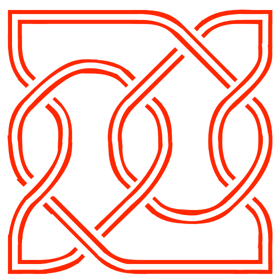
CS E6204 Lectures 7,8a
A Celtic Framework for Knots and Links*



A Celtic knot.

Abstract

Q. What's the definition of a Celtic knot?

Traditional — I know it when I see it.
Bain — via diagrams for graphic artists.
Topological — ambiant isotopic to a Bain knot.

* Extracted from "A Celtic Framework for Knots and Links" by Jonathan L. Gross and Thomas W. Tucker, soon to appear in the online edition of *Discrete and Combinatorial Geometry*.

Objectives of this Talk

- Deriving nec and suff conditions to be Celtic.
  All Celtic links are alternating.
  All alternating links are Celtic.

- Celtic approach to traditional knot invariants.
  Alexander-Conway polynomials
  Jones polynomials

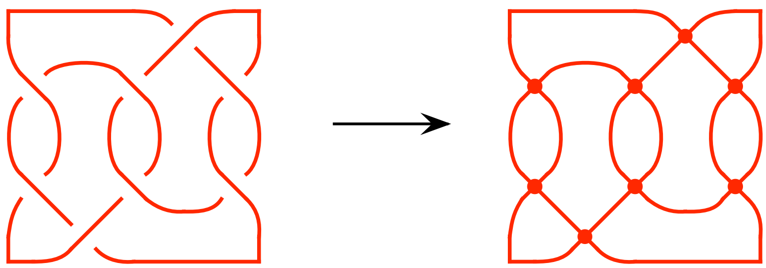
Outline*

1. Introduction
2. Drawing a Celtic Knot
3. Every Alternating Link is Celtic
4. Some Geometric Invariants of Knots and Links
5. Knot Polynomials
6. Computer Graphics Related to Celtic Knots
7. Conclusions

* Relevant background in knot theory is given, for example, by [Ad94], [Kau83], [Man04], and [Mu96]. Our topological graph theory terminology is consistent with [GrTu87] and [BWGT09].

Preliminaries

A ***normal projection*** of a link (aka ***shadow***) is a 4-regular graph imbedded in the plane. Graph imbeddings are taken to be cellular and graphs to be connected, unless the alternative is declared or evident from context.



For simplicity of exposition, we may sometimes say "knot" when our meaning is either a knot or a link.

1 Introduction

In general, the art works of authentic Celtic origin that are called "Celtic knots" are topologically recognizable as alternating links. Similar figures have occurred among Romans, Saxons, and Vikings, and also in some Islamic art and African art.

Figure 1.1: Artwork containing Celtic knots.

Our main concerns:

- **topological properties of knots** given as Celtic designs
- a **"Celtic framework"** for organizing the study of knots

2 Drawing a Celtic Knot

Step 1: grid and dots.

- Choose $m, n \in \mathbb{Z}^+$
- Place a dot at each lattice-point (x, y) such that $x + y$ is odd, where $0 \leq x \leq 2n$ and $0 \leq y \leq 2m$.

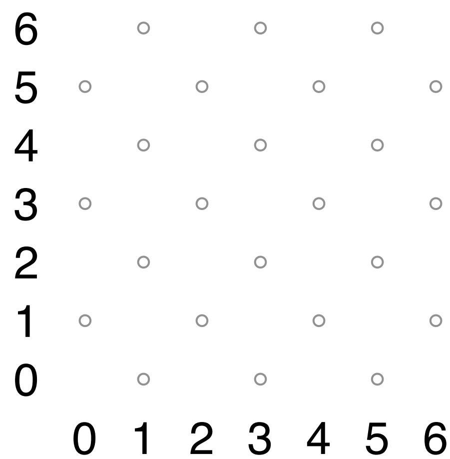


Figure 2.1: A 6×6 grid with dots.

DEF. ***barrier*** = a horiz or vert line segment of length two, whose center is one of the dots, drawn within the grid boundaries.

Step 2: outside borders.

- Install a vertical barrier centered on each dot on the grid lines $x = 0$ and $x = 2n$.
- Install a horizontal barrier centered on each dot on the grid lines $y = 0$ and $y = 2m$.

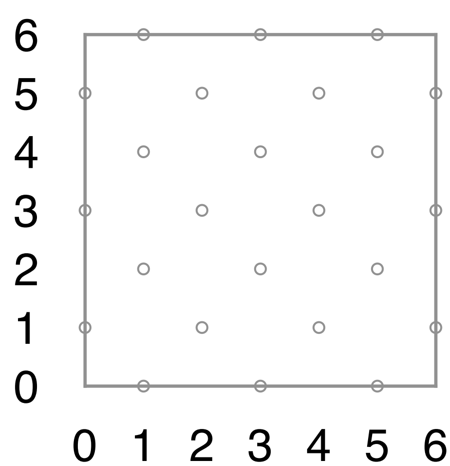


Figure 2.2: A 6×6 grid with dots and borders.

Step 3: interior barriers. Select some dots in the interior of the grid, and install at each such dot a vertical barrier or a horizontal barrier, but not both. Figure 2.3 shows the outside borders and a selection of interior barriers, with some symmetry.

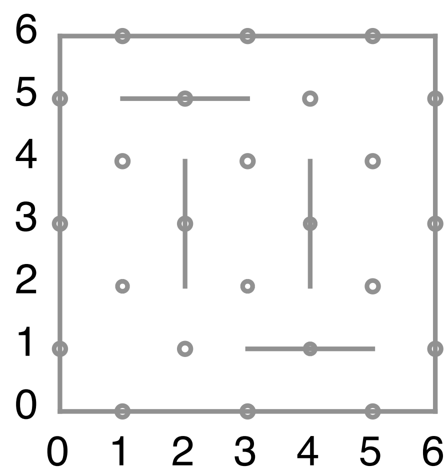


Figure 2.3: Celtic grid with interior barriers.

This grid with barriers completely determines the Celtic link that is to be drawn.

Step 4: crossings. Through each interior dot that does **not** lie on a barrier, draw two small line-segments that cross. If the x-coordinate is even, the over-crossing is southwest to northeast; if odd, the overcrossing is northwest to southeast.*

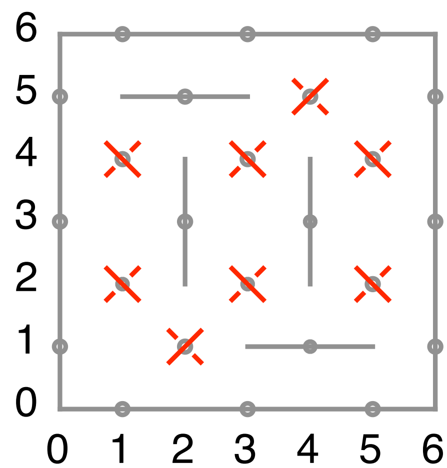


Figure 2.4: Celtic grid with link crossings installed.

* Also, the mirror image of a Celtic design (which switches overcrossings to undercrossings, and vice versa) is a Celtic design.

Step 5: extend the crossing-lines.

- If a straight extension of a line-segment thus a crossing would arrive next at another interior dot, then make that extension;
- if it would arrive next at a barrier, then extend with a short curve to the open edge of the grid-square, halfway between the two corners of the grid-square.

