11.11 LECTURE 11: MAPS – PART B

Adapted from §7.6 in HBGT, by

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11.11.5 Automorphisms and Coverings

Every map M has a universal cover that is a classical tiling of the sphere, of the Euclidean plane, or of the hyperbolic plane (unit disc). We also address the relation between a group acting as automorphisms of a map and a group acting as homeomorphisms of the surface.

In this section, the classical Euclidean and hyperbolic tessellations are regarded as infinite maps, even though, by our definition, a map is a *finite* cell-complex.

DEFINITIONS

D1: An ***automorphism*** of a map M is an isomorphism of M onto itself. The automorphisms form a group under composition, denoted $\mathcal{A}ut(M)$.

D2: A ***map covering*** $f : M_1 \to M_2$ is a topological covering (see §7.4) of the respective surfaces that takes the graph of M_1 onto the graph of M_2, with ramification points possible only at vertices and face centers.

D3: The ***tessellation*** $\{p, q\}$ is the tesselation (unique, if it exists) of the sphere or plane into

regular p-gons, q-valence.

Tessalation $\{p,\, q\}$ tiles the sphere if $\dfrac{1}{p} + \dfrac{1}{q} > \dfrac{1}{2}$, i.e.

$$\begin{aligned}\{p,\, q\} \;=\;& \{3,\, 3\} \text{ tetrahedron}\\ & \{4,\, 3\} \text{ octahedron}\\ & \{5,\, 3\} \text{ dodecahedron}\\ & \{3,\, 4\} \text{ cube}\\ & \{3,\, 5\} \text{ icosahedron}\end{aligned}$$

tiles the Euclidean plane if $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{2}$

$$\begin{aligned}\{p,\, q\} \;=\;& \{4,\, 4\} \text{ rectangular}\\ & \{6,\, 3\} \text{ hexagonal}\\ & \{3,\, 6\} \text{ triangular}\end{aligned}$$

or tiles the hyperbolic plane (unit disc) if $\dfrac{1}{p} + \dfrac{1}{q} < \dfrac{1}{2}$.

(See Fact F36 below for an example.)

Automorphisms of Tesselations

JG: triangle groups – a natural model

The automophisms of a tesselation by a triangle whose angles are

$$\frac{\pi}{r},\ \frac{\pi}{s},\ \frac{\pi}{t} \quad \text{where} \quad \frac{1}{r}+\frac{1}{s}+\frac{1}{t} \ = \ 1$$

form the triangle group $\triangle(r, s, t)$. As a first example, we consider $r = s = t = 3$

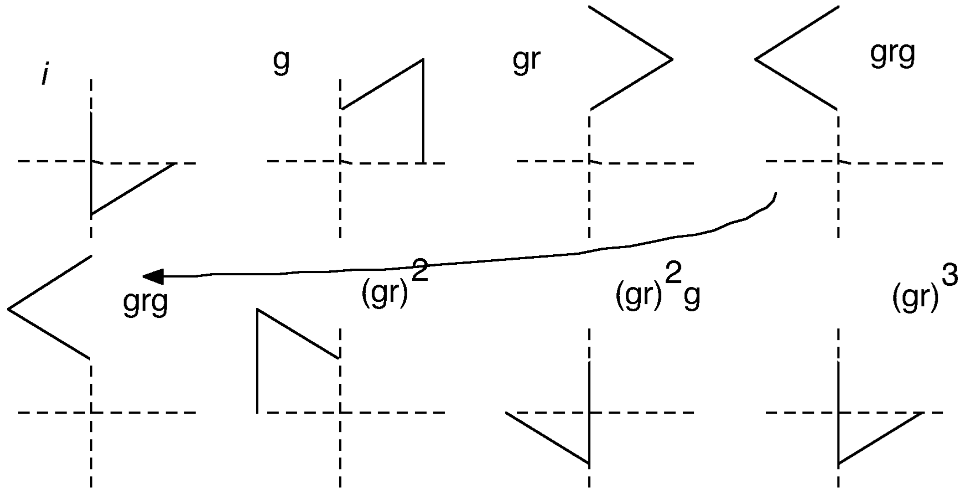


Fig. JG-1: The orbit of an equilateral triangle under alternating reflections through two sides.

