CS E6204 Lecture 2
Minimum and Maximum Genus

Abstract

Minimum graph imbeddings have always been a dominant concern of topological graph theory. Maximum graph imbeddings were among the first of the new topics to emerge at the onset of the modern era. The theoretical importance of these topics has been enhanced by impressive connections to such areas as VLSI design, computer algorithms and complexity, and computer graphics.

* This lecture is based on chapter [Ch04] contributed by Jianer Chen of Texas A&M to the Handbook of Graph Theory.

1 Fundamentals

The graphs in our discussion may have **multiple adjacencies** or **self-adjacencies**. All graphs in our discussion are assumed implicitly to be **connected**. We recall some definitions from Lecture 1 (on weaving, links, and general rotation systems)

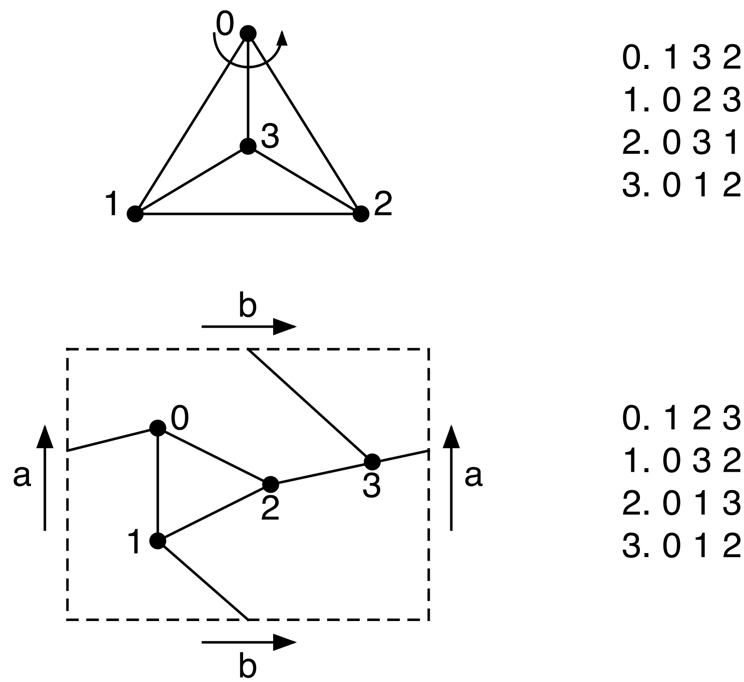
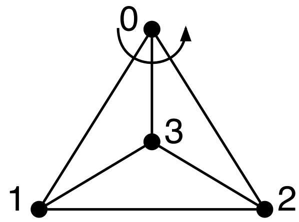


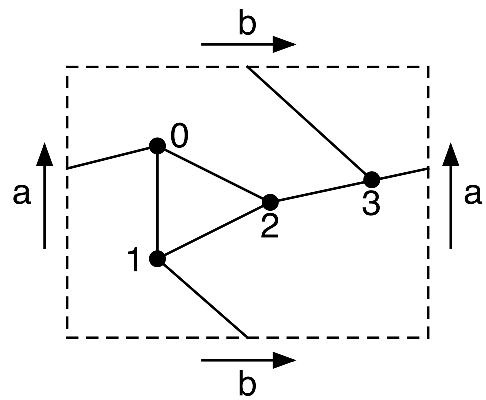
Figure 1: Two inequivalent rotation systems for K_4.

An ***imbedding of a graph*** G in an orientable surface S is a continuous one-to-one function $\rho : G \to S$ from a topological representation of the graph G into the surface S.

The ***image of an imbedding*** $\rho : G \to S$ is the subspace $\rho(G)$ of the surface S. Sometimes, one refers to $\rho(G)$ either as "the graph" or as "the imbedding".



0. 1 3 2
1. 0 2 3
2. 0 3 1
3. 0 1 2



0. 1 2 3
1. 0 3 2
2. 0 1 3
3. 0 1 2

Each connected component of $S - \rho(G)$ is called a ***face*** of the imbedding $\rho(G)$.

The imbedding is ***cellular*** if the interior of each face of the imbedding is homeomorphic to a 2-dimensional open disk. (Our discussion will be restricted to cellular graph imbeddings.)

The ***genus of the imbedding*** $\rho : G \to S_g$ is the genus g of the imbedding surface.