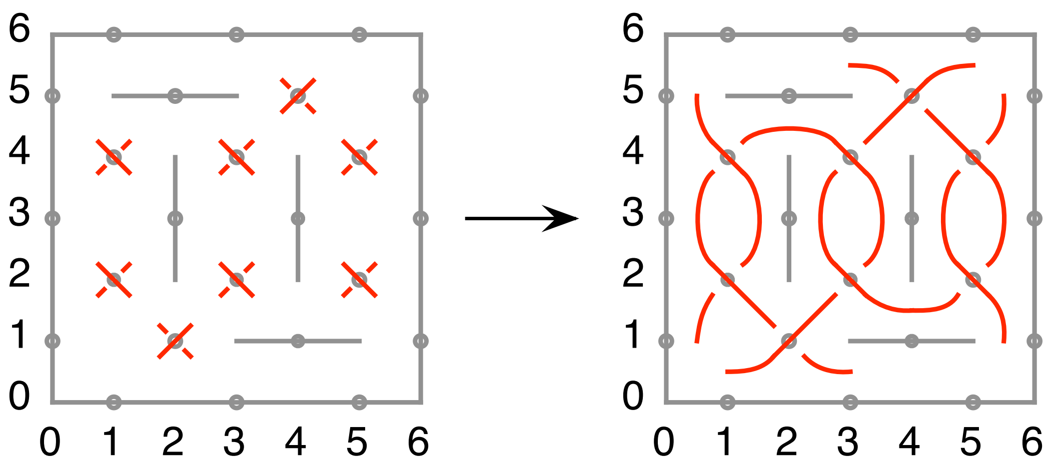


Figure 2.5: Celtic grid with link crossings extended.

Step 6: add corners. In each grid-square where two barriers meet at right angles, install a corner of the link.

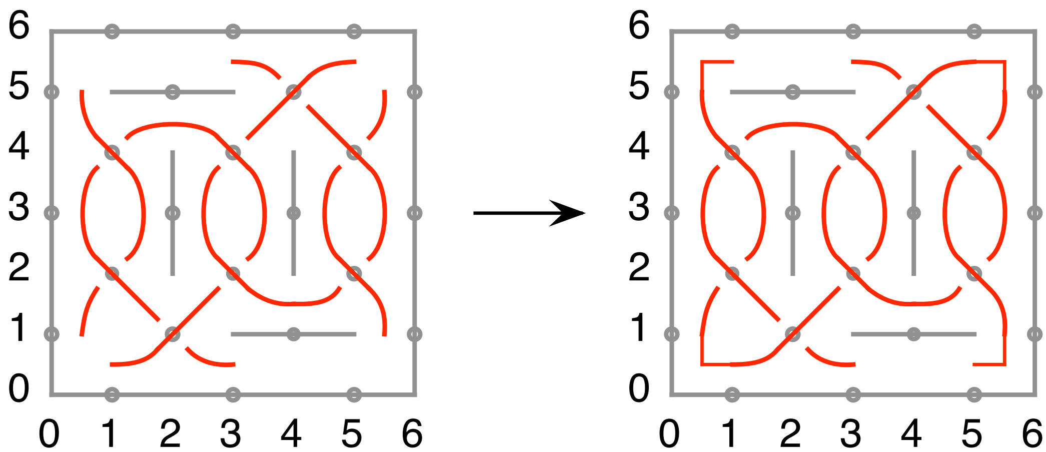


Figure 2.6: Celtic grid with extended link crossings and corners.

Step 7: add long lines. Install a straight line-segment wherever doing so would join an open end of a path to another open end of that same path or of another path.

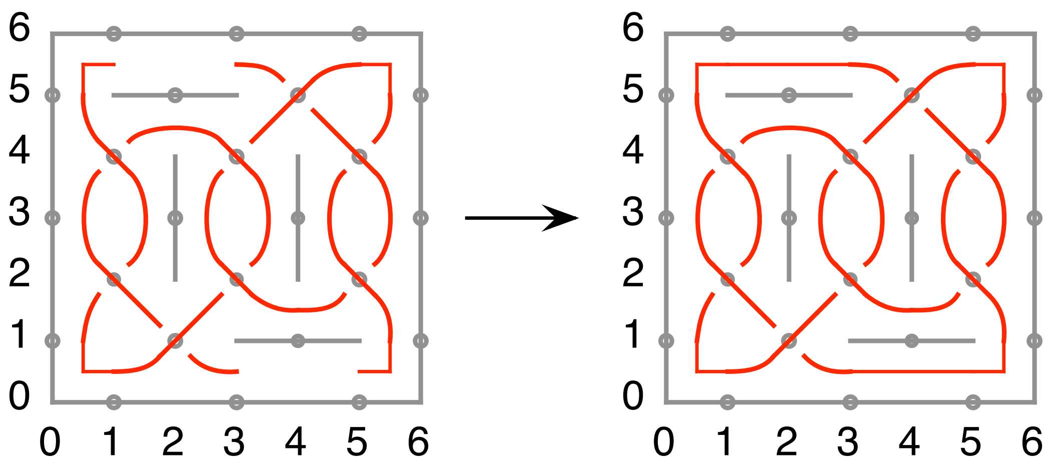


Figure 2.7: Long lines complete the Celtic link.

Step 8: delete the dots and barriers. Finish the drawing by deleting the dots and barriers.

3 Every Alternating Link is Celtic

For the sake of completeness, we include a simple proof of the converse of the section title, i.e., that every Celtic link, as we have defined it here, is alternating.

Theorem 3.1 *Every Celtic diagram specifies an alternating link.*

Proof Before installing a barrier at a construction dot, the local pattern for an alternating link is as illustrated by Figure 3.1 (left) or by a reflection of that figure. After installing the barrier, the local configuration is as in Figure 3.1 (right) or its reflection. Thus, the link that results from splitting and reconnecting an alternating link remains alternating. ♢

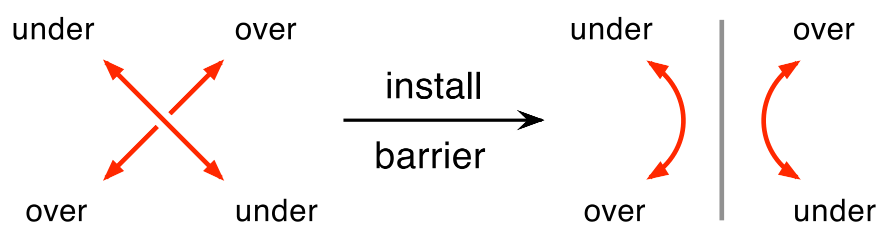


Figure 3.1: Installing a barrier in an alternating link diagram.

One possible way to prove that every alternating link is topologically equivalent to some Celtic link is by **induction on the number of crossings**. The proof is reasonably straightforward, but involves numerous details and cases. Accordingly, we present a proof that draws on some basic concepts from topological graph theory, specifically medial graphs and graph minors.

Medial Graphs and Inverse-Medial Graphs

Given a cellular imbedding $\iota : G \to S$ of a graph in a closed surface, the ***medial graph*** M_ι is defined as follows:

- The vertices of M_ι are the **barycenters** of the edges of G.
- For each face f of the imbedding $\iota : G \to S$ and for each vertex v of G on $bd(f)$, install an edge joining the vertex of M_ι that immediately precedes v in an fb-walk for f to the vertex of M_ι that immediately follows v on that fb-walk. (If the face f is a monogon, then that edge is a self-loop.)

The imbedding $M_\iota \to S$ is called the ***medial imbedding*** for the imbedding $\iota : G \to S$.

