CS E6204 Lecture 5
Automorphisms

Abstract

An automorphism of a graph is a permutation of its vertex set that preserves incidences of vertices and edges. Under composition, the set of automorphisms of a graph forms what algbraists call a *group*. In this section, graphs are assumed to be simple.

1. The Automorphism Group
2. Graphs with Given Group
3. Groups of Graph Products
4. Transitivity
5. s-Regularity and s-Transitivity
6. Graphical Regular Representations
7. Primitivity
8. More Automorphisms of Infinite Graphs

* This lecture is based on chapter [**?**] contributed by Mark E. Watkins of Syracuse University to the *Handbook of Graph Theory*.

1 The Automorphism Group

Given a graph X, a permutation α of $V(X)$ is an ***automorphism*** of X if for all $u, v \in V(X)$

$$\{u, v\} \in E(X) \Leftrightarrow \{\alpha(u), \alpha(v)\} \in E(X)$$

The set of all automorphisms of a graph X, under the operation of composition of functions, forms a subgroup of the symmetric group on $V(X)$ called the ***automorphism group*** of X, and it is denoted $\mathrm{Aut}(X)$.

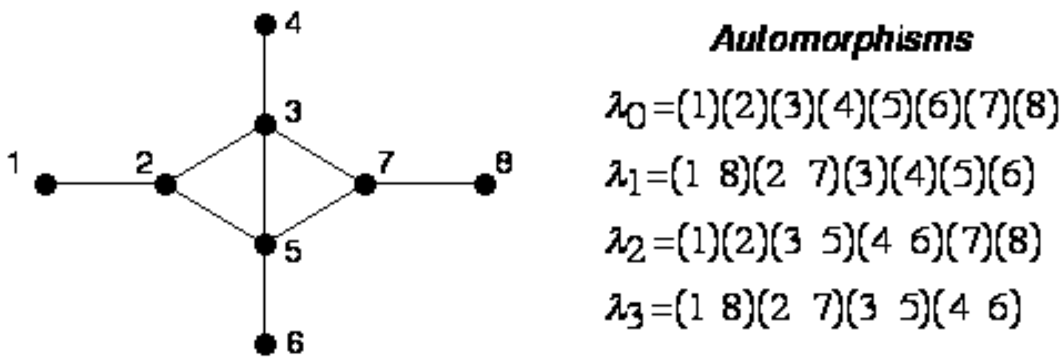


Figure 1: Example 2.2.3 from GTAIA.

Notation The identity of any permutation group is denoted by ι.

(JG) Thus, Watkins would write ι instead of λ_0 in Figure 1.

Rigidity and Orbits

A graph is ***asymmetric*** if the identity ι is its only automorphism. Synonym: ***rigid***.

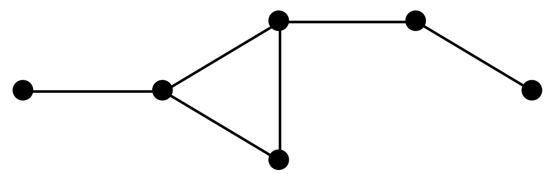


Figure 2: The smallest rigid graph.

(JG) The ***orbit*** of a vertex v in a graph G is the set of all vertices $\alpha(v)$ such that α is an automorphism of G.

Remark While all vertices in the same *orbit* of $\mathrm{Aut}(X)$ must have the same valence, there exist rigid graphs all of whose vertices have the same valence.

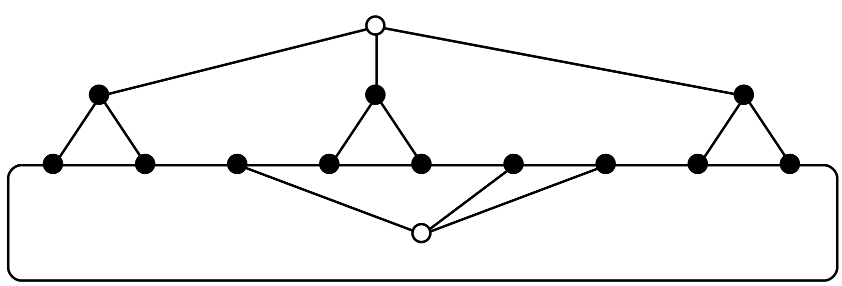


Figure 3: Is this 3-regular graph rigid?