EXPLANATION

r is reflection thru the red edge

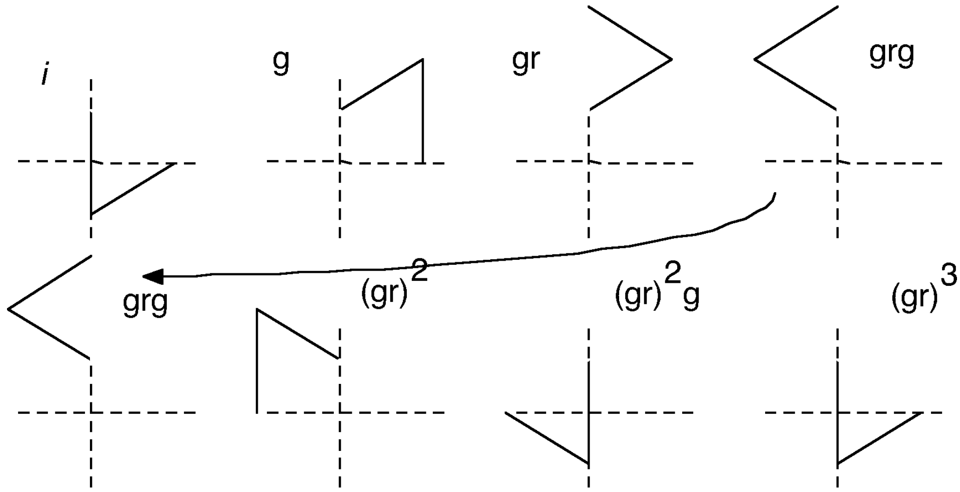
g is reflection thru the green edge

b is reflection thru the blue edge

r is reflection thru the red edge

g is reflection thru the green edge

b is reflection thru the blue edge



We observe in the figure that $(gr)^3 = \iota$. Similarly $(bg)^3 = \iota$ and $(rb)^3 = \iota$. We also observe that the angle between any two sides is $\pi/3$. The group with presentation

$$\langle b,\, g,\, r \;:\; b^2 = g^2 = r^2 = \iota,\; (bg)^3 = (gr)^3 = (rb)^3 = \iota \rangle$$

is known as the ***triangle group*** $\triangle(3,3,3)$.

Remark: Since there are sequences of flips that move the principal triangle arbitrarily far from the origin, it follows that this triangle group is infinite.

The automophisms of a tesselation by a triangle whose angles are

$$\frac{\pi}{2} \quad \frac{\pi}{4} \quad \frac{\pi}{4}$$

form the triangle group $\triangle(4,4,2)$, with presentation

$$\langle b,\, g,\, r \ :\ b^2 = g^2 = r^2 = \iota,\ (bg)^4 = (gr)^2 = (rb)^4 = \iota \rangle$$

The automophisms of a tesselation by a triangle whose angles are

$$\frac{\pi}{2} \quad \frac{\pi}{3} \quad \frac{\pi}{6}$$

form the triangle group $\triangle(6,3,2)$, with presentation

$$\langle b,\, g,\, r \ :\ b^2 = g^2 = r^2 = \iota,\ (bg)^6 = (gr)^2 = (rb)^3 = \iota \rangle$$

D4: The ***Coxeter group*** $W(p,q)$ is the group with presentation by three generators ρ_0, ρ_1, ρ_2 and the relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_2\rho_0)^2 = 1 \qquad (3)$$

EXAMPLES

E1: Both torus maps M and M^* in Figure 7.6.1 are coverings of the tetrahedral map in Figure 7.6.4. The covering by M is ramified at vertices and the covering by M^* is ramified at face centers. Both are 2-fold coverings, that is, each unramified point of the sphere is covered by two points of the torus.

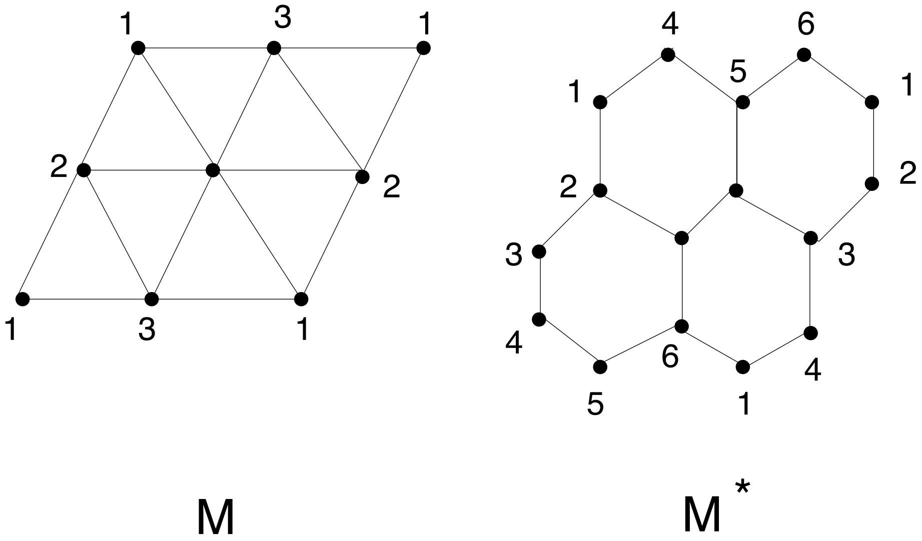


Fig 11.11.1 Review: A torus map and its dual.

The map M^* is obtained by assigning voltage 1 mod 2 to every edge of $K_4 \to S_0$. The map M is obtained as the dual of map M^*.

FACTS

F4: Every map M of type $\{p, q\}$ has an unramified covering by the tessellation $\{p, q\}$. For example, the map on N_5 of type $\{5, 5\}$ in Figure 11.11.2, is covered by the tessellation $\{5, 5\}$ of the hyperbolic plane. (The map is obtained by identifying like-labeled edges in the figure.)