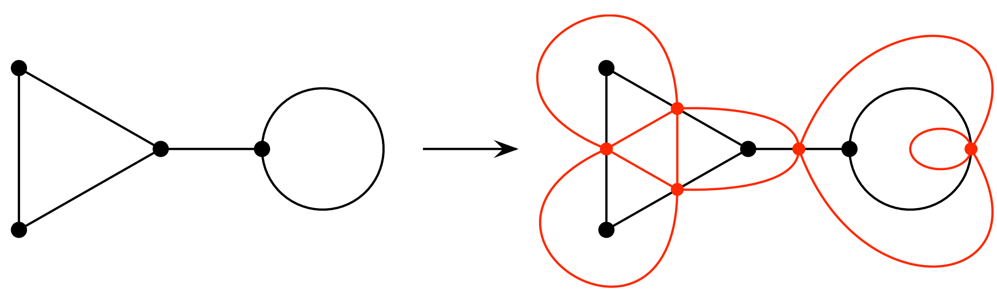
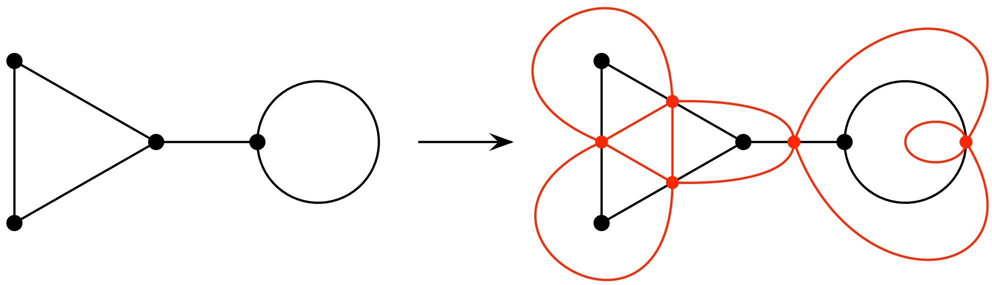


Figure 3.2: An imbedded graph and its medial imbedding.

Two properties of a medial graph and its imbedding:

- A medial graph is **4-regular**.
- The **dual graph is bipartite**.



These two properties characterize completely which 4-regular imbeddings are medial imbeddings.

Proposition 3.2 *If the imbedding $M \to S$ is 4-regular with bipartite dual, then it has an inverse medial imbedding.*

Proof

1. Color the faces of the imbedding $M \to S$ gray and white.
2. Place a vertex at the barycenter of each gray face.
3. Through each vertex v of M, draw an edge between the barycenters of the two gray faces incident to v.

The resulting graph imbedding $G \to S$ has $M \to S$ as its medial. Note that if we had placed the new vertices at the barycenters in the white faces instead, we would have the dual imbedding $G^* \to S$. $\diamondsuit$

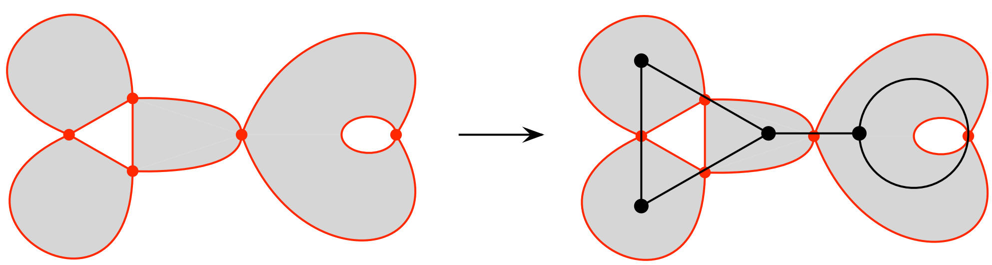


Figure 3.3: An imbedded graph and its inverse medial.

The following well-known fact identifies the characteristic of an imbedded 4-regular plane graph that permits it to have an inverse-medial graph.

Proposition 3.3 *The dual of a 4-regular plane graph* $G \to \mathbb{R}^2$ *is bipartite.*

Proof Since each vertex of G is 4-valent, it follows that each face of the dual graph is 4-sided. Since every cycle of a planar graph is made up of face-cycles, it follows that all cycles in the dual graph have even length, making the graph bipartite. ♢

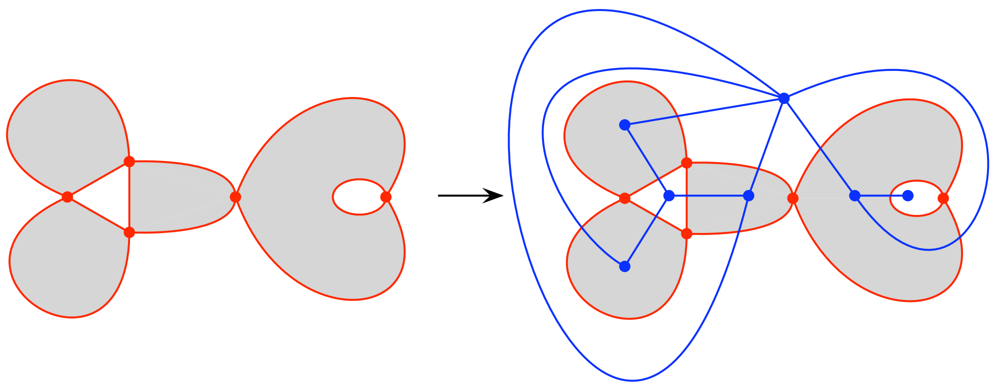


Figure 3.4: A 4-regular plane graph and its dual.

Theorem 3.4 *Every 4-regular plane graph has two inverse-medial graphs.*

Proof We observe that an imbedded graph and its dual have the same medial graph. Thus, this Theorem follows from Propositions 3.3 and 3.2. ♢

Inverse-Medial Graphs for Celtic Shadows

The link specified by the barrier-free $2m \times 2n$ Celtic diagram is denoted CK_{2m}^{2n}.

The ***inner grid*** of a Celtic diagram is formed by

$$\begin{array}{ll} \text{horizontals:} & y = 1, 3, 5, \ldots, 2m-1 \\ \text{verticals:} & x = 1, 3, 5, \ldots, 2n-1 \end{array}$$

As a graph, that inner grid for CK_{2m}^{2n} is isomorphic to the cartesian product $P_m \times P_n$ of the path graphs P_m and P_n.

In Figure 3.5, we observe that the 1×2 inner grid (in black) is an inverse-medial graph for the shadow of the Celtic link CK_4^6. We observe that every interior dot of the diagram lies at the midpoint of some edge of this grid.

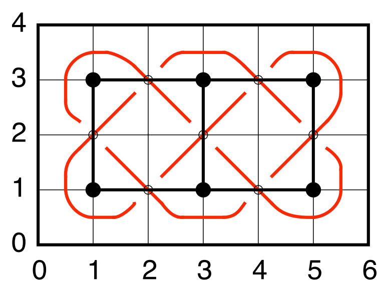


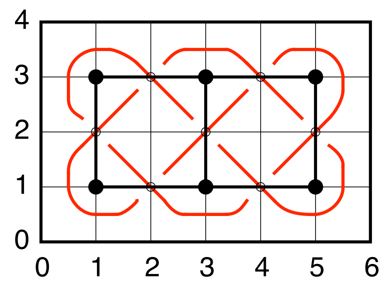
Figure 3.5: An inverse-medial for the shadow of the link CK_4^6.

The ***outer grid*** is formed by

$$\begin{array}{ll} \text{horizontals:} & y = 0, 2, 4, \ldots, 2m \\ \text{verticals:} & x = 0, 2, 4, \ldots, 2n \end{array}$$

Proposition 3.5 *The shadow of the Celtic link* CK_{2m}^{2n} *has as one of its two inverse-medial graphs the inner grid for the* $2m \times 2n$ *Celtic diagram.*

Proof A formal approach might use an easy double induction on the numbers of rows and columns. ◊



REMARK In general, the other inverse-medial graph of the Celtic link CK_{2m}^{2n} is obtained by contracting the border of the outer grid to a single vertex.

Clearly, every interior dot in a Celtic diagram is the midpoint of some edge of the inner grid, and every interior barrier in the diagram either coincides with an edge of the inner grid or lies orthogonal to the edge of that grid whose midpoint it contains.

Theorem 3.6 *Given an arbitrary Celtic diagram (i.e., perhaps with interior barriers) we can construct an inverse-medial graph for the shadow of the link it specifies as follows:*

1. *Start with the inner grid M.*
2. *Delete every edge of M that meets a barrier orthogonally at its midpoint.*
3. *Contract every edge of M that coincides with a barrier.*

Proof Use induction on the number of barriers. This result follows from the given method for constructing the link specified by a Celtic diagram. ♢

In view of Theorem 3.6, we can characterize the interior barriers in a Celtic diagram as follows;

- a ***deletion barrier*** meets an edge of the inner grid orthogonally
- a ***contraction barrier*** coincides with an edge of the inner grid.