Exercise Must an automorphism map each white vertex to itself. Is the graph in Figure 3 rigid?

Automorphism Group of a Subgraph

Remark If Y is a subgraph of X, then except in special cases there is no relationship between $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$.

- (JG) For instance, if X is the rigid graph of Figure 2, and if Y is the 3-cycle, then $\mathrm{Aut}(X)$ is trivial and $\mathrm{Aut}(Y)$ is the symmetric group Σ_3.
- (JG) Alternatively, if Y is the rigid graph with 6 vertices, and it is a subgraph of K_6, then $\mathrm{Aut}(X)$ is Σ_6 and $\mathrm{Aut}(Y)$ is the trivial group.

Remark When saying that the automorphism group of a graph X "is isomorphic to" a group G, it is ambiguous whether we mean that the isomorphism is between *abstract* groups or between *permutation* groups (see §2). In the examples immediately below, the automorphism groups $\mathrm{Aut}(X)$ are abstractly isomorphic to the given groups G.

Examples of Automorphism Groups of Graphs

NOTATION We will use the formalistic notation $\{u, v\}$ (rather than uv) to represent an edge with endpoints u and v, for reasons that become clear in §3.

EXAMPLE Let $V(K_4) = \{a, b, c, d\}$ and let $X = K_4 - \{a, c\}$, the result of removing edge $\{a, c\}$ from K_4. Then

$$\mathrm{Aut}(X) = \{\iota,\ \alpha,\ \beta,\ \alpha\beta\}$$

where α interchanges a and c but fixes both b and d, while β fixes a and c but interchanges b and d. Thus,

$$\mathrm{Aut}(X) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

REMARK (JG) $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the additive group on 2-tuples of numbers mod 2, with component-wise addition mod 2.

***** REVIEW FROM W4203*****

$\mathrm{Aut}(K_n) \cong \Sigma_n$.

$\mathrm{Aut}(C_n) \cong \mathbb{D}_n$.

EXAMPLE The automorphism group of the n-cycle C_n is the dihedral group $\mathbb{D}_n$ with $2n$ elements.

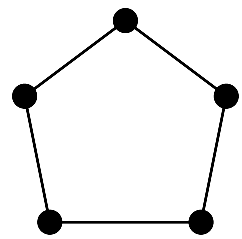


Figure 4: $\mathrm{Aut}(C_5) = \mathbb{D}_5$.

EXAMPLE [Fr37] The automorphism group of the Petersen graph is isomorphic to Σ_5, the symmetric group on five objects.

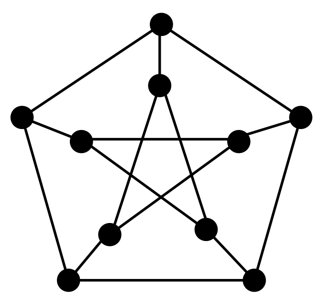


Figure 5: The Petersen graph.

Some Elementary Facts about Automorphism Groups

FACT Let the components of X be $X_1, \ldots, X_k$. Then

$$\mathrm{Aut}(X) \;=\; \prod_{i=1}^{k} \mathrm{Aut}(X_i)$$

(JG) This "fact" is not true except when the components are mutually non-isomorphic.

FACT For a simple graph X with edge-complement $\overline{X}$, we have

$$\mathrm{Aut}(X) \;=\; \mathrm{Aut}(\overline{X})$$

FACT Given any finite tree, either there is a unique vertex or there is a unique edge that is fixed by every automorphism.

EXERCISE Use induction on the diameter of the tree to prove Fact F3.