The ***minimum genus*** $\gamma_{\min}(G)$ (or, simply, the ***genus*** $\gamma(G)$) ***of a graph*** G is the minimum integer g such that there exists an imbedding of G into the orientable surface S_g of genus g.

The ***maximum genus*** $\gamma_{\max}(G)$ ***of a graph*** G is the maximum integer g such that there exists a (cellular) imbedding of G into the orientable surface of genus g.

The number $|E| - |V| + 1$ is called the ***cycle rank*** (or the ***Betti number***) of the graph G, denoted $\beta(G)$. This is best regarded conceptually as the number of non-tree edges relative to a spanning tree for G.

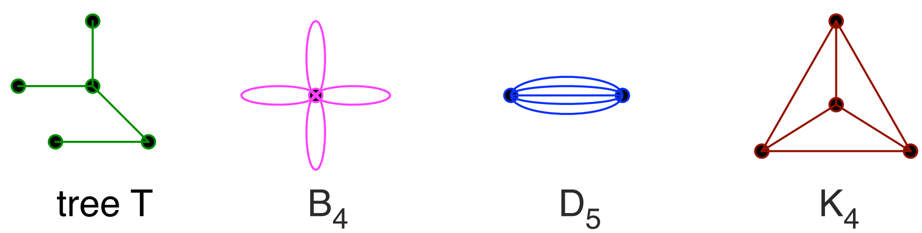


Figure 2: Tree, bouquet, and dipole.

Example 1.1 For a graph of any of the types in Figure 2

graph	$\gamma_{\min}$	$\gamma_{\max}(T) = 0$	$\beta(T) = 0$
tree T	0	0	0
bouquet B_n	0	$\lfloor \frac{n}{2} \rfloor$	n
dipole D_n	0	$\lfloor \frac{n-1}{2} \rfloor$	$n-1$
complete graph K_n	$\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$	$\left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor$	$\binom{n-1}{2}$

Some General Facts

We recall from Lecture 1 the operation of adding a handle to a surface.

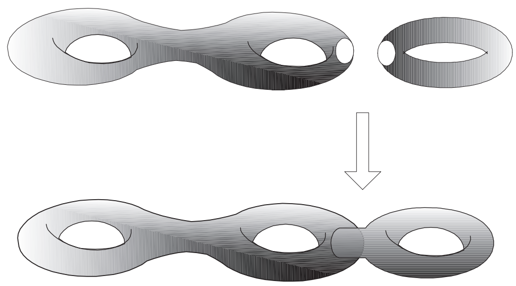


Figure 3: Adding a handle.

Theorem 1.2 *Any orientable surface is homeomorphic to one of the surfaces S_g obtained by adding g handles to the sphere.*

Proof. [Br23]. □

• The genus of any imbedding of a graph G is an integer between 0 and $\lfloor \beta(G)/2 \rfloor$, where $\beta(G)$ is the cycle rank of the graph G.

• Inserting an edge into a graph imbedding can never decrease the imbedding genus, and deleting an edge can never increase the imbedding genus.

Theorem 1.3 (Euler Polyhedral Equation) *An orientable imbedding $\rho : G \to S$ of a graph G with vertex set V, edge set E, face set F, and genus g satisfies the relation*

$$|V| - |E| + |F| \;=\; 2 - 2g$$

Proof. (e.g., see [GrTu87]). □

Theorem 1.4 (Additivity of Minimum Genus) *Let*

$$\{B_1,\, B_2,\, \cdots ,\, B_k\}$$

be the set of ***2-connected*** *components of a graph G. Then*

$$\gamma_{\min}(G) = \sum_{i=1}^{k} \gamma_{\min}(B_i)$$

Proof. [BHKY62]. □

Theorem 1.5 (Additivity of Maximum Genus) *Let*

$$\{B_1,\, B_2,\, \cdots ,\, B_k\}$$

be the set of ***2-edge-connected*** *components of a graph G. Then*

$$\gamma_{\max}(G) = \sum_{i=1}^{k} \gamma_{\max}(B_i)$$

Proof. [NoStWh71]. □

Accordingly, in most cases, we may concentrate on the minimum genus of 2-connected graphs and on the maximum genus of 2-edge-connected graphs.

2 Planarity and Upper-Imbeddability

There has been extensive research on graphs of min genus 0 and on graphs G of max genus $\lfloor \beta(G)/2 \rfloor$. Such graphs have special names.

