signed Plane Graphs

The shadow of a projection of an inseparable link is a connected 4-regular plane graph. The following proposition estrablishes that we can properly bi-color the map of a shadow. Our convention is to color the exterior region white.

Proposition 3.1 *The dual of a connected 4-regular plane graph is bipartite.*

Proof Every cycle of the dual graph separates the plane, by the Jordan curve theorem, and thus, it is a boundary cycle. Each fb-walk in the dual imbedding has length 4, because the primal graph is 4-regular. Every boundary cycle in any imbedding is a sum modulo 2 of the edges in a set of bd-walks, which implies that its length is even. $\diamondsuit$

We define the ***black face-graph*** as follows:

- We place a black vertex in the interior of each black region.
- Through each vertex v of the shadow, we draw a black edge joining the black vertices in the two black regions incident on v. If one black region is twice incident on v, we draw a self-loop.

We define the ***white face-graph*** similarly.

Example 3.1 Figure 3.1 shows a knot projection with 9 crossings. Its shadow is a 4-regular plane graph with 9 vertices and 11 faces.

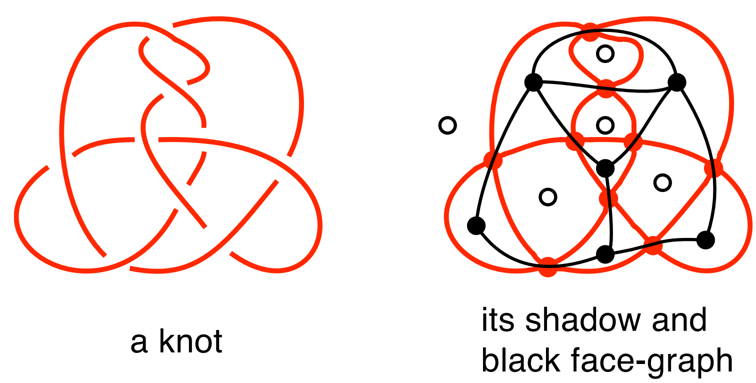


Figure 3.1: A knot, its shadow, and its black face-graph.

We observe the following:

- The shadow (red graph) is the medial graph of the black face-graph.
- The shadow is also the medial graph of the white face-graph.

If a link projection has k crossings, then there are 2^k links that share the corresponding shadow. We now describe a way to assign labels $+$ and $-$ to the k edges of the face-graphs.

At each vertex of the shadow, if the angular direction from the overcrossing strand to the black edge to the undercrossing strand is counterclockwise, than assign $+$ to the black edge; if is is clockwise, then assign the label -. The sign on a white edge is similarly assigned, which implies that it is the opposite sign from that of the black edge that it crosses.

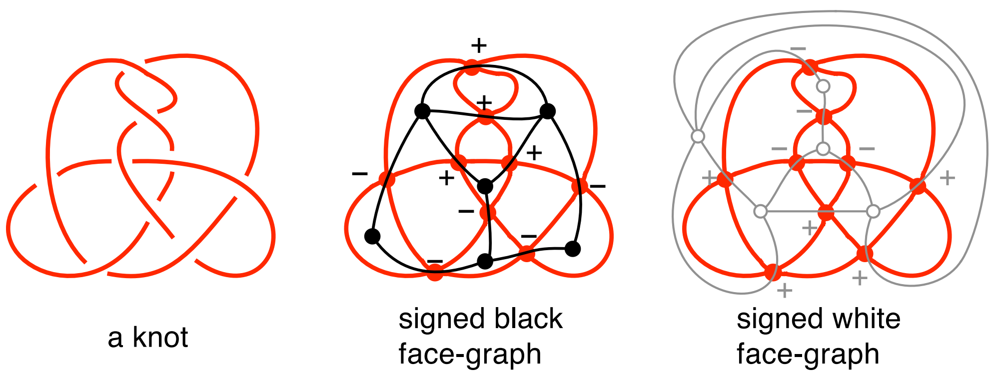


Figure 3.2: A knot and and its two signed face-graphs.