2 Graphs with Given Group

Definitions

A group G of perms of a set S

- ***acts transitively*** or ***is transitive*** on S if for every $x, y \in S$, there exists $\alpha \in G$ such that $\alpha(x) = y$.
- is ***vertex-transitive*** if $\mathrm{Aut}(X)$ acts transitively on $V(X)$.
- ***acts doubly transitively*** on S if for any two ordered pairs of distinct elements $(x_1, x_2), (y_1, y_2) \in S \times S$ there exists $\alpha \in G$ such that $\alpha(x_1) = y_1$ and $\alpha(x_2) = y_2$.

EXAMPLE Σ_5 acts transitively and doubly transitively on the vertex set of the complete graph K_5.

EXAMPLE Σ_5 acts transitively, but NOT doubly transitively, on the vertex set of the Petersen graph.

Permutation Group Isomorphism

For $i = 1, 2$, let G_i be a group of permutations of the set S_i. We say that G_1 and G_2 are ***isomorphic as permutation groups*** if there exist a group-isomorphism $\Phi : G_1 \to G_2$ and a bijection $f : S_1 \to S_2$ such that

$$f(\alpha(x)) = [\Phi(\alpha)](f(x)) \text{ for all } \alpha \in G_1,\ x \in S_1,$$

i.e., the diagram in Figure 6.1.1 commutes.

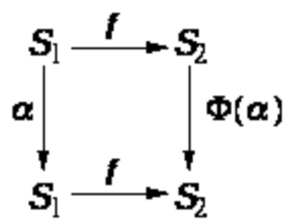


Figure 6: Isomorphism of permutation groups.

EXAMPLE (revised JG) $\mathrm{Aut}(C_3)$ and $\mathrm{Aut}(K_{1,3})$ are abstractly isomorphic, but not isomorphic as permutation groups.

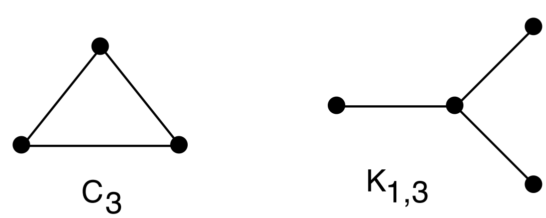


Figure 7: C_3 and $K_{1,3}$.

The ***order*** of a permutation group G acting on a set S is $|G|$, and the ***degree*** is $|S|$.

Edge-Isomorphism and Edge-Automorphism

An ***edge-isomorphism*** from a graph X_1 to a graph X_2 is a bijection $\eta : E(X_1) \to E(X_2)$ such that edges e_1 and e_2 are incident with a common vertex of X_1 if and only if $\eta(e_1)$ and $\eta(e_2)$ are incident with a common vertex of X_2.

An ***edge-automorphism*** is an edge-isomorphism from a graph to itself.

The set of edge-automorphisms forms a subgroup of the symmetric group on $E(X)$; it is called the ***edge-group*** of X.

Basic Fact

Every automorphism α of a graph X induces a unique edge-automorphism η_α; namely, if $\{u, v\} \in E(X)$, then $\eta_\alpha(\{u, v\}) = \{\alpha(u), \alpha(v)\}$. The converse is not true, for some non-connected graphs.

More Complicated Facts

FACT [HarPa68] The edge-group of a graph X and $\mathrm{Aut}(X)$ are (abstractly) isomorphic if and only if X has at most one isolated vertex and K_2 is not a component of X.

FACT [Wh32] Let X_1 an X_2 be connected graphs, neither of which is isomorphic to $K_{1,3}$. If there exists an edge-automorphism from X_1 to X_2, then X_1 and X_2 are isomorphic graphs.

Frucht's Theorem

FACT [Fr38] **Frucht's Theorem:** Given any group G, there exist infinitely many connected graphs X such that $\mathrm{Aut}(X)$ is (abstractly) isomorphic to G. Moreover, X may be chosen to be 3-valent [Fr49].

(JG) The trivial group and $\mathbb{Z}_2$ are represented by K_1 and K_2. What about $\mathbb{Z}_3$? For a directed graph, the answer is easy.

A key idea in Frucht's proof is the construction of "virtual arrows".