- A ***planar graph*** is a graph of minimum genus 0.
- An ***upper-imbeddable graph*** is a graph G of maximum genus $\lfloor \beta(G)/2 \rfloor$.

Theorem 2.1 (Kuratowski's theorem) *A graph is planar if and only if it contains no subgraph homeomorphic to either K_5 or $K_{3,3}$.*

PROOF. [Ku30] $\square$

- A ***minor*** of a graph G is a graph H that can be obtained from G by a sequence of edge-deletions and contractions.

Theorem 2.2 *A graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor.*

PROOF. [Wa37] $\square$

Theorem 2.3 *There is essentially only one way to imbed a 3-connected planar graph in the plane, up to homeomorphism of pairs.*

PROOF. [Wh33] $\square$

Theorem 2.4 *Every planar graph has a planar imbedding in the geometric plane in which every edge is a straight line segment.*

PROOF. [Fa48] $\square$

Theorem 2.5 *Every 3-connected planar graph has a planar imbedding in the geometric plane in which every face, except the outer face, is a convex polygon.*

PROOF. [Tu60] $\square$

Theorem 2.6 *A graph G is upper-imbeddable if and only if G has a spanning tree T such that the co-tree $G - T$ has at most one odd component.*

PROOF. [NoStWh71] □

EXAMPLE It follows that the dumbbell graph has maximum genus 0.

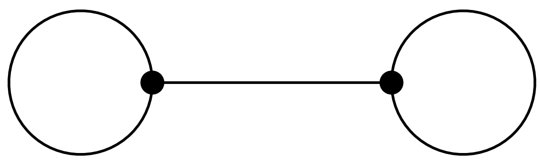


Figure 4: The dumbbell graph.

CLASSSROOM EXERCISE

Show that the circular ladder CL_4 is upper imbeddable.

Theorem 2.7 (Ku74) *Every 4-edge-connected graph contains two edge-disjoint spanning trees.*

CLASSROOM EXERCISE

Show that $C_4 \times C_4$ has two disjoint spanning trees.

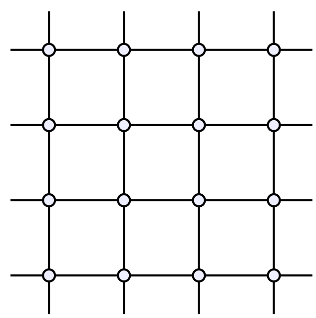


Figure 5: $C_4 \times C_4$.

Corollary 2.8 *Every 4-edge-connected graph is upper-imbeddable.*

PROOF. This follows immediately from the two preceding facts, and is commonly ascribed to [Ku74]. $\Box$

REMARK The study of upper-imbeddability has been focused on graphs that are not 4-edge connected, due to the preceding corollary.

CLASSROOM EXERCISE

Show that $K_{3,3}$ does not have two disjoint spanning trees, but that it is upper-imbeddable.

Exercise The complete bipartite graph $K_{3,n}$ is an upper-imbeddable graph. Construct a spanning tree for $K_{3,5}$ whose complement is connected.

Remark See [ChArGr96] for a general construction of 3-edge connected graphs that are not upper-imbeddable.

Deficiency

Early study of upper-imbeddability was focused on derivation of the upper-imbeddability of special graph classes. It became clear later that most of these results could be obtained from effective characterizations of maximum genus. There have been several productive characterizations of maximum genus.

• The ***co-tree*** for a spanning tree T of a graph G is the edge complement $G-T$. (The co-tree $G-T$ need not be connected.)

• A connected component H of the co-tree $G-T$ is called an ***even component*** (resp. ***odd component***) if the number of edges in H is even (resp. odd).

• The ***deficiency*** $\xi(G,T)$ of a spanning tree T is defined to be the number of odd components of the co-tree $G-T$.