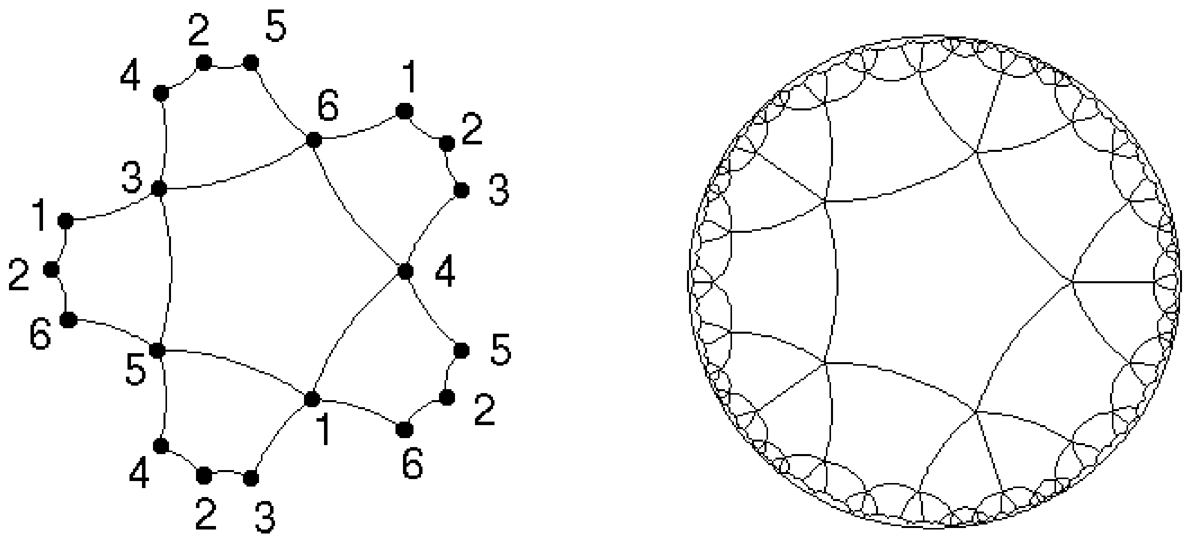


Fig 11.11.2 The regular self-dual map $\{5, 5\}_3$ and its universal cover $\{5, 5\}$.

F6: [Hu1892] ***Hurwitz formula:*** If a group Γ acts on a surface of Euler characteristic $c < 0$, then

$$|\Gamma| \leq -84c$$

F8: [Tu83] If a group Γ acts on an orientable surface S, then some Cayley graph G of Γ imbeds in S, and the natural action of Γ on G (by left multiplication) extends to an action of Γ on S.

11.11.6 Combinatorial Schemes

The definition of a map in §10.1 as a cell complex is topological. A strictly combinatorial description, although less intuitive, is often easier to apply. Three such schemes are described: rotation scheme, permutation scheme, and graph encoded map.

DEFINITIONS

D5: A ***rotation scheme*** (G, ρ) consists of a graph G and a set of cyclic permutation (called rotations)

$$\rho = \{\rho_v\}_{v \in V(G)}$$

of the edge-ends incident on each vertex v. Any map with graph G imbedded on an orientable surface (this can be extended to include nonorientable imbeddings) is representable by such a scheme.

D6: The ***map of a rotation scheme*** is obtained as follows. Given a directed edge $e_1 = (v_0, v_1)$ of G, consider the cycle consisting of successive directed edges

$$e_1, e_2, \ldots, e_m = e_1$$

where $e_i = (v_{i-1}, v_i)$ and

$$e_{i+1} = \rho_{v_i}(e_i)$$

Each (undirected) edge lies on exactly two such cycles. Regarding each cycle as the boundary of a polygonal 2-cell and gluing together 2-cells along paired edges results in an orientable surface in which G is imbedded.

Conversely, the ***rotation scheme of a map*** M on an orientable surface is (G, ρ), where G is the graph of M and ρ_v is the cyclic permutation of the edge-ends incident on vertex v induced by the orientation of the surface, say clockwise.

Remark: Any process of pasting polygons together along edges yields a 2-complex.

- Having each edge occur exactly twice in the set of polygon boundaries implies that the boundary of that 2-complex is empty.
- Having a cyclic rotation of the edges incident at each vertex guarantees that there are no singularities of dimension zero.

That is, the 2-complex specified by a rotation scheme with cyclic permutations of the edge-ends at each vertex is a surface, rather than a pseudosurface.

D7: An ***abstract permutation scheme*** (π, σ) ***on a finite set*** X consists of permutations π and σ acting on X, such that each orbit of π has length 2 and such that the permutation group $H\langle \pi, \sigma \rangle$ generated by σ and π is transitive on X.

D8: The ***vertices, edges,*** and ***faces*** of the permutation scheme (π, σ) are the cycles of σ, π and $\sigma \circ \pi$, respectively.

Example 11.11.3: Let $X = \{1, 2, \ldots, 8\}$.

$$\begin{aligned} \sigma &= (1,2)(3,4)(5,6)(7,8) \\ \pi &= (1,3,5,8)(2,4,6)(7) \\ \sigma \circ \pi &= (1,4,5,2,3,6,8,7) \end{aligned}$$

Thus, the permutation group generated by σ, π, and $\sigma \circ \pi$ is transitive.

Remark: This abstract nonsense verion of a scheme represents the following toroidal map.

FIGURE GOES HERE

11.11.7 Symmetry of Maps

Regular maps, those enjoying the greatest symmetry, are the surface analogues of the Platonic solids. Also discussed are symmetrical and vertex-transitive maps.