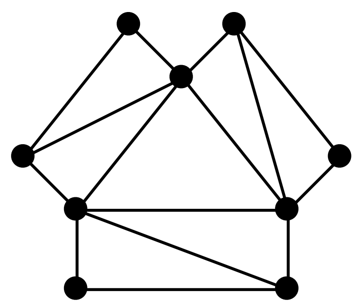


Figure 8: A virtual directed 3-cycle.

(JG) The example above is due to Harary and Palmer (1966), who proved it is a smallest graph with automorphism group $\mathbb{Z}_3$.

(JG) Frucht's proof begins with a directed Cayley graph for the given group. It then uses a slightly different kind of arrow for the arcs corresponding to each generator.

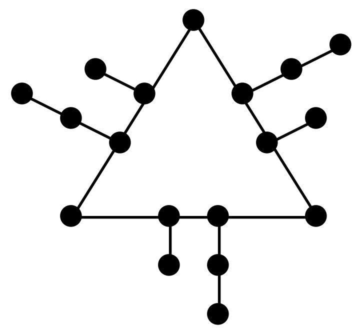


Figure 9: Another virtual directed 3-cycle.

Sabidussi's Extensions of Frucht's Theorem

FACT [Sa57] In Frucht's Theorem, in addition to having $\mathrm{Aut}(X)$ isomorphic to a given group G, one may further impose that X

• has connectivity κ for any integer $\kappa \geq 1$, *or*

• has chromatic number c for any integer $c \geq 2$ (see §5.1), *or*

• is r-valent for any integer $r \geq 3$, *or*

• is spanned by a graph $\tilde{Y}$ homeomorphic to a given connected graph Y.

Minimum Graph Representations of a Group

A ***minimum graph representation*** of a finite group G is a graph X such that $|V(X)|$ is a smallest graph with $\mathrm{Aut}(X) \cong G$. Then $\mu(G)$ denotes $|V(X)|$.

FACT (JG modified) The rigid graph with the fewest vertices is shown in Figure 2. Thus $\mu(\{\iota\}) = 6$.

FACT [Bab74] If G is a nontrivial finite group different from the cyclic groups of orders 3, 4, and 5, then

$$\mu(G) \leq 2\,|G|$$

FACT $\mu(\mathbb{Z}_3) = 9$; $\mu(\mathbb{Z}_4) = 10$; $\mu(\mathbb{Z}_5) = 15$. (See [Sa67].)

EXERCISE Consider the automorphism groups of the 11 simple graphs with 4 vertices. What is the largest group among them?

EXERCISE Construct a simple graph G with six vertices such that $\mathrm{Aut}(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Hint: start with a 4-vertex graph whose automorphism group is $\mathbb{Z}_2 \times \mathbb{Z}_2$.

JG: Cartesian Products of 2-Complexes

A graph is topologically a 1-dim ***cell complex***. The cartesian product $Z = X \square Y$ of two graphs is a reduction from the product of their underlying cell complexes. The generalization to higher dimensional cell complexes becomes clear from consideration of the 2-dimensional case.

Example

$$\begin{array}{cc} X_0 : u, v, w, x & Y_0 : 1, 2, 3 \\ X_1 : uv, vw, vx, wx & Y_1 : 12, 13, 23 \\ X_2 : vwx & Y_2 : 123 \end{array}$$

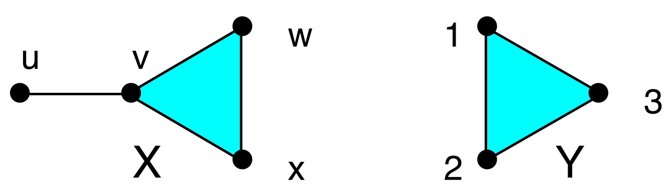
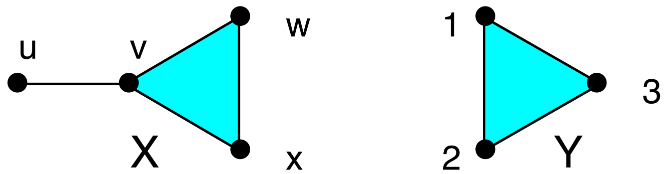


Figure 10: Two 2-dimensional complexes.