Proposition 3.2 *Every signed plane graph corresponds to a unique link projection.*

Proof Draw the medial graph and implement the overcrossings and undercrossings associated with the respective signs. $\diamondsuit$

4 Reidemeister Moves on Graphs

Each Reidemeister move on a link projection is representable by an operation on the corresponding pair of signed graphs.

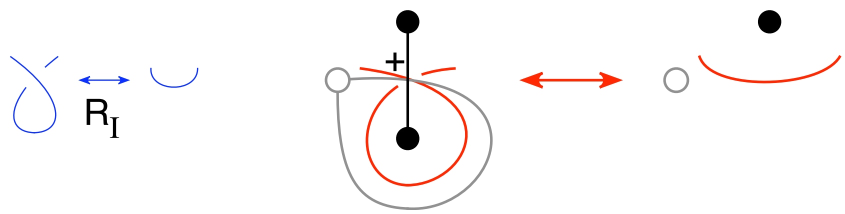


Figure 4.1: RGM1: delete a self-loop in one signed graph, and contract a spike in the other.

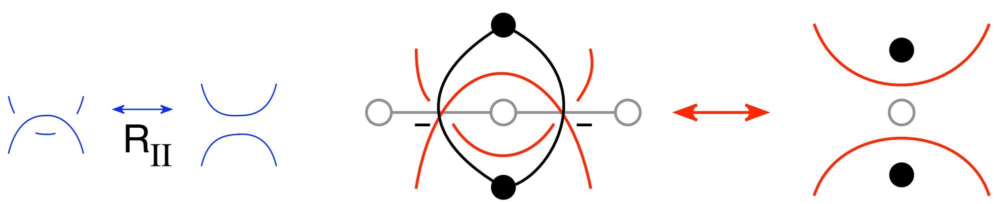


Figure 4.2: RGM2: contract a 2-path in one signed graph, and delete two "parallel" edges in the other.

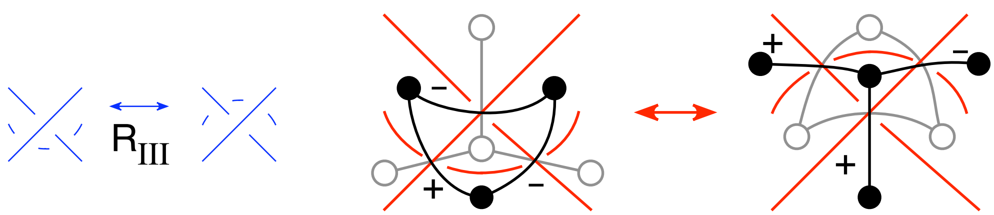


Figure 4.3: RGM3: change $K_{1,3}$ in one graph into C_3, and C_3 into $K_{1,3}$ in the other.

7 The Jones Polynomial

We skip §5 of the text, because it disgresses into an unneeded matroidal derivation of link invariants. We skip §6, because we have already defined the Kauffman bracket polynomial.

NOTATION We switch from $< L >$ to $[L]$ to denote the bracket polynomial of a link.

Consider traversing the undercrossing strand at a crossing of an oriented link, in the direction of orientation. If the oriented overcrossing strand goes left to right, we say that the crossing is ***left-handed***. Otherwise we say the crossing is ***right-handed***.

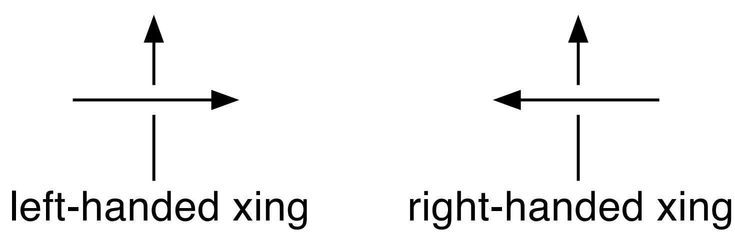


Figure 7.1: Left-handed and right-handed crossings.

Kauffman Bracket Polynomials

We recall the axioms for Kauffman's bracket polynomial.