DEFINITIONS

D15: A ***flag*** of a map M is an ordered triple (F_0, F_1, F_2) of mutually incident faces of dimensions $0, 1$ and 2, respectively.

(JG) A concrete perspective is that a flag of M is a face of the barycentric subdivision of M, as illustrated below.

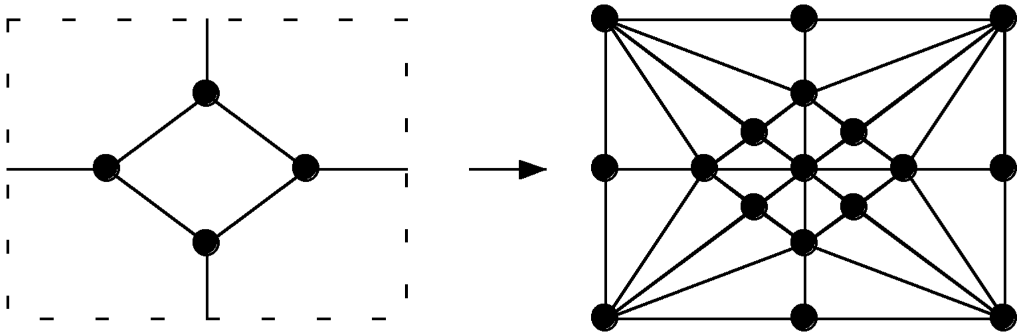
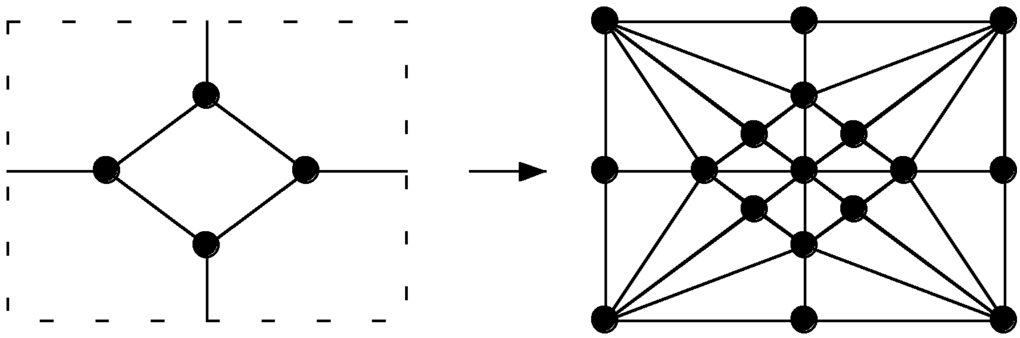


Fig. JG-2: A toroidal map and its flags.

Exer 11.11.1 (JG) What is the face-width of the map of Fig JG-2? What is the edge-width?

D16: A map M is a ***regular map*** if $\mathcal{A}ut(M)$ acts transitively on the set of flags.

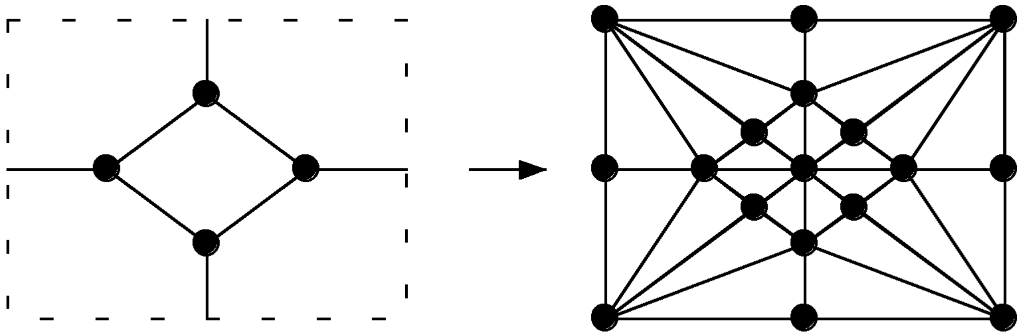
(JG) Although $\mathcal{A}ut(M)$ acts transitively on the (black) vertices of $G(M)$, it does not act transitively on the regions, since it cannot map an 8-gon to a 4-gon. Thus, it does not map a flag in the 8-gon to a flag in the 4-gon. Thus, it is not regular.



(JG) By way of contrast, all five Platonic maps are regular. For any fixed flag F, every automorphism is determined by the choice of a flag onto which F is to be mapped. It follows, for instance, that the cube map has 48 automorphisms, because it has 48 flags.

D17: A map M is a ***symmetrical map*** if $\mathcal{A}ut(M)$ has at most two orbits in its action on the set of flags.

(JG) The map M of JG-2 is not symmetrical. Since $\mathcal{A}ut(M)$ does not map an edge that occurs twice on the 8-gon to an edge on the 4-gon, it cannot map a flag with one kind of edge to a flag with the other kind.



D18: A ***Cayley map for a group*** Γ with generator set Δ, is an imbedding of the Cayley graph $G_\Delta\Gamma$, using a rotation scheme. The cyclic permutation on the edges $\Delta^* = \Delta \cup \Delta^{-1}$ incident at each vertex must be the same at each vertex (see Example 21).

REMARKS

R5: For a symmetrical map M, the automorphism group $\mathcal{A}ut(M)$ acts transitively on the set of vertices, on the set of edges, and on the set of faces.