Example 3.1 We apply Theorem 3.6 to the Celtic link in Figure 3.6. We delete the edge of the inner grid that is crossed by barriers, in the lower left corner of the diagram, and we contract the three edges of the mesh that coincide with barriers. The result is an inverse-medial for the shadow of the link, whose vertices are the black dots.

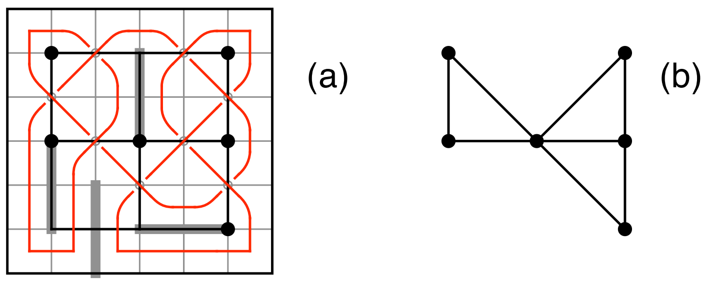


Figure 3.6: (a) A Celtic link, its inner grid, and the barriers.
(b) Inverse-medial of the shadow of that Celtic link.

To obtain the other inverse-medial of the shadow of the given link, we would contract the edges of the outer grid that cross barriers and delete the edges that coincide with barriers. We would also contract the border of the diagram to a single vertex.

Corollary 3.7 *One inverse-medial graph for the shadow of any link specified by a* $2m \times 2n$ *Celtic diagram is a minor of the graph* $P_m \times P_n$*, and the other is a minor of* $P_{m+1} \times P_{n+1}$*.*

Proof This follows easily from Theorem 3.6 and the Remark that follows it. ♢

Constructing a Celtic Diagram for an Alternating Link

By ***splitting a vertex*** of a graph, we mean inverting the operation of contracting an edge to that vertex.

Proposition 3.8 *Let $\iota : G \to S$ be a graph imbedding such that some vertex of G has degree greater than 3. Then it is possible to split that vertex so that the resulting graph is imbedded in S and that the result of contracting the new edge is to restore the imbedding $\iota : G \to S$.*

Proof This is a familiar fact that follows from elementary considerations in topological graph theory. $\diamondsuit$

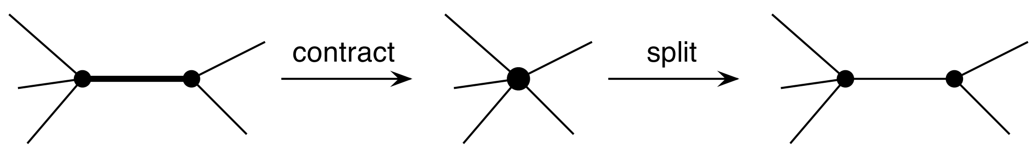


Figure 3.7: Contracting an edge and splitting.

Example 3.2 Contracting an edge cannot increase the minimum genus of a graph. However, Figure 3.8 illustrates that splitting the hub of the planar wheel graph W_4 can yield the non-planar graph $K_{3,3}$.

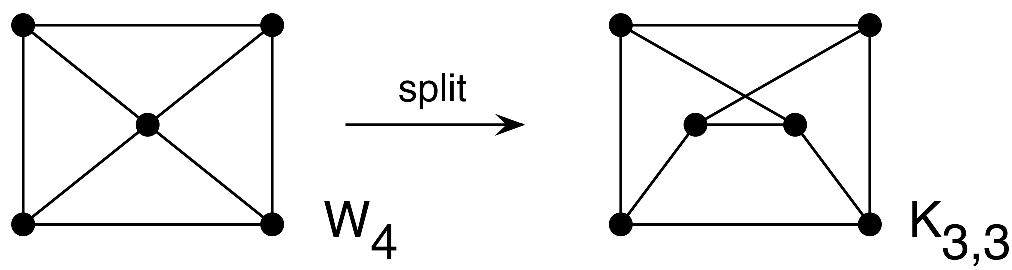


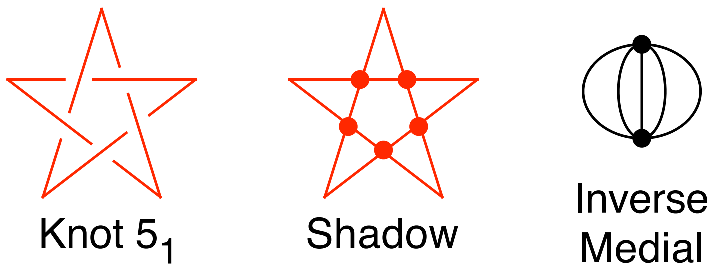
Figure 3.8: Splitting can increase the minimum genus.

Proposition 3.9 *Let G be any planar graph with maximum degree at most 4. Then G is homeomorphic to a subgraph of some orthogonal grid.*

Proof As explained, for instance, by Chapter 5 of [DETT99] or by [Sto80], every planar graph with maximum degree at most 4 has a subdivision that can be drawn as a subgraph of some orthogonal planar mesh. $\diamondsuit$

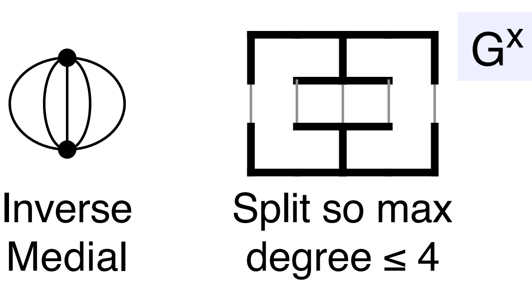
The following ***Celtification algorithm*** constructs a Celtic diagram for any alternating link L supplied as input.

1. Construct an inverse-medial graph G for shadow of link L.



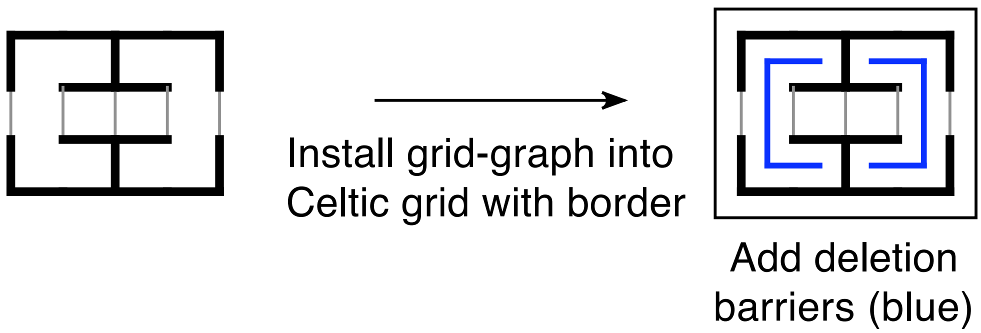
   N.B. Observe that the other inverse medial is C_5.

2. Iteratively split vertices of G as needed, so that every split graph is planar, until max degree ≤ 4. After each such split, install a contraction barrier on the *newly created edge*. Then subdivide as needed until resulting graph $G^{\times}$ is a subgraph of an orthgonal grid (as per Prop. 3.9). After each subdivision, install a contraction barrier on one of the two "new" edges.



   N.B. Some other splittings would allow a smaller eventual Celtic diagram.

3. Enclose the represention of $G^\times$ as a subgraph of an orthgonal grid by a border, and install a deletion barrier on each edge of the grid that is not in the image of $G^\times$.



4. The resulting orthogonal grid with its contraction and deletion barriers is a Celtic design for the given link L.

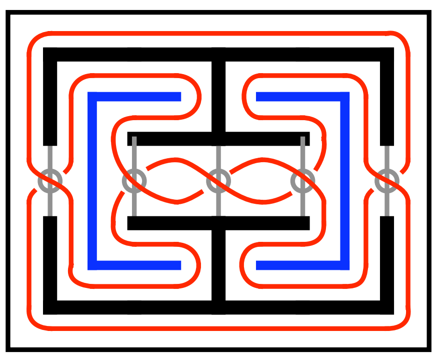


Figure 3.9: Reconstruct the given knot 5_1.