CARTESIAN PRODUCT of TWO COMPLEXES

$$
\begin{aligned}
Z_0 &= X_0 \times Y_0 \\
Z_1 &= X_0 \times Y_1 \cup X_1 \times Y_0 \\
Z_2 &= X_0 \times Y_2 \cup X_1 \times Y_1 \cup X_2 \times Y_0 \\
Z_3 &= X_1 \times Y_2 \cup X_2 \times Y_1 \\
Z_4 &= X_2 \times Y_2
\end{aligned}
$$

$$
\begin{aligned}
|Z_0| &= 4 \times 3 = 12 \\
|Z_1| &= 4 \times 3 + 4 \times 3 = 24 \\
|Z_2| &= 4 \times 1 + 4 \times 3 + 1 \times 3 = 19 \\
|Z_3| &= 4 \times 1 + 1 \times 3 = 7 \\
|Z_4| &= 1 \times 1 = 1
\end{aligned}
$$

If we discard X_2 and Y_2, then $Z_2 = X_1 \times Y_1$, and then

$$|Z_2| = 4 \times 3 = 12$$

What does $X \times Y$ look like?

Let's temporarily exclude edge $uv \in X$. Then we get these 9 0-cells, 18 1-cells, and nine 2-cells.

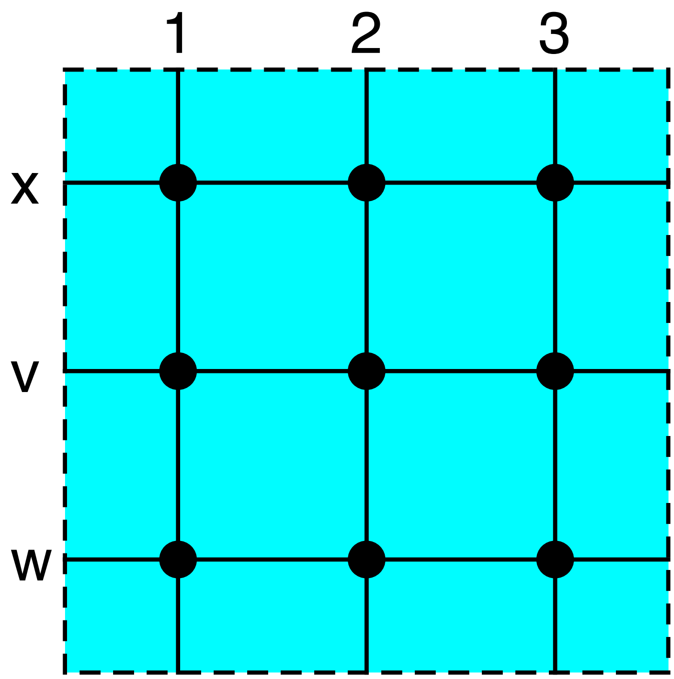


Figure 11: Partial product.

The part resulting from edge uv yields 3 other 0-cells, 6 other 1-cells, and 3 other 2-cells.

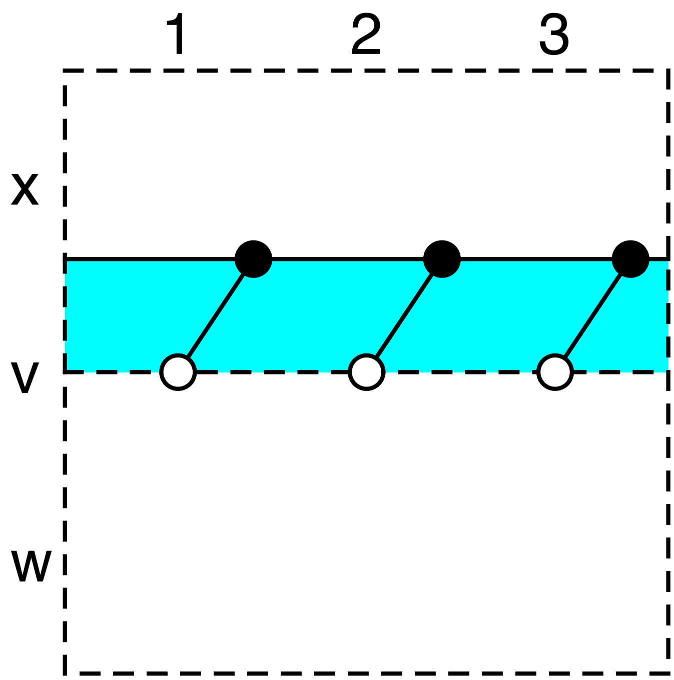


Figure 12: Other part.