EXAMPLES

E5: The regular maps on the sphere are the boundary complexes of the five ***Platonic solids*** which have types

$$\{3,3\},\ \{3,4\},\ \{4,3\},\ \{3,5\},\ \{5,3\}$$

respectively, plus the infinite families of (non-polyhedral) maps of types $\{p,2\}, \{2,p\},\ p > 0$.

E6: Since every map on the projective plane has a 2-fold covering by a map on the sphere (Fact 51), it follows from Example 11.11.5 that there are four regular maps on the projective plane of types $\{3,4\}, \{4,3\}, \{3,5\}, \{5,3\}$ and infinite families of types $\{p,2\}$ and $\{2,p\}$, where $p \equiv 2 \bmod 4$.

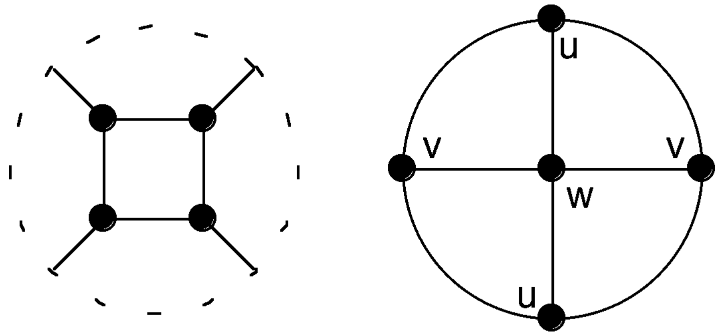


Fig. JG-3: Regular maps of types $\{4,3\}$ and $\{3,4\}$ in the projective plane.

Exer 11.11.2 (JG) Draw a regular map of type $\{2p,2\}$ and another of type $\{2,2p\}$, both in the projective plane.

E7: There are 3 infinite families of regular torus maps of types $\{3,6\}, \{6,3\}$ and $\{4,4\}$. For example, the maps in Figure 11.11.1 are of types $\{3,6\}$ and $\{6,3\}$.

Exer 11.11.3 (JG) Use voltage graphs to construct infinite families of regular torus maps of types $\{3,6\}$ and $\{4,4\}$.

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E8: [CoDo01] used a network of computers to determine all regular maps on orientable surfaces of genus 2 to 15 and all regular maps on nonorientable surfaces from genus 4 to 30.

E9: The ***Kepler-Poinsot regular star-polyhedra*** — see Figure 11.11.3 — are self-intersecting realizations of regular maps. In the notation of Fact 60 below, these maps are $\{5,5\,|\,3\}$ (12 pentagons on a surface of genus 4 — great dodecahedron and small stellated dodecahedron), $\{5,3\}_{10}$ (12 pentagons on the torus — great stellated dodecahedron) and $\{3,5\}_{10}$ (20 triangles on the torus — great icosahedron).

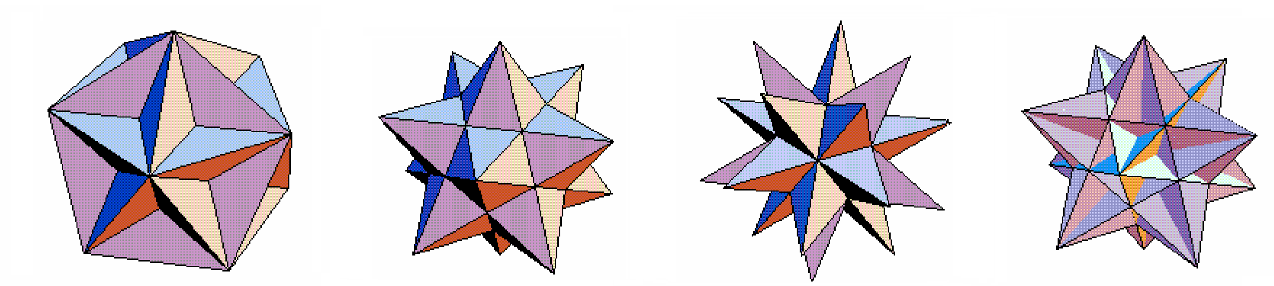

Fig 11.11.3 **Star polyhedra.**

E10: [ScWi85, ScWi86] From the history of automorphic functions come two regular maps of genus 3, the 1879 ***Klein map*** $\{7,3\}_8$ composed of 24 heptagons with automorphism group $PGL(2,7)$, and the 1880 ***Dyck map*** $\{8,3\}_6$ composed of 12 octagons (shown in dual form in Figure 11.11.4).

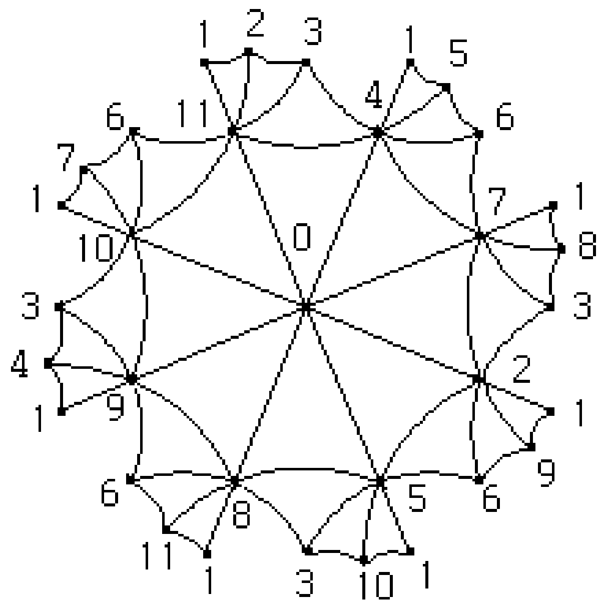

Fig 11.11.4 Dyck's map $\{3,7\}_8$.