Theorem 3.10 *Let L be an alternating link. Then there is a Celtic diagram that specifies L.*

Proof We verify that each of the steps of the Celtification algorithm is feasible.

1. Thm 3.4 establishes that Step 1 is possible.
2. Prop 3.8 verifies the possibility of Step 2.
3. Prop 3.9 ensures that Step 3 is possible.
4. Thm 3.6 is the basis for Step 4.

♢

Celtistic Link Diagrams

To generalize our scope, we define a ***Celtistic diagram*** to be otherwise like a Celtic diagram, except that we are permitted to specify at each dot whether the overcrossing is northwest to southeast or southwest to northeast. The Celtistic perspective on links simplifies the derivation of some kinds of general results and also facilitates the application of our methods of calculating knot polynomials for infinite sequences of knots and links.

Theorem 3.11 *Every link is Celtistic.*

Proof The proof that every alternating link is Celtic depends only on the shadow of the link, not on the overcrossings and undercrossings. Accordingly, we may apply the same argument here. ♢

In a imbedded 4-regular graph, a ***straight-ahead walk*** is a walk that goes neither left nor right.

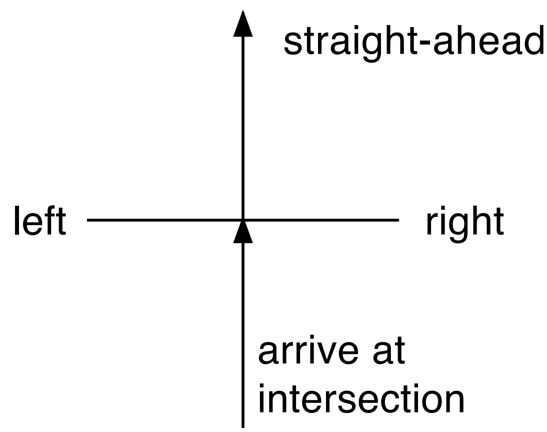


Figure 3.10: Going straight-ahead.

Corollary 3.12 *Every 4-regular plane graph G is the shadow of an alternating link.*

Proof The straight-ahead walks) in G form the components of a link L. Of course, overcrossings and undercrossings could be assigned arbitrarily. By Theorem 3.11, the link L could be specified by a Celtistic design. Changing the intersections so that they all follow the rules for a Celtic diagram produces a Celtic link L' whose shadow is the plane graph G. ◇

REMARK An interpretation of Corollary 3.12 within Kauffman's terminology ([Kau83]) is that every knot universe corresponds to some alternating knot. There are many different proofs of this widely known fact.

4 Some New Geometric Invariants of Links

Blending the theory of graph drawings (for an extensive survey, see [LT04]) with Theorems 3.10 and 3.11 suggests some interesting new geometric invariants of links. These four come immediately to mind:

- The ***Celtic area of a projection of a link*** L is the minimum product mn such that an equivalent projection L is specifiable by a Celtistic diagram with m rows and n columns. The ***Celtic area of a link*** L is the min Celtic area taken over all projections of L. **Notation**: $C_A(L)$.
- The ***Celtic depth of a link projection*** is the mininum number m such that a Celtistic diagram with m rows specifies an equivalent projection. The ***Celtic depth of an alternating link*** L is the minimum Celtic depth taken over all projections of L. **Notation**: $C_D(L)$.

  N.B. Celtic depth is the most important of these invariants, because for any given Celtic depth at least 4, there are infinitely many links.

- The ***Celtic edge-length of a projection of a link*** L is the minimum number of grid-squares traversed by the link in a Celtistic diagram for that projection. If a sublink of components of the link specified by the diagram splits off from the projection of L, then the edge-length of that sublink is not counted. The ***Celtic edge-length of a link*** is the minimum Celtic edge-length taken over all projections. **Notation**: $C_L(L)$.
- The ***Celtic perimeter of a projection of a link*** L is the minimum sum $4m + 4n$, such that there is a $2m \times 2n$ Celtic diagram for L. **Notation**: $C_P(L)$.

This simple proposition is helpful in deriving values of these geometric invariants for specific links. Its proof is omitted. We use $cr(\mathcal{D})$ for the number of crossings in a Celtistic diagram.

Proposition 4.1 *Let $\mathcal{D}$ be a $2m \times 2n$ Celtistic diagram with h horizontal interior barriers and v vertical interior barriers.*

(a). $cr(\mathcal{D}) + h + v = 2mn - m - n.$

(b). The Celtic depth of a non-trivial knot is at least 4.

(c). The Celtic area of a link with x crossings is at least $2x + 8$.

Proof (a) The numbers of dots in the odd-numbered rows and the even-numbered rows of the grid are, respectively

$$m \times (n-1) \quad \text{and} \quad n \times (m-1)$$

Part (a) follows immediately.

(b) A 2×2 grid specifies a trivial knot. Use induction on the number of columns to demonstrate (b).

(c) From part (a) we have

$$x \ \leq \ 2mn - m - n$$

Thus,

$$C_A \ = \ 4mn \ \geq \ 2x + 2m + 2n \ \geq \ 2x + 8 \qquad \diamondsuit$$

(a). $cr(\mathcal{D}) + h + v = 2mn - m - n$.

(b). The Celtic depth of a non-trivial knot is at least 4.

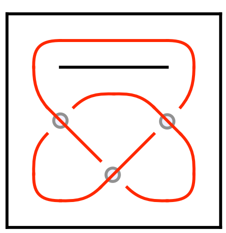
(c). The Celtic area of a link with x crossings is at least $2x+8$.

Example 4.1 Using Proposition 4.1, we calculate lower bounds for some geometric invariants for the trefoil knot 3_1 and for the figure-eight knot 4_1.

Knot	C_A	C_D	C_L	C_P
3_1	16	4	16	16
4_1	24	4	24	20

If the figure-eight knot could be drawn in a 4×4 grid, there would have to be no barriers, because it has 4 crossings. However, the barrier-free 4×4 grid grid specifies a link with two unknotted components.

Upper bounds for the geometric invariants follow from the following two figures.


3_1

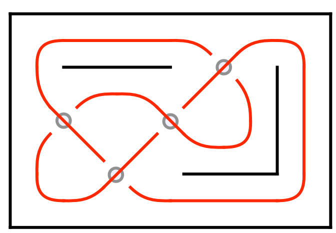

4_1

Figure 4.1: Trefoil and figure-eight knots.

Total Curvature

We observe that the total curvature $\kappa(L)$ of a link in $\mathbb{R}^3$ (see [Mi50]) can be bounded using Celtic invariants. For example, for the barrier-free Celtic knot in the $2m \times 2n$ grid, each corner supplies π to the total curvature, and each of the

$$2(m-2)+2(n-2)$$

curves at the sides adds $\pi/2$. Thus, the total curvature is at most

$$\kappa(CK_{2m}^{2n}) \;\leq\; (m+n)\pi$$

The addition of each barrier increases the total curvature by at most π; however, adding barriers can also reduce the curvature. Since there at most $2mn - m - n$ barriers, we have the following upper bound:

Proposition 4.2 *The total curvature of a link L satifies the inequality*

$$\kappa(L) \;\leq\; C_A(L) \cdot \frac{\pi}{2} \qquad \diamondsuit$$

The total curvature invariant also provides upper bounds on other physical invariants of knots, such as thickness (see [LSDR99]). Accordingly, Celtic invariants can be related to those invariants as well.

5 Knot Polynomials

Celtic diagrams can be helpful when calculating knot polynomials for a recursively specifiable family of links. In this section, we derive recursions for the Alexander-Conway polynomials and the Kauffman bracket polynomials of the links specified by $4 \times 2n$ barrier-free Celtic diagrams.