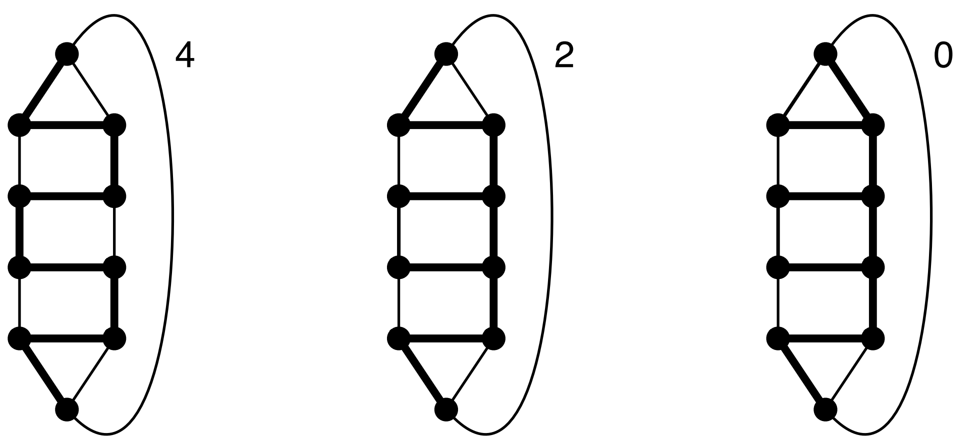


Figure 6: spanning trees of deficiency 4, 2, and 0.

• [Xu79a] The ***deficiency*** $\xi(G)$ ***of the graph*** G is defined to be the minimum of $\xi(G, T)$ over all spanning trees T of the graph G.

• A spanning tree T of G is called a ***Xuong tree*** if its deficiency $\xi(G, T)$ of T is equal to the deficiency $\xi(G)$ of the graph G.

Theorem 2.9 (Xu79a) *The maximum genus* $\gamma_{\max}(G)$ *of a graph* G *is equal to* $(\beta(G) - \xi(G))/2$.

REMARK (JG) The number of spanning trees for K_n is n^{n-2}. Therefore, the Xuong deficiency should not be calculated by exhaustion.

• Two edges are ***adjacent*** if they share a common end.

Theorem 2.10 *Two adjacent edges* e_1 *and* e_2 *can be inserted into an imbedding* $\rho(G)$ *of a graph* G *so thaat the imbedding genus is increased.*

Theorem 2.11 (ChKa99) *A graph* G *has a spanning tree such that the co-tree* $G-T$ *contains at least* $\gamma_{\max}(G)$ *pairs of adjacent edges.*

Example The complete graphs K_n are upper-imbeddable for all $n \geq 1$.

Proof. Use $K_{1,n-1}$ as a spanning tree. The cotree is K_{n-1}. $\square$

Example The complete bipartite graphs $K_{n,m}$ are upper-imbeddable for all $n, m \geq 1$.

Proof. Use the bar-amalgamation $K_{1,n-1}$—$K_{1,m-1}$ as a spanning tree. The cotree is $K_{n-1,m-1}$. $\square$

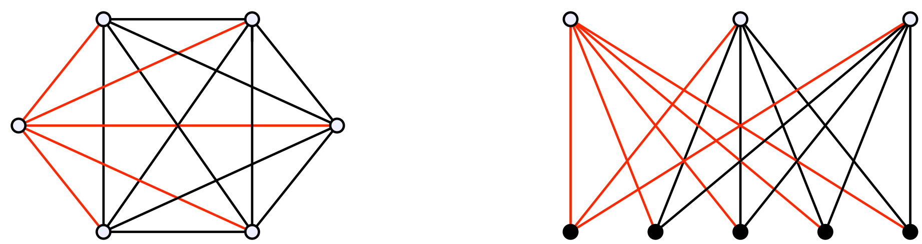


Figure 7: K_n and $K_{m,n}$.

Theorem 2.12 (SkNe89) *k-regular vertex-transitive graphs of girth g such that $k \geq 4$ or $g \geq 4$ are upper-imbeddable.*

Theorem 2.13 (Skov91) *Loopless graphs of diameter 2 are upper-imbeddable.*

Theorem 2.14 (HuLi00a) *$(4k+2)$-regular graphs and $(2k)$-regular bipartite graphs are upper-imbeddable.*

3 Lower Bounds on Min and Max Genus

The *Euler Polyhedral Equation*

$$|V| - |E| + |F| \;=\; 2 - 2g$$

implies that

- a minimum genus imbedding of a graph G is an imbedding with the largest number of faces; and
- a maximum genus imbedding of G is one with the minimum number of faces.

Lower Bounds for Minimum Genus

• The ***size of a face*** f, denoted $size(f)$ is the number of edge-steps in its boundary walk. (If f is an n-gon, then $size(f) = n$; that is, each edge in the boundary walk of f that occurs twice is counted twice.)

• The ***girth*** of a graph G is the length of a shortest cycle in G. It is undefined for a tree.