D19: A map is a ***chiral map*** if it is symmetrical, but not regular.

(JG) Fig 14 contains a chiral map.

E11: Figure 11.11.5 is a chiral map on the torus. Opposite sides of the square are to be identified. This map is presented as the Cayley map of the cyclic group $Z_5 = \{0, 1, 2, 3, 4\}$ with generating set $\Delta = \{1, 2\}$ and cycle $\pi = (-1\ 2\ 1\ -2)$ on Δ^*.

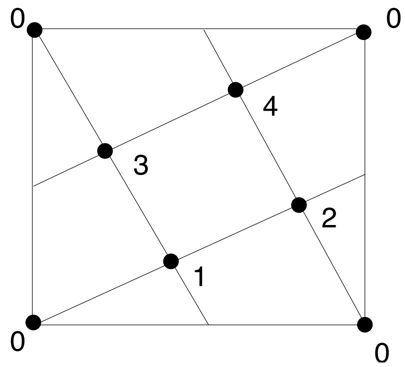

Fig 11.11.5 A chiral map on the torus given as a Cayley map of Z_5.

Denoting an edge by a pair of vertices and a face by its four vertices, the flags $(1, 12, 1234)$ and $(2, 12, 1234)$ are in two different orbits under the action of the automorphism group acting on the set of flags. There is no automorphism that leaves edge 12 and face 1234 fixed and takes vertex 1 to vertex 2.

E12: Coxeter and others noticed that regular maps frequently occur as coverings of smaller regular maps on other surfaces. For example, the regular torus maps $\{3,6\}_4$ and $\{6,3\}_4$ in Figure 11.11.1 are 2-fold coverings of the tetrahedral map $\{3,3\}$ on the sphere. Constructions of families of regular maps using coverings appear in [JoSu00], [Si00], [Vi84], and [Wi78] among others.

FACTS

F12: A map M with f_1 edges has exactly $4f_1$ flags.

F13: In $\mathcal{A}ut(M)$, the stabilizer of any flag is trivial.

F14: For any map M with f_1 edges, the two immediately preceding facts imply that

$$|\mathcal{A}ut(M)| \leq 4f_1$$

with equality if and only if M is regular. In this sense, the regular maps have the largest possible automorphism group.

F15: On each orientable surface there is a regular map.

F16: For a regular map on an orientable surface, half the automorphisms act as orientation-preserving homeomorphisms of the surface and half as orientation-reversing.

F17: Not every nonorientable surface has a regular map; for example, there are no regular maps on the surfaces with nonorientable genus 2 and 3.

F18: [Vi83b], [Wi78a] Every nonorientable regular map has a unique 2-fold unramified covering by a regular orientable map.

F19: No chiral map exists on a nonorientable surface.

F20: For any surface S with Euler characteristic

$$c(S) < 0$$

there are at most finitely many regular maps. This follows from the Hurwitz formula.

F21: [Vi83b] For any pair (p, q) such that

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$$

there are infinitely many regular maps of type $\{p, q\}$. [NeŠk01] subsequently showed that these maps may be chosen to have arbitrarily large face-width.

F22: [Wi89] There is a regular map with complete graph K_n if and only if $n = 2, 3, 4, 6$.

F23: [Bi71] There is a symmetrical map with complete graph K_n if and only if n is a prime power and, for each prime power, the symmetrical map is unique.

F27: Any Cayley map of a group Γ is vertex transitive, Γ acting as a group of automorphism of the Cayley map by left multiplication.

F28: The double torus S_2 has the interesting property that only finitely many groups act (as a group of homeomorphisms) on S_2, but there are infinitely many vertex-transitive (Cayley graphs) with genus 2.

F29: [Th91],[Ba91] For each $g \geq 3$, there are only finitely many vertex-transitive graphs of orientable genus g while there are infinitely many of genus $0, 1$ and 2.

11.11.8 Enumeration

W. T. Tutte [Tu68] pioneered map enumeration in the 1960's. Explicit results for maps on the sphere appear below.

DEFINITIONS

D20: A **rooted map** is a map in which a flag has been distinguished.

D21: A rooted map is a ***near-triangulation*** if every nonroot face is a 3-gon.

EXAMPLES

E15: For the sphere, the 2-connected rooted maps with 4 edges are shown in the first row of Figure 11.11.6. The first four of these comprise all 2-connected rooted maps with 3 vertices and 3 faces. The root face is the outer face, the root vertex and edge are in boldface.