The three smallest $4 \times 2n$ barrier-free Celtic links are shown in Figure 5.1. We see that CK_4^6 is the knot 7_4.

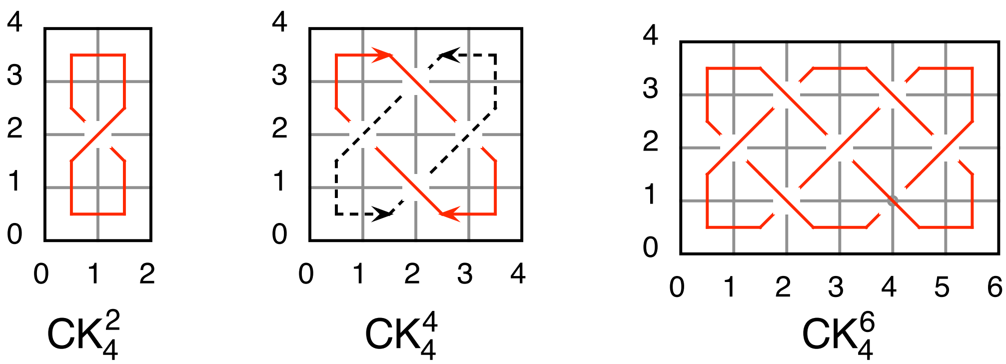


Figure 5.1: Some small barrier-free $4 \times c$ Celtic knots.

Alexander-Conway Polynomials

DEF. The link diagrams L and L' are ***equivalent link diagrams*** if one can be derived from the other by a sequence of Reidemeister moves.
NOTATION $L \sim L'$.

We calculate the Alexander-Conway polynomial, denoted either by ∇_K or by $\nabla(K)$, of a knot (or link) K by using the following three axioms.

Axiom 1. If $K \sim K'$, then $\nabla_K = \nabla_{K'}$.

Axiom 2. If $K \sim 0$, then $\nabla_K = 1$.

Axiom 3. If L^+, L^-, and L^0 are related as in Figure 5.2, then $\nabla_{L^+} - \nabla_{L^-} = z\nabla_{L^0}$.

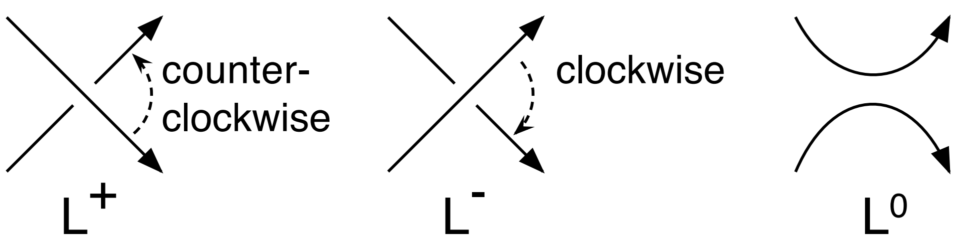


Figure 5.2: Switch and elimination operations.

REMARK Axiom 1 means that the value of the Alexander-Conway polynomial is **invariant under Reidemeister moves**.

A link L is said to be ***split*** if there is a 2-sphere in 3-space that does not intersect the link, such that at least one component of L is on either side of the separation.

Remark The orientations of the components of the link matter quite a lot, in particular, when calculating the Alexander-Conway polynomial or the genus of a link.

Proposition 5.1 *We give the Alexander-Conway polynomials of several well-known links:*

(a) Any split link — 0.

(b) Hopf link — z.

(c) The trefoil knot — $1 + z^2$.

Proof On the next few pages, we do each of these calculations.

Prop 5.1(a): A split link L^0 has $\nabla(L^0) = 0$

Tying two knots consecutively along a cycle is called a ***link sum***.

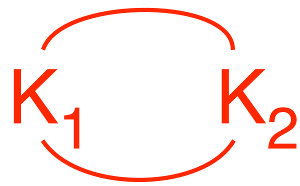
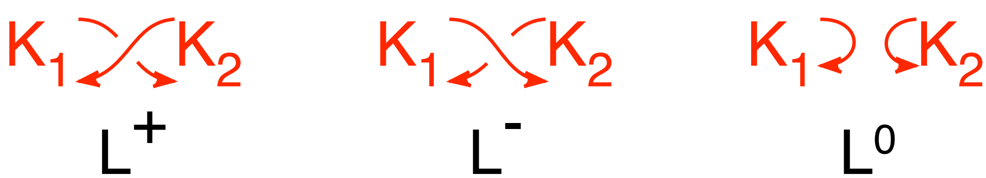


Figure 5.3: Knot sum $K_1 + K_2$.

By Axiom 3, we have

$$\nabla(L^+) \;=\; \nabla(L^-) + z\nabla(L^0)$$



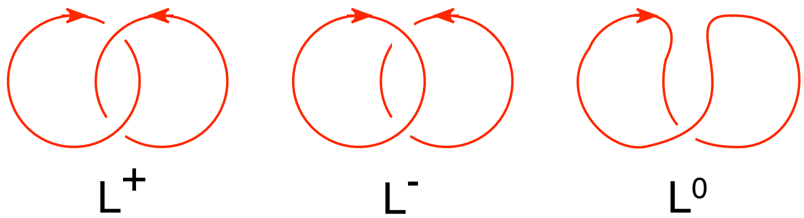
Since $L^+ \sim L^-$, it follows from Axiom 1 that $\nabla(L^+) = \nabla(L^-)$. Accordingly, we have

$$z\nabla(L^0) \;=\; 0$$

and thus,

$$\nabla(L^0) \;=\; 0 \qquad\qquad \Diamond(a)$$

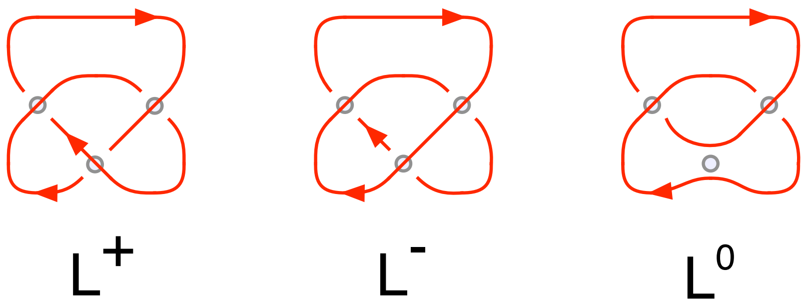
Prop 5.1(b): The Hopf link $\mathbf{2_1^2}$ has $\nabla(\mathbf{2_1^2}) = \mathbf{z}$



We observe that L^- is a split link and that L^0 is the unknot. Therefore,

$$\begin{aligned} \nabla(L^+) &= \nabla(L^-) + z\nabla(L^0) && \text{by Axiom 3} \\ &= z\nabla(L^0) && \text{by Part (a)} \\ &= z && \text{by Axiom 2} \qquad \diamondsuit(b) \end{aligned}$$

Prop 5.1(c): The trefoil knot 3_1 has $\nabla(3_1) = 1 + z^2$



We observe that L^- is an unknot (by Reidemeister moves) and that L^0 is the Hopf link. Therefore,

$$\begin{aligned} \nabla(L^+) &= \nabla(L^-) + z\nabla(L^0) && \text{by Axiom 3} \\ &= 1 + z\nabla(L^0) && \text{by Axiom 2} \\ &= 1 + z^2 && \text{by Part b} \qquad \Diamond(c) \end{aligned}$$

NOTATION The notation S_r^c means switch the crossing of a Celtic link K at row r, column c.

NOTATION The notation E_r^c means eliminate the crossing at row r, column c.