Kauffman's ***bracket polynomial*** is defined by three axioms:

Axiom 1. $[\bigcirc] \; = \; 1$

Axiom 2u. $[\times] \; = \; A[\,)\,(\,] \, + \, A^{-1}[\asymp]$ $\qquad$ $\diagup$-overcross

Axiom 2d. $[\times] \; = \; A[\asymp] \, + \, A^{-1}[\,)\,(\,]$ $\qquad$ $\diagdown$-overcross

Axiom 3. $[L \cup \bigcirc] \; = \; (-A^2 - A^{-2})[L]$

Both parts of Axiom 2 can be combined into a single axiom:

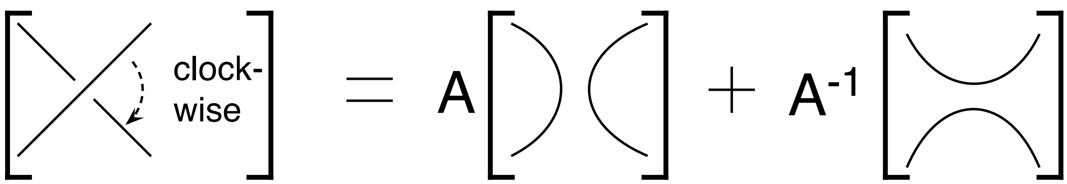


Figure 7.2: Unified skein for bracket polynomial.

X-Polynomials

The ***writhe of a link diagram*** L is the number of left-handed crossings minus the number of right-handed crossings. It is denoted $wr(L)$.

Example 7.1 Figure 7.3 shows that the right trefoil has writhe 3, and that the left trefoil has writhe -3.

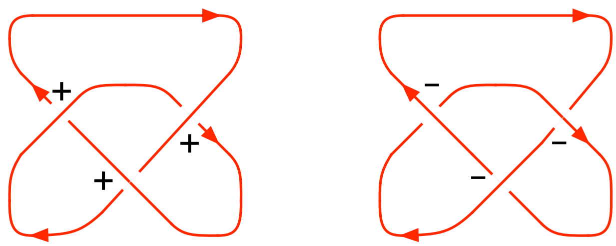


Figure 7.3: Left and right trefoil knots.

We next define the ***normalized bracket polynomial***

$$X(L) \; = \; (-A^3)^{-wr(L)}[L]$$

which is also called the X***-polynomial***.

Remark There is a typo in the text. The negative sign in the exponent is omitted.

Theorem 7.1 *The X-polynomial is invariant under the three Reidemeister moves.*

Proof for RM1

$$X(L) = (-A^3)^{-wr(L)-1}[L] = (-A^3)^{-wr(L)}(-A^3)^{-1}\{A[L,O] + A^{-1}[L\supset]\}$$
$$= (-A^3)^{-wr(L)}(-A^{-3})\{A(-A^2-A^{-2})[L] + A^{-1}[L]\}$$
$$= (-A^3)^{-wr(L)}\{(1+A^{-4})[L] - A^{-4}[L]\} = (-A^3)^{-wr(L)}[L] = X[L]$$

Proof for RM2

$$[L] = A[L] + A^{-1}[L]$$
$$= A\{A[L]+A^{-1}[L]\} + A^{-1}\{A[L)(]+A^{-1}[L]\}$$
$$= (A^2+A^{-2})[L] + [L)(] + (-A^2-A^{-2})[L]\} = [L)(]$$
$$\therefore X[L] = X[L)(]$$

Proof for RM2

$$[L] = A[L] + A^{-1}[L] \quad \text{now apply invariance under RM2}$$
$$= A[L] + A^{-1}[L] = [L]$$
$$\therefore X[L] = X[L]$$

The ***Jones polynomial*** $V_L(t)$ of an oriented link is obtained by substituting $t^{1/4}$ for A in the X-polynomial $X_L(A)$. It is a ***Laurent polynomial***, which means that negative degrees are allowed.