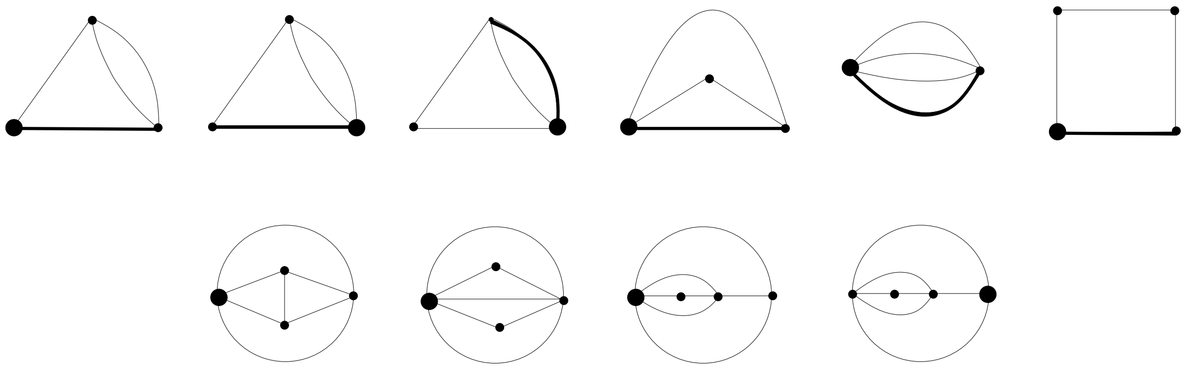

Fig 11.11.6 Counting maps on the sphere.

E16: On the second row of Figure 11.11.6 are the rooted near triangulations with 4 inner faces and a root face with 2 edges. The root face is the outer face; the root edge is the bottom edge; and the root vertex is in boldface.

FACTS

F30: [Tut63] The number of rooted maps on the sphere with $n \geq 0$ edges is

$$g(n) = \frac{2 \cdot 3^n (2n)!}{n!(n+2)!}$$

F31: [Tut63] The number of 2-connected rooted maps on the sphere with $n \geq 1$ edges is

$$\frac{2(3n-3)!}{n!(2n-1)!}$$

F32: [N. Wormald] (see [GoJa83]) The number of 2-edge-connected rooted maps on the sphere with $n \geq 0$ edges is

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}$$

F33: [BrTu64] The number of 2-connected rooted maps on the sphere with $n \geq 1$ vertices and $k \geq 2$ faces is

$$\frac{(2n+k-5)!(2k+n-5)!}{(n-1)!(k-1)!(2n-3)!(2k-3)!}$$

11.11.9 Paths and Cycles in Maps

This section covers three topics involving paths and cycles:

- the Lipton-Tarjan separator theorem;
- the existence of nonrevisiting paths in polyhedral maps;
- the decomposition of maps along cycles in the graph.

The third topic is related to a result of Robertson and Seymour on minors.

DEFINITIONS

D22: A path p in the graph of a map M is said to be ***nonrevisiting*** if $p \cap F$ is connected for each face F of M.

D23: A surface S has the ***nonrevisiting path property*** if, for any polyhedral map M on S, any two vertices of M are joined by a nonrevisiting path.

D24: A map M is a ***map minor*** of a map M' if M can be obtained from M' by a sequence of edge contractions and deletions. The operations of edge deletion and edge contraction on a graph can be extended to a surface imbedding of the graph in an obvious way.

EXAMPLE

E17: A polyhedral map on the surface S_2 that lacks the nonrevisiting path property appears in Figure 11.11.7 below. (The map is obtained by gluing along like labeled edges.) There is no nonrevisiting path from x to y.