Lemma 5.2 *The following three relations hold for operations on Celtic links.*

$$S_1^{2n-2} S_2^{2n-1} CK_4^{2n} \sim CK_4^{2n-4} \tag{5.1}$$
$$E_1^{2n-2} S_2^{2n-1} CK_4^{2n} \sim CK_4^{2n-2} \tag{5.2}$$
$$E_2^{2n-1} CK_4^{2n} \sim CK_4^{2n-2} \tag{5.3}$$

Proof These relations follow from the diagrams in Figure 5.4. Retracting the dashed parts of the links corresponds to Reidemeister moves. ♢

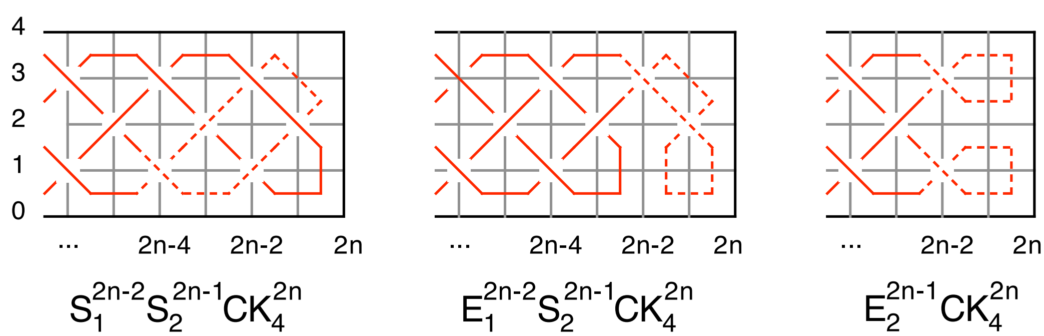


Figure 5.4: Iterative operations on CK_4^{2n}.

Theorem 5.3 *The coefficients of the Alexander-Conway polynomial for the barrier-free link sequence*

$$CK_4^2,\ CK_4^4,\ CK_4^6,\ \ldots$$

are given by this formula:

$$\begin{aligned}\nabla(CK_4^{2n}) &= \sum_{k=0}^{n-1} b_k^{2n} z^k \\ \textit{where}\quad b_k^{2n} &= \begin{cases} 0 & \textit{if } k \equiv n \mod 2 \\ \binom{(n+k-1)/2}{k} 2^k & \textit{otherwise} \end{cases}\end{aligned}$$

Proof We first establish this recursion:

$$\nabla(CK_4^0) = 0 \tag{5.4}$$

$$\nabla(CK_4^2) = 1 \tag{5.5}$$

$$\nabla(CK_4^{2n}) = \nabla(CK_4^{2n-4}) + 2z\nabla(CK_4^{2n-2}) \quad \text{for } n \geq 2 \tag{5.6}$$

Eq (5.4) is a normalization constant.

Since CK_4^2 is an unknot, Eq (5.5) follows from Axiom 2.

We now verify Eq (5.6):

$$\nabla(CK_4^{2n}) \;=\; \nabla(CK_4^{2n-4}) + 2z\nabla(CK_4^{2n-2}) \quad \text{for } n \geq 2$$

$$\begin{aligned}
\nabla(CK_4^{2n}) &= \nabla(S_2^{2n-1}CK_4^{2n}) + z\nabla(E_2^{2n-1}CK_4^{2n}) \quad \text{(Axiom 3)} \\
&= \left[\nabla(S_1^{2n-2}S_2^{2n-1}CK_4^{2n}) + z\nabla(E_1^{2n-2}S_2^{2n-1}CK_4^{2n})\right] \\
&\quad +z\nabla(E_2^{2n-1}CK_4^{2n}) \quad \text{(Axiom 3)} \\
&= \nabla(CK_4^{2n-4}) + 2z\nabla(CK_4^{2n-2}) \quad \text{(Lemma 5.2)}
\end{aligned}$$

To obtain b_k^{2n} as the coefficient of $t^{2n}u^k$, the generating function is

$$\frac{t^2}{1 - t^2(t^2 + 2u)}$$

The conclusion follows. $\diamondsuit$

Example 5.1 Applying the recursion in Theorem 5.4 gives these Alexander-Conway polynomials:

$$\begin{aligned}
\nabla(CK_4^4) &= \nabla(CK_4^0) + 2z\nabla(CK_4^2) \\
&= 0 + 2z \cdot 1 = \mathbf{2z} \\
\nabla(CK_4^6) &= \nabla(CK_4^2) + 2z\nabla(CK_4^4) \\
&= 1 + 2z \cdot 2x = \mathbf{1} + \mathbf{4z^2}, \quad \text{and} \\
\nabla(CK_4^8) &= \nabla(CK_4^4) + 2z\nabla(CK_4^6) \\
&= 2z + 2z \cdot (1 + 4z^2) = \mathbf{4z} + \mathbf{8z^3}
\end{aligned}$$

Kauffman Bracket Polynomials

Prior to 1984, the main invariants used to distinguish knots and links prior to 1984 were derivable from an algebraic object called the Seifert matrix. In 1984, while exploring operator algebras, Vaughn Jones discovered a new invariant of knots, now called the ***Jones polynomial***.

Fortunately for anyone not already expert in operator algbras, other mathematicians were soon able to construct a purely combinatorial approach to the calculation of Jones polynomials. The approach presented here is due to Louis Kauffman.

Kauffman's ***bracket polynomial*** is defined by three axioms:

Axiom 1. $<\bigcirc> \ = \ 1$

Axiom 2u. $<\times> \ = \ A<)(> + A^{-1}<\asymp>$ /-overcross

Axiom 2d. $<\times> \ = \ A<\asymp> + A^{-1}<)(>$ \\-overcross

Axiom 3. $<L \cup \bigcirc> \ = \ (-A^2 - A^{-2})<L>$

Both parts of Axiom 2 can be combined into a single axiom:

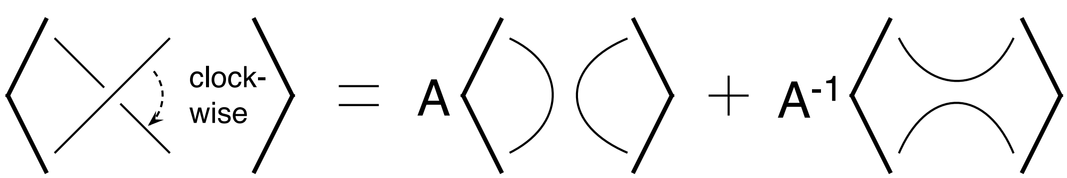


Figure 5.5: Unified skein for bracket polynomial.

Basic Facts about Jones and Bracket Polynomials

1. Calculating the Jones polynomial of a link is known to be $\#P$-hard ([JVW90]).
2. The coefficients of a Jones polynomial can be calculated by making a substitution into a bracket polynomial.
3. Thus, the general problem calculating the bracket polynomial of links is computationally intractible.
4. Nonetheless, there remains the possibility of calculating bracket polynomials for an infinite sequence of links, as we now illustrate.

Notation The notation H_r^c means replace the crossing of a Celtic link K at row r, column c by a horizontal pair.

Notation The notation V_r^c means replace the crossing at row r, column c by a vertical pair.