Trivial link with μ Components

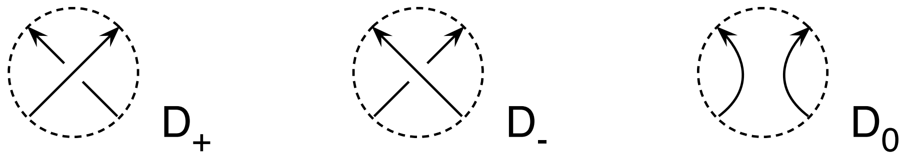
Let O_μ be a trivial link with μ components. Then be applying induction, we obtain

$$\begin{aligned} [O_1] &= 1 \\ [O_m] &= \left(-A^2 - A^{-2}\right)[O_{m-1}] \\ \therefore [O_\mu] &= (-1)^{\mu-1}\left(A^2 + A^{-2}\right)^{\mu-1} \end{aligned}$$

and we continue

$$\begin{aligned} \therefore X(O_\mu) &= (-1)^{\mu-1}\left(A^2 + A^{-2}\right)^{\mu-1} \\ \therefore V_{O_\mu}(t) &= (-1)^{\mu-1}\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{\mu-1} \end{aligned} \tag{7.1}$$

To reduce the number of steps in the calculation of Jones polynomials, we follow Murasugi [Mu96] and use two relations:

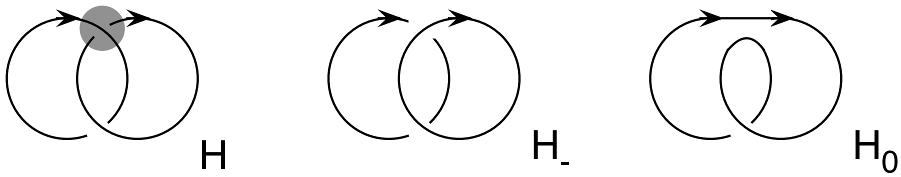


$$V_{D_+}(t) = t^2 V_{D_-}(t) + tzV_{D_0}(t) \tag{7.2}$$
$$V_{D_-}(t) = t^{-2} V_{D_+}(t) - t^{-1}zV_{D_0}(t) \tag{7.3}$$

where

$$z = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) \tag{7.4}$$

The Hopf link

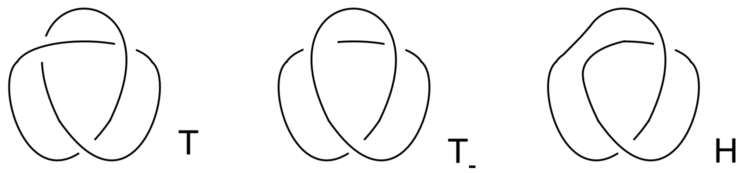


$$\begin{aligned} V_H(t) &= t^2 V_{H_-}(t) - tzV_{H_0}(t) && \text{by (7.2)} \\ &= t^2(-1)\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) + tz && \text{by (7.1)} \\ &= -t^{2.5} - t^{1.5} + t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) && \text{by (7.4)} \\ &= -t^{0.5} - t^{2.5} \end{aligned}$$

Exercise: Prove that if the orientation is changed on one (but not both) of the components, then the Jones polynomial is

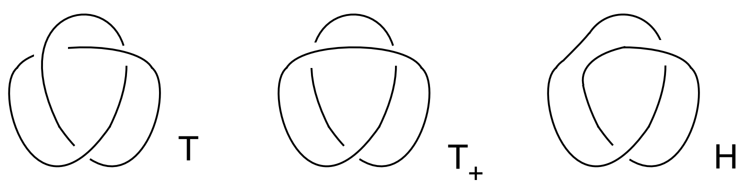
$$-t^{-0.5} - t^{-2.5}$$

Right trefoil knot



$$\begin{aligned}
V_T(t) &= t^2 V_{T_-}(t) + tzV_H(t) && \text{by (7.2)} \\
&= t^2 + t\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\left(-t^{0.5} - t^{2.5}\right) && \text{by (7.4), Hopf} \\
&= t + t^3 - t^4
\end{aligned}$$

Left trefoil knot



$$\begin{aligned}
V_T(t) &= t^{-2} V_{T_+}(t) - t^{-1}zV_H(t) && \text{by (7.3)} \\
&= t^{-2} - t^{-1}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\left(-t^{-0.5} - t^{-2.5}\right) && \text{by (7.4), Hopf*} \\
&= t^{-1} + t^{-3} - t^{-4}
\end{aligned}$$

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