FACT *Edge-Face Equality*: For any imbedded graph $G = \langle V, E \rangle$ with face set F,

$$2|E| \;=\; \sum_{f \in F} size(f)$$

This is because each edge of G is counted twice on the right size.

FACT The girth of a simple graph is at least 3.

FACT The girth of a simple bipartite graph is at least 4.

FACT For any imbedded graph $G = \langle V, E \rangle$ with face set F, and for any face $f \in F$,

$$girth(G) \;\leq\; size(f)$$

FACT *Edge-Face Inequality*: For any imbedded graph $G = \langle V, E \rangle$ with face set F,

$$2|E| \;\geq\; girth(G) \cdot |F|$$

This follows from the Edge-Face Equality and the previous Fact.

Fact For any graph $G = \langle V, E \rangle$,

$$\gamma_{\min}(G) \ \geq \ \frac{(girth(G) - 2) \cdot |E|}{2girth(G)} - \frac{|V|}{2} + 1$$

Fact Let G be a simple graph. Then

$$\gamma_{\min}(G) \ \geq \ \frac{|E|}{6} - \frac{|V|}{2} + 1$$

Fact Let G be a simple bipartite graph. Then

$$\gamma_{\min}(G) \ \geq \ \frac{|E|}{4} - \frac{|V|}{2} + 1$$

Fact If a simple graph $G = \langle V, E\rangle$ has a triangulated orientable imbedding, then the imbedding is min-genus and

$$\gamma_{\min}(G) \;=\; \frac{|E|}{6} - \frac{|V|}{2} + 1$$

Fact A quadrangulated orientable imbedding for a simple bipartite graph $G = \langle V, E\rangle$ is min-genus imbedding and

$$\gamma_{\min}(G) \;=\; \frac{|E|}{4} - \frac{|V|}{2} + 1$$

Remark A classical approach to computing the minimum genus of a simple (non-bipartite) graph is to try to construct a triangulated imbedding of the graph, or for a simple bipartite graph, a quadrangulated imbedding.

Example Construct $K_7 \to S_1$ with a voltage graph.

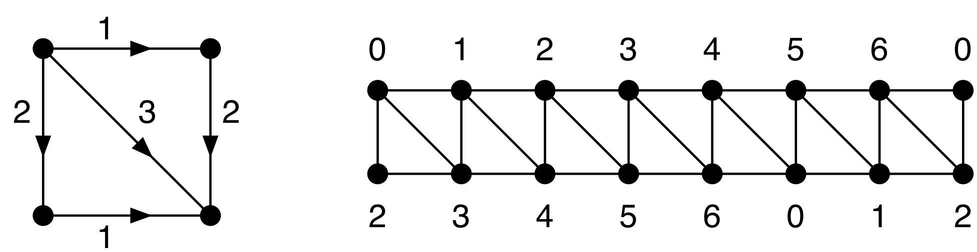


Figure 8: Constructing $K_7 \to S_1$ with a voltage graph.

Remark This approach has been very successful in deriving minimum genus of well-known graph classes. Voltage graphs are presently the main tool for constructing triangulations and quadrangulations.

JG-insert: min genus of $K_{10} - K_6$

When the standard lower bound is unattainable for a graph G, it can be less straightforward to calculate the minimum genus. One of the simpler approaches is to find a subgraph of G whose minimum genus exceeds the standard lower bound. We consider the graph $K_{10} - K_6$.

We obtain the standard lower bound

$$\gamma_{min}(K_{10} - K_6) \;\geq\; \frac{45 - 15}{6} - \frac{10}{2} + 1 \;=\; 1$$

Trying to draw an imbedding of $K_{10} - K_6$ into S_1 is doomed to failure. However, we can increase the lower bound by observing that $K_{6,4}$ is a subgraph of $K_{10} - K_6$. It follows that

$$\gamma_{min}(K_{10} - K_6) \;\geq\; \gamma_{min}(K_{6,4}) \;=\; \left\lceil \frac{(6-2)(4-2)}{4} \right\rceil \;=\; 2$$

Of course, this improved lower bound might not be achievable, but this is progress.

Proofs (or sketches of proofs) of most of the following facts appear in [GrTu87].

FACT [RiYo68] For the complete graph K_n, with $n \geq 3$,

$$\gamma_{\min}(K_n) \;=\; \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

(JG) This is the standard lower bound. The imbeddings were constructed with combinatorial current graphs, which are predecessors of topological current graphs, which are the duals of voltage graphs.