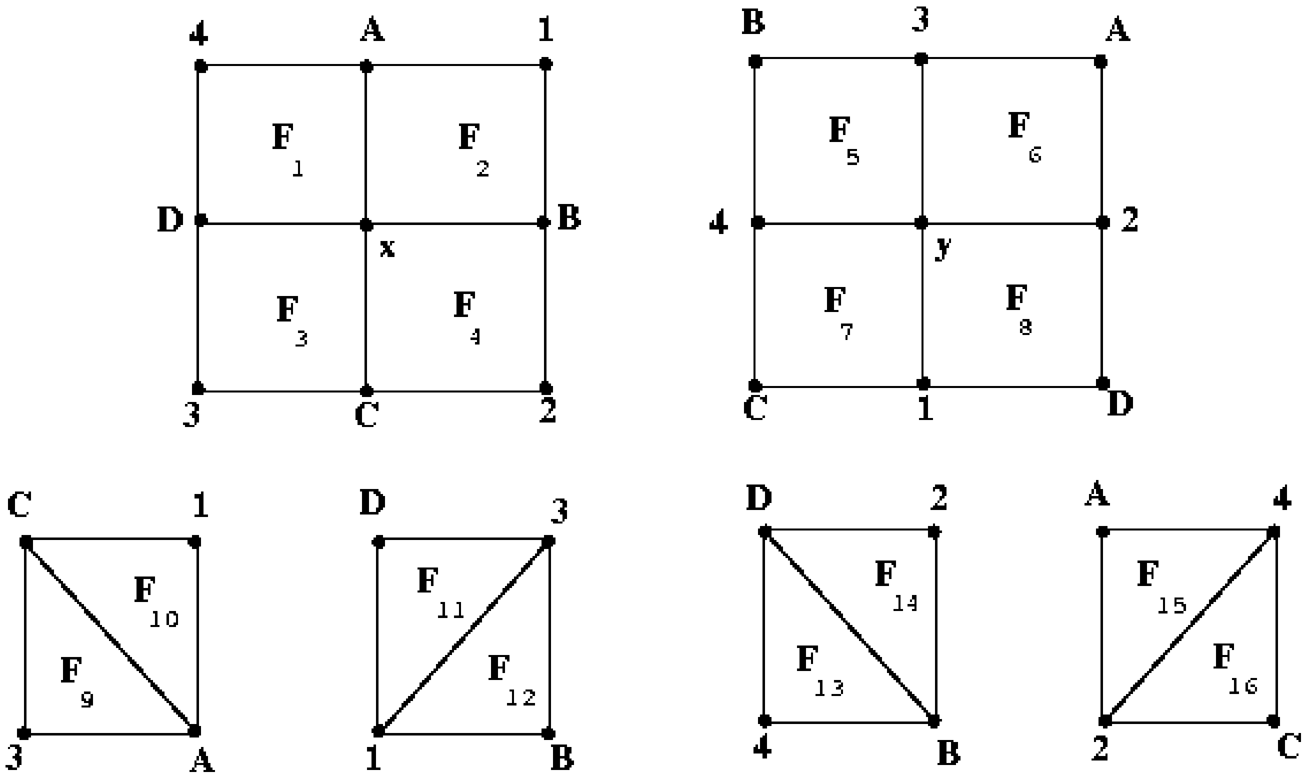

Fig 11.11.7 A map on S_2 that does not satisfy the non-revisiting path property.

(JG) To show that the path x, B, 4, y revisits face F_1, it suffices to show that $x, 4 \in F_1$, but $B \notin F_1$.

Exer 11.11.4 (JG) What face does the path x, A, 2, y revisit?

FACTS ABOUT SEPARATORS

F35: [LiTa79] **Planar Separator Theorem**: In a planar graph with n vertices, there is a set of at most

$$2\sqrt{2n}$$

vertices whose removal leaves no component with more than

$$2n/3$$

vertices.

F36: [AlSeTh94] Let M be a loopless map on the sphere with n vertices. Then there is a simple closed curve τ on the surface of the sphere passing through at most

$$k \leq \frac{2}{3}\sqrt{2n}$$

vertices (and no other points of the graph) such that each of the two open disks bounded by τ contain at most

$$2n/3 - k/2$$

vertices. This result slightly improves the Lipton-Tarjan separator theorem.

F37: [GiHuTa84] A map of genus g contains a set of at most

$$O(\sqrt{gn})$$

vertices whose removal leaves no component of the graph with more than

$$2n/3$$

vertices. This generalizes the Lipton-Tarjan theorem to maps on orientable surfaces of higher genus.

FACTS ABOUT NONREVISITING PATHS

F38: [PuVi98] For polyhedral maps, the nonrevisiting path property holds for the

sphere, torus, projective plane and Klein bottle.

It fails for all other surfaces except possibly the nonorientable surface of genus 3 (see [PuVi96] and Example E27).

F39: The nonrevisiting path property holds for every polyhedral map with face-width at least 4.

FACTS ABOUT DISJOINT CYCLES

F40: [RoSe88] Let M_0 be a map on a surface S other than the sphere. There exists a constant k such that, for any map M on S with $fw(M) \geq k$, M_0 is a map minor of M. The following two results provide values for the constant k when the given map M_0 consists of certain sets of disjoint cycles.

F41: [Sc93] A map M on the torus with face-width w contains $\lfloor 3w/4 \rfloor$ disjoint noncontractible cycles.

F42: [BrMoRi96] For general surfaces there exist $\lfloor w/2 \rfloor$ pairwise disjoint contractible cycles in the graph of any map M, all containing a particular face, $\lfloor (w-1)/2 \rfloor$ pairwise disjoint, pairwise homotopic, surface nonseparating cycles, and $\lfloor (w-1)/8 \rfloor - 1$ pairwise disjoint, pairwise homotopic, surface separating, noncontractible cycles. (It is unknown whether any map of orientable genus $g \geq 2$ with face-width at least 3 must contain a noncontractible surface separating cycle.)

F43: [Bar88] Every polyhedral map on the torus (projective plane, Klein bottle) is isomorphic to the complex obtained by identifying the boundaries of two faces of a 3-polytope (cross-identifying one face of a 3-polytope, cross-identifying two faces of a 3-polytope).

F44: [Yu97] (see also [Th93]) If d is a positive integer and M is a map on S_g of face-width at least $8(d+1)(2^g-1)$, then the graph of M contains a collection of induced cycles $C_1, C_2, \ldots, C_g$ such that the distance between distinct cycles is at least d and cutting along the cycles results in a map on the sphere. This generalizes Fact 11.11.43.

F45: [Sc91] Schrijver proved necessary and sufficient conditions (conjecture by Lovasz and Seymour) for the existence of pairwise disjoint cycles $\tilde{C}_1, \ldots, \tilde{C}_k$ on the graph of a map M homotopic to given closed curves $C_1, \ldots, C_k$ on the surface.

REMARK

R7: The Lipton-Tarjan separator theorem has applications to divide-and-conquer algorithms. Nonrevisiting paths $\blacksquare$ arise in complexity issues for edge following linear programming algorithms like the simplex method.

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