Locally, when you cross $K_{1,n}$ with K_2, you get a book with n pages, whose spine corresponds to the K_2 and the n-valent vertex. Since vertex v has degree 3, you get books with 3 pages.

• The graph-theoretic cartesian product simply discards the 2-cells. In other words, it is the 1-skeleton of the topological cartesian product.

3 Groups of Graph Products

In general, a ***graph product*** of graphs X and Y is a graph with vertex set $V(X) \times V(Y)$, whose edge set is determined in a prescribed way by (and only by) the adjacency relations in X and in Y. The symbol & indicates a generic graph product $X\&Y$ of graphs X and Y. It has been shown (see [ImIz75]) that there exist exactly 20 graph products that satisfy this definition.

A graph product & is ***associative*** if

$$(W\&X)\,\&\,Y \;\cong\; W\,\&\,(X\&Y)$$

for all graphs W, X, Y. Interest is usually restricted to products that are *associative*.

Remark In this expository article, Watkins uses notation for each products that looks something like the result of applying that product to two copies of the path P_3, as shown below in Figure 13. Elsewhere, the cartesian product of two graphs is almost always denoted $X \times Y$. The other three products described below occur only in specialized works.

Four commonly used associative graph products

Let Z be a graph product of arbitrary graphs X and Y. Let x_1, x_2 be (not necessarily distinct) vertices of X, and let y_1, y_2 be (not necessarily distinct) vertices of Y. Suppose that

$$\{(x_1, y_1), (x_2, y_2)\} \in E(Z)$$

In the ***cartesian product*** $Z = X \Box Y$
$\big[\{x_1, x_2\} \in E(X) \wedge y_1 = y_2\big]$ or $\big[\{x_1 = x_2\} \wedge \{y_1, y_2\} \in E(Y)\big]$.

In the ***strong product*** $Z = X \boxtimes Y$
$\big[\{x_1, x_2\} \in E(X) \wedge y_1 = y_2\big]$ or $\big[x_1 = x_2 \wedge \{y_1, y_2\} \in E(Y)\big]$
or $\big[\{x_1, x_2\} \in E(X) \wedge \{y_1, y_2\} \in E(Y)\big]$.

In the ***weak product*** $Z = X \times Y$
$\{x_1, x_2\} \in E(X)$ and $\{y_1, y_2\} \in V(Y)$.

In the ***lexicographic product*** $Z = X[Y]$
$\{x_1, x_2\} \in E(X)$ or $[x_1 = x_2 \wedge \{y_1, y_2\} \in E(Y)]$.

These four products are illustrated in Figure 6.1.2, when both X and Y denote the path of length 2.

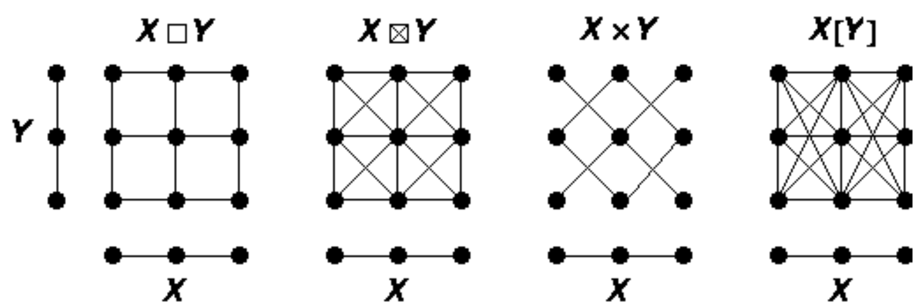


Figure 13: The four products of the 2-path by the 2-path.