- The ***Heawood number*** of a surface S with Euler characteristic c and chromatic number $chr(S)$ is

$$H(S) = \left\lfloor \frac{7 + \sqrt{49 - 24c}}{2} \right\rfloor$$

FACT [RiYo68] (Formerly called the Heawood Conjecture) The chromatic number of every surface except the Klein bottle N_2 is equal to its Heawood number.

FACT Let G be a graph of chromatic number c, then

$$\gamma_{\min}(G) \geq \frac{(c^2 - 7c + 12)}{12}$$

FACT [Ri65] For $m, n \geq 2$,

$$\gamma_{\min}(K_{m,n}) = \left\lceil \frac{(n-2)(n-2)}{4} \right\rceil$$

FACT [Ri55] For the cube graph Q_n of n vertices, with $n \geq 3$,

$$\gamma_{\min}(Q_n) = 1 + 2^{n-3}(n-4)$$

Lower Bounds on Maximum Genus

By the additivity theorem for maximum genus, we may confine our interest to graphs that are 2-connected. Subdivision and smoothing do not change the maximum genus. The maximum genus of a 4-edge connected graph G is $\lfloor \beta(G)/2 \rfloor$. This leads us to focus on lower bounds for the maximum genus of graphs of minimum degree at least 3 and are not 4-edge-connected.

- [GrKlRi93] A ***necklace of type*** (r, s) is a graph obtained from an $(r+s)$-cycle by doubling r disjoint edges and then attaching a self-loop at each of the s vertices that is not an endpoint of a doubled edge.

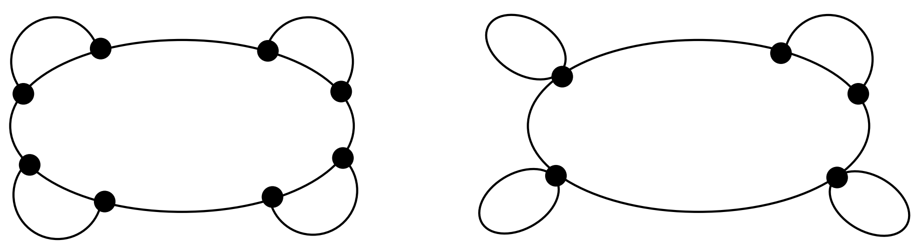
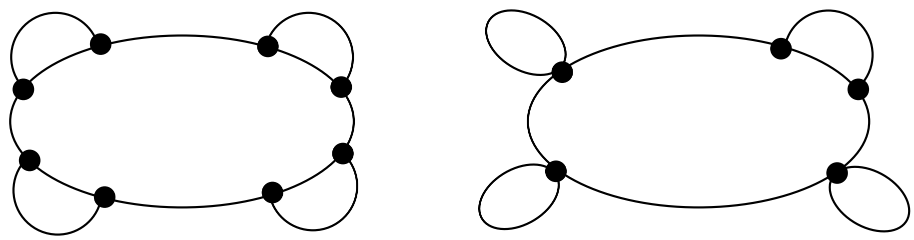


Figure 9: a type-$(4, 0)$ necklace and a type-$(1, 3)$ necklace.



FACT The cycle rank of a necklace of type (r, s) is $r + s + 1$.

FACT [GrKlRi93] The max genus of every necklace is 1.

FACT [ChGr95] The necklace construction is essentially the only way to construct graphs of large cycle rank and small maximum genus.

FACT [ChKaGr96] Let G be a simple graph of minimum degree at least 3. Then

$$\gamma_{\max}(G) \geq \left\lceil \frac{\beta(G)}{4} \right\rceil$$

This bound is tight, since there are infinitely many simple graphs G of minimum degree 3 whose maximum genus is arbitrarily close to $\beta(G)/4$.

FACT Let G be a 2-edge connected simple graph of minimum degree at least 3. Then

$$\gamma_{\max}(G) \geq \left\lceil \frac{\beta(G)}{3} \right\rceil$$

(See [Ar+02].) The next Fact indicates that this result cannot be improved for 3-edge-connected graphs.

FACT [ChKaGr96] There exists an infinite class of 3-edge-connected simple graphs G whose maximum genus is equal to $\lceil \beta(G)/3 \rceil$.