Lemma 5.4 *These four relations hold for bracket polynomials:*

$$< H_2^{2n-1} CK_4^{2n} > \;=\; A^{-6} < CK_4^{2n-2} > \quad (5.7)$$

$$< V_1^{2n-2} V_2^{2n-1} CK_4^{2n} > \;=\; -A^{-3} < CK_4^{2n-2} > \quad (5.8)$$

$$< H_3^{2n-2} H_1^{2n-2} V_2^{2n-1} CK_4^{2n} > \;=\; -A^{3} < V_2^{2n-3} CK_4^{2n-2} > \quad (5.9)$$

$$< V_3^{2n-2} H_1^{2n-2} V_2^{2n-1} CK_4^{2n} > \;=\; < CK_4^{2n-2} > \quad (5.10)$$

Proof Equations (5.7), (5.8), (5.9), and (5.10), follow from the diagrams (a), (b), (c), and (d), respectively in Figure 5.6. ♢

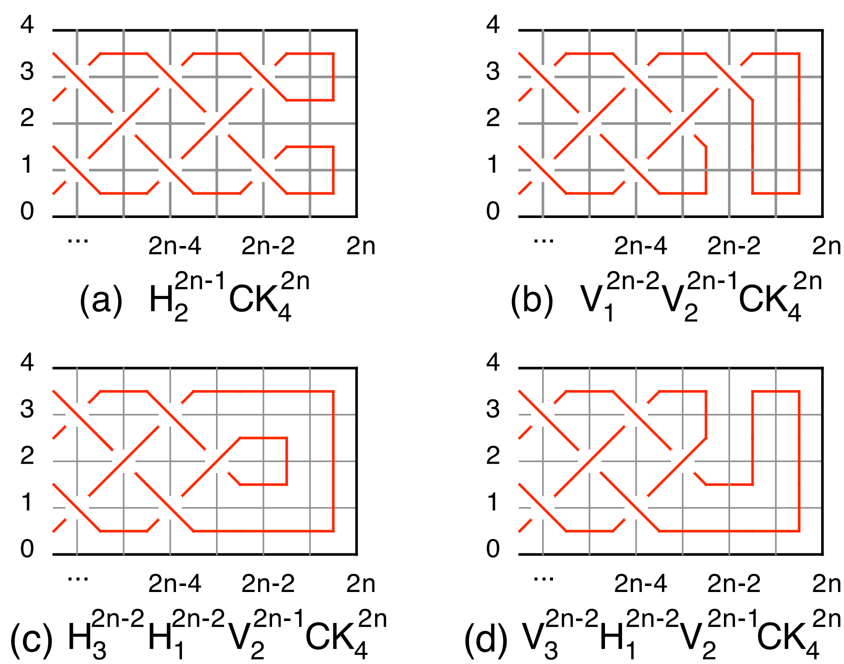


Figure 5.6: Celtic diagrams for bracket polynomial relations.

Remark Equations (5.7), (5.8), and (5.9) reflect the fact that the bracket polynomial is not preserved by the first Reidemeister move. Indeed, the first Reidemeister move changes the bracket polynomial by A^3 or A^{-3}, depending on the direction of the twisting or untwisting.

Theorem 5.5 *The bracket polynomial for the barrier-free link sequence*

$$CK_4^0,\, CK_4^2,\, CK_4^4,\, CK_4^6,\, \ldots$$

is given by the following recursion:

$$\begin{aligned}
<CK_4^0> &= 0 && (5.11)\\
<V_2^1 CK_4^2> &= 1 && (5.12)\\
<CK_4^2> &= -A^{-3} && (5.13)\\
<V_2^{2n-1} CK_4^{2n}> &= (1 - A^{-4})\,<CK_4^{2n-2}> - A^5 <V_2^{2n-3} CK_4^{2n-2}> &&\\
&\qquad \textit{for } n \geq 2 && (5.14)\\
<CK_4^{2n}> &= A <V_2^{2n-1} CK_4^{2n}> + A^{-7} <CK_4^{2n-2}> &&\\
&\qquad \textit{for } n \geq 2 && (5.15)
\end{aligned}$$

Proof Eq (5.11) is a normalization constant, and Eq (5.12) and Eq (5.13) are easily verified from fundamentals.

We now confirm Eq (5.14) and Eq (5.15):

$$
\begin{aligned}
<V_2^{2n-1}CK_4^{2n}> &= (1-A^{-4})<CK_4^{2n-2}> - A^5 <V_2^{2n-3}CK_4^{2n-2}> \\
&\qquad \text{for } n \geq 2 \qquad\qquad \text{Eq (5.14)} \\
<CK_4^{2n}> &= A<V_2^{2n-1}CK_4^{2n}> + A^{-7}<CK_4^{2n-2}> \\
&\qquad \text{for } n \geq 2 \qquad\qquad \text{Eq (5.15)}
\end{aligned}
$$

$$
\begin{aligned}
<V_2^{2n-1}CK_4^{2n}> &= A<H_1^{2n-2}V_2^{2n-1}CK_4^{2n}> + A^{-1}<V_1^{2n-2}V_2^{2n-1}CK_4^{2n}> \\
&\qquad \text{(by Ax. 2d)} \\
&= A<H_1^{2n-2}V_2^{2n-1}CK_4^{2n}> + A^{-1}(-A^{-3})<CK_4^{2n-2}> \\
&\qquad \text{(by Eq. (5.8))} \\
&= -A^{-4}<CK_4^{2n-2}> + A\big[A<H_3^{2n-2}H_1^{2n-2}V_2^{2n-1}CK_4^{2n}> \\
&\qquad + A^{-1}<V_3^{2n-2}H_1^{2n-2}V_2^{2n-1}CK_4^{2n}>\big] \quad \text{(by Ax. 2d)} \\
&= -A^{-4}<CK_4^{2n-2}> + A^2(-A^3)<V_2^{2n-3}CK_4^{2n-2}> \\
&\qquad + A\cdot A^{-1}<CK_4^{2n-2}> \quad \text{(by Eqs. (5.9, 5.10))} \\
&= (1-A^{-4})<CK_4^{2n-2}> - A^5<V_2^{2n-3}CK_4^{2n-2}>
\end{aligned}
$$

$$
\begin{aligned}
<CK_4^{2n}> &= A<V_2^{2n-1}CK_4^{2n}> + A^{-1}<H_2^{2n-1}CK_4^{2n}> \text{ (by Ax. 2u)} \\
&= A<V_2^{2n-1}CK_4^{2n}> + A^{-1}A^{-6}<CK_4^{2n-2}> \text{ (by Eq. (5.7))} \\
&= A<V_2^{2n-1}CK_4^{2n}> + A^{-7}<CK_4^{2n-2}> \qquad \diamondsuit
\end{aligned}
$$

Example 5.2 Thm 5.5 yields these bracket polynomials:

$$
\begin{aligned}
<V_2^3 CK_4^4> &= A^{-7} - A^{-3} - A^5 && \text{knot } 3_1 \\
<CK_4^4> &= -A^{-10} + A^{-6} - A^{-2} - A^6 && \text{link } 4_1^2 \\
<V_2^5 CK_4^6> &= A^{-14} - 2A^{-10} + 2A^{-6} - 2A^{-2} && \\
&\quad +2A^2 - A^6 + A^{10} && \text{knot } 6_2 \\
<CK_4^6> &= -A^{-17} + 2A^{-13} - 3A^{-9} + 2A^{-5} && \\
&\quad -3A^{-1} + 2A^3 - A^7 + A^{11} && \text{knot } 7_4
\end{aligned}
$$

Complexity The time to calculate $< CK_{2n}^4 >$ by the usual skein relations or by any other universally applicable method is exponential in n. By way of contrast, each iteration of the recursions (5.14) and (5.15) increases the span of the bracket polynomial by at most 12. Thus, the time needed to calculate $<CK_{2n}^4>$ by applying these recursions is quadratic in n.

6 Computer Graphics Connections

M. Wallace [Wa] has posted a method in which the barriers are drawn first, based on publications of G. Bain [Ba51] and I. Bain [Ba86], intended for graphic artists, by which anyone capable of following directions can hand-draw Celtic knots, and which lends itself to implementation within a graphics system for creating computer-assisted art.

In computer-graphics research on Celtic knots by [KaCo03], [Me01] and others, the primary concern has been the creation of computer-assisted artwork that produces their classical geometric and stylistic features.

Cyclic plain-weaving is a more general form of computer-assisted artwork, and, as observed by [ACXG09], the graphics it creates are alternating projections of links onto various surfaces in 3-space. From a topological perspective, Celtic knots and links are a special case of cyclic plain-weaving.

7 Conclusions

Celtic design can be used to specify any alternating link and that Celtistic design can be used to specify any link. We have seen that Celtic representation suggests some new geometric invariants, which can yield information about some well-established knot invariants. It can also be used to calculate knot polynomials for infinite families of knots and links. Moreover, the computation time for bracket polynomials (or Jones polynomials) by the methods given here is quadratic in the number of crossings, in contrast to the standard exponential-time skein-based recursive algorithm.

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