Primality

A graph X is a ***divisor*** of a graph Z (with respect to a product &) if there exists a graph Y such that

$$Z = X \& Y \;\vee\; Z = Y \& X$$

A graph Z is ***prime*** (w.r.t. a given product &) if Z has no ***proper divisor***, i.e., no divisor other than itself and the graph consisting of a single vertex.

Graphs X and Y are ***relatively prime*** (w.r.t. a product &) if they have no common proper divisor.

EXAMPLE CL_3 and Q_3 are not relatively prime w.r.t. Cartesian product. They both have K_2 as a factor.

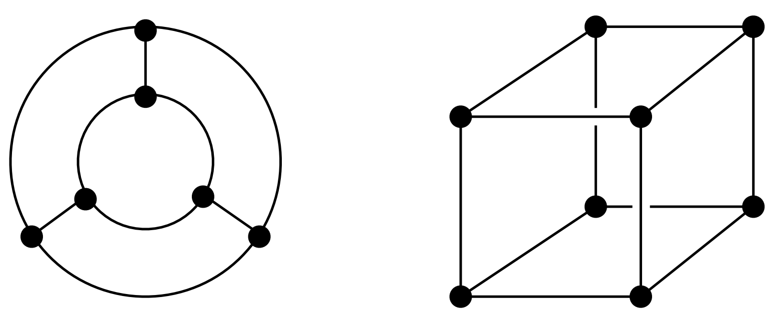


Figure 14: CL_3 and Q_3.

Fact about decomposition

FACT [Sa60] Every connected graph has a unique prime decomposition with respect to the cartesian product and with respect to the strong product.

(JG) Let's not take unique decompositions for granted. In the multiplicative group of even numbers, a number is prime if and only if it is not divisible by 4. However,

$$36 \;=\; 6 \times 6 \;=\; 2 \times 18$$

Facts about automorphisms

FACT The cartesian (respectively, strong) product of two graphs X and Y is connected if and only if both X and Y are connected.

FACT [Sa60] If X is connected, then $\mathrm{Aut}(X)$ is generated by the automorphisms of its prime divisors with respect to the cartesian product and the transpositions interchanging isomorphic prime divisors.

The next Fact is an important corollary.

FACT Let X be the cartesian product

$$X = X_1 \square \cdots \square X_k$$

of *relatively prime* connected graphs. Then $\mathrm{Aut}(X)$ is the direct product

$$\prod_{i=1}^{k} \mathrm{Aut}(X_i)$$

(JG) For instance,

$$\mathrm{Aut}(C_5 \square C_6) \;=\; \mathbb{D}_5 \times \mathbb{D}_6$$

FACT Each of the four products $X \& Y$ is vertex-transitive if and only if both X and Y are vertex-transitive.

Further reading

For a comprehensive and up-to-date treatment of all graph products, see [ImKl00].

4 Transitivity

Definitions

A graph X is ***edge-transitive*** if given

$$e_1,\, e_2 \in E(X)$$

there exists an automorphism $\alpha \in \mathrm{Aut}(X)$ such that

$$\alpha(e_1) = \alpha(e_2)$$

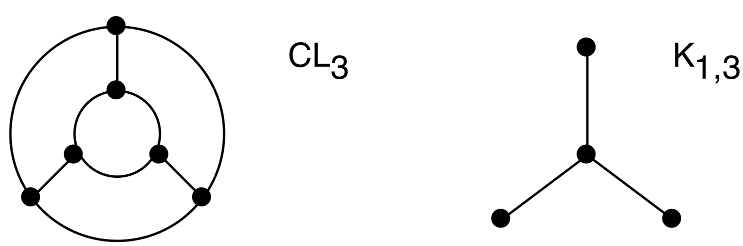


Figure 15: Basic v-trans and e-trans examples.

(JG) We observe that the graph $K_{1,3}$ is edge-transitive but not vertex-transitive. Also, CL_3 is vertex-transitive but not edge-transitive, since six edges lie on a 3-cycle and three do not.