FACT [Ar+02] Let G be a 3-edge connected graph. Then

$$\gamma_{\max}(G) \geq \left\lceil \frac{\beta(G)}{3} \right\rceil$$

This bound is tight, by the previous Fact.

4 Kuratowski-Type Theorems

Erdós and König [Koni36] asked whether there is a Kuratowski-type theorem for the class of graphs that are imbeddable (not necessarily 2-cell imbeddable) in a fixed surface S, by which they meant a finite set of forbidden subgraphs. Due to the work of Robertson and Seymour, we now discuss forbidden minors.

Complete Forbidden Sets for Minimum Genus

- A class $\mathcal{F}$ of graphs is ***closed under minors*** if for each graph G in $\mathcal{F}$, all minors of G are also in $\mathcal{F}$.
- Let $\mathcal{F}$ be a graph class closed under minors. A graph G is a ***minimal forbidden minor*** for $\mathcal{F}$ if G is not in $\mathcal{F}$ but every proper minor of G is in $\mathcal{F}$.
- Let $\mathcal{F}$ be a graph class that is closed under minors. A set $\mathcal{M}$ of minimal forbidden minors is a ***complete set of forbidden minors*** for $\mathcal{F}$ if for every graph G that is not in $\mathcal{F}$, there exists a graph in $\mathcal{M}$ that is a minor of G.

FACT The graphs K_5 and $K_{3,3}$ make a complete set of forbidden minors for the class of planar graphs. (This is an alternative version of the Kuratowski Theorem.)

FACT [GlHu78] There exists a finite complete set of forbidden minors for the class of graphs that can be imbedded in the projective plane N_1. A complete list can be found in [GlHuWa79] or [Ar81].

FACT [ArHu89] For every non-orientable surface N, there is a finite set $\mathcal{F}_N$ of forbidden minors for the class of graphs that are imbeddable in N.

FACT [RoSe88] (Formerly known as Wagner's Conjecture) Any class of graphs closed under minors has a finite complete set of forbidden minors. An extraordinary series of papers ([RoSe85, RoSe88, RoSe90a, RoSe90b, RoSe95]) led to this result.

FACT For every integer $g \geq 0$, the class of graphs whose minimum genus is bounded by g is closed under minors.

FACT [RoSe90b] For every integer $g \geq 0$, there is a finite complete set of minimal forbidden minors for the class of graphs of minimum genus bounded by g.

FACT [Se93] The size of a minimum complete set of minimal forbidden minors for graphs of min genus bounded by g is at most $2^{2^{(6g+9)^9}}$.

REMARK Contracting an edge e of G on a planar imbedding $\rho(G)$ can be accomplished by continuously "shrinking" the edge e on the plane until its two ends are identified. This yields a planar imbedding of the contracted graph G/e. Moreover, edge deletion does not increase imbedding genus. Thus, the class of planar graphs is closed under minors. Using a similar argument, one can show that the minimum genus of a minor of a graph G can never be larger than $\gamma_{\min}(G)$.

REMARK A constructive proof for Seymour's theorem (immediately above) was developed by Mohar [Mo99]. There has been further effort to simplify the proof [Th97b]. On the other hand, it has remained as a challenge, even for very small g such as $g = 1$, to give a good estimation on the number of graphs or the size of the graphs in the set of minimal forbidden minors.

Complete Forbidden Sets for Maximum Genus

FACT Classes of graphs of given maximum genus are not closed under minors.

EXAMPLE The bouquet B_2 of two self-loops (i.e., the graph with a single vertex and two self-loops) is a minor of the dumbbell graph DB (i.e., the graph consisting of an edge $[u, v]$ plus two self-loops on u and v, respectively). However, it is easy to verify that

$$\gamma_{\max}(B_2) = 1 \qquad \text{and that} \qquad \gamma_{\max}(DB) = 0$$

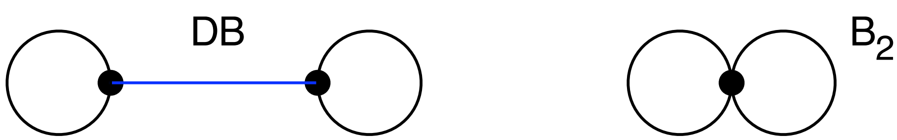


Figure 10: Contracting DB to B_2.