Cartesian products (e.g., $CL_3 = C_3 \square K_2$) are not the only graphs that are vertex-transitive, but not edge-transitive. For instance, all circulant graphs are vertex-transitive, but some are not edge-transitive.

Exercise Prove that $circ\,(8:1,2)$ is not edge-transitive.

Exercise However, prove that $circ\,(8:1,3)$ is edge-transitive.

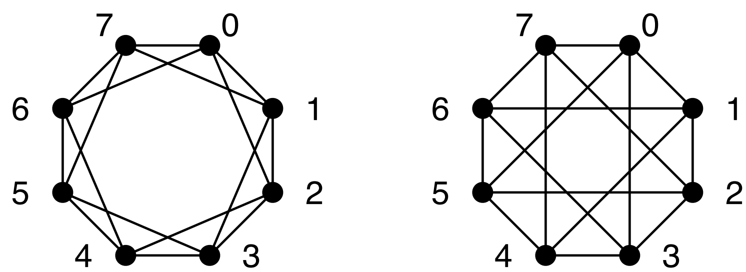


Figure 16: $circ\,(8:1,2)$ and $circ\,(8:1,3)$.

FACT If a vertex-transitive graph is not connected, then all of its components are isomorphic and vertex-transitive.

FACT When $m \neq n$, the graph $K_{m,n}$ is edge-transitive but not vertex-transitive.

FACT If a graph X is edge-transitive but not vertex-transitive, then it is bipartite. In this case, $\mathrm{Aut}(X)$ induces exactly two orbits in $V(X)$, namely, the two sides of the bipartition.

(JG) However, it need not be complete bipartite, as illustrated by the following bipartite graph.

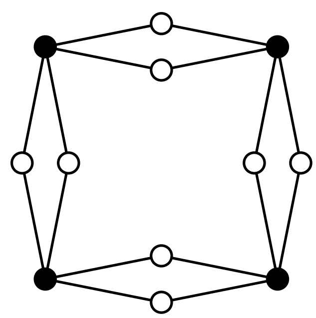


Figure 17: E-transitive but not V-transitive.

Stabilizers

If G is a group of permutations of a set S and $x \in S$, then the ***stabilizer*** of x (in G) is the subgroup

$$G_x = \{\alpha \in G : \alpha(x) = x\}$$

NOTATION The stabilizer of a vertex $u \in V(X)$ in $\mathrm{Aut}(X)$ will be denoted by $\mathrm{Aut}_u(X)$.

Facts

FACT If X is vertex-transitive, then for any $u, v \in V(X)$ we have:

- $\mathrm{Aut}_u(X)$ and $\mathrm{Aut}_v(X)$ are conjugate subgroups of $\mathrm{Aut}(X)$;
- $|\mathrm{Aut}_u(X)| = |\{\alpha \in \mathrm{Aut}(X) : \alpha(u) = v\}|$;
- If X is finite, then $|\mathrm{Aut}(X)| = |\mathrm{Aut}_u(X)| \cdot |V(X)|$.

4.1 s-Regularity and s-Transitivity

Definitions

Let G be a group of permutations of a set S. We say that G ***acts freely*** on S if $G_x = \{\iota\}$ for all $x \in S$.

A permutation group G ***acts regularly*** or ***is regular*** if G acts both transitively and freely.

Facts

FACT If G is regular on S, then

- for all $x, y \in S$, there is a unique $\alpha \in G$ such that $\alpha(x) = y$.
- If S is finite, then $|G| = |S|$.

Examples

An m-***cage*** is a smallest 3-valent graph with girth m.

Example The complete graph K_4 is the unique 3-cage; $K_{3,3}$ is the unique 4-cage.

Example The Petersen graph is 3-regular. It is the unique 5-cage.

Example The unique 6-cage is the ***Heawood graph*** H, which is 4-regular and defined as follows. Let $V(H)$ be the cyclic group $\mathbb{Z}_{14}$. For $j = 0, \ldots, 6$, let the vertex $2j$ be adjacent to the three vertices $2j-1$, , $2j+1$, and $2j+5$.

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