• Let G be a graph and let v be a degree-2 vertex with two neighbors u and w in G (u and w could be the same vertex). We say that a graph G' is obtained from G by ***smoothing*** the vertex v if G' is constructed from G by removing the vertex v then adding a new edge connecting the vertices u and w.

• Two graphs G_1 and G_2 are ***homeomorphic*** if they become isomorphic after smoothing all degree-2 vertices. It is easy to see that two homeomorphic graphs have the same minimum genus and the same maximum genus.

FACT A 2-edge-connected graph G has maximum genus 0 if and only if G is a cycle.

FACT [NoStWh71] A graph G has maximum genus 0 if and only if no vertex is contained in two different cycles in G. Such a graph has been called a ***cactus***.

FACT [ChGr93] A 2-edge connected graph G has maximum genus 1 if and only if G is homeomorphic either to a necklace or to one of the graphs in Figure 11.

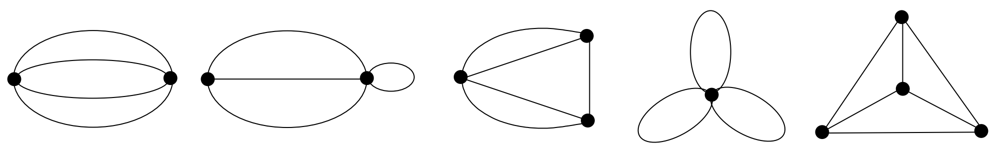


Figure 11: Graphs of maximum genus 1 that are not necklaces.

FACT A graph G has maximum genus 1 if and only if all but one of its 2-edge connected components are either a cycle or a single vertex, and the exceptional 2-edge connected component of G is homeomorphic either to a necklace or to one of the graphs in Figure 11.

5 Algorithmic Issues

Fellows and Langston [FeLa88] indicated that the min genus problem for graphs of **bounded minimum genus** can be solved in polynomial time, based on Robertson and Seymour's results in graph minor theory. In fact, they showed a much stronger result, that for any graph class $\mathcal{C}$ closed under minors, there is a polynomial-time algorithm that tests the membership for the class $\mathcal{C}$.

Minimum Genus Testing

FACT [HoTa74] There is a linear-time algorithm that either constructs a planar imbedding for a given graph or reports that the graph is not planar.

FACT [HoWo74] There is a linear-time algorithm that tests the isomorphism of planar graphs.

FACT [RoSe95] Let H be a fixed graph. There is a polynomial-time algorithm that for a given graph G decides whether H is a minor of G.

FACT For any closed under minors graph class $\mathcal{C}$, there is a polynomial time algorithm that tests the membership for the class $\mathcal{C}$.

FACT [Mo99] For each fixed integer g, there is a linear-time algorithm that, for a given graph G, either constructs an imbedding of genus bounded by g for G or reports that no such an imbedding exists. (This constructive result of Mohar significantly improves the corollary to [RoSe95] that a polynomial-time algorithm exists.)

FACT [Th89] The following problem is NP-complete: given a graph G and an integer k, decide whether $\gamma_{\min}(G) \leq k$.

FACT [Th97a] The problem of deciding whether a graph of maximum degree 3 has its minimum genus bounded by a given integer k is NP-complete.

Maximum Genus Testing

FACT [FuGrMc88] There is a polynomial-time algorithm that constructs a maximum genus imbedding for a given graph.

REMARK Direct calculation from the Xuong and Nebeský characterizations requires exponential time. The polynomial-time algorithm is based on a reduction to the linear matroid parity problem, which is solvable in polynomial time [GaSt85].

FACT [Ch94] For any fixed integer g, there is a linear time algorithm that decides whether a given graph has maximum genus g, and if so, the algorithm constructs a maximum genus imbedding for the graph.

FACT [Ch94] For any fixed integer g, there is a linear time isomorphism algorithm for graphs of maximum genus bounded by g.

Fact [GrRi91] Starting from any imbedding of a graph, there is a sequence of edge deletion-then-reinsertion operations that never decreases the imbedding genus and eventually leads to a maximum genus imbedding of the graph. Thus, there are no graph imbeddings that are "strictly locally maximal" but not globally maximum for imbedding genus.

Fact [GrRi91] There exist "strictly locally minimal" graph imbeddings that are not minimum genus imbeddings, that are traps of arbitrarily great depth, serving as obstructions.

Remark The two facts immediately above together explain the difference in complexity of the calculations of minimum genus and maximum genus.

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