10.10 LECTURE 10: MAPS – PART A

Adapted from §7.6 of HBGT, by

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INTRODUCTION

Among the many contributors to the theory of maps are Archimedes, Kepler, Euler, Poinsot, de Morgan, Hamilton, Dyck, Klein, Heawood, Hurwitz, Steinitz, Whitney, Koebe, Tutte, Coxeter, and Grünbaum. General references on maps include [BoLi95], [BrSc97], [CoMo57], [GrTu87], [MoTh01], and [Wh01].

TERMINOLOGY: In ***topological graph theory***, we study invariants of a fixed graph, whose imbeddings may range over many surafces. In ***topological map theory***, we study the invariants of a fixed surface, usually from a polygonal perspective in which the 1-skeleton may vary.

10.10.1 Maps and Polyhedra Maps

Some basic notions are map and polyhedral map, duality, isomorphism, face and edge-width. The existence and uniqueness of a map with a given graph is addressed.

DEFINITIONS

D1: A ***map*** M ***on a surface*** S is a finite cell-complex whose underlying topological space is S. The surface of a map M is denoted $|M|$.

D2: The ***graph of the map*** M is its 1-skeleton. It is denoted $G := G(M)$.

Remark: The 1-skeleton must be connected, because each 2-cell is homeomorphic to the unit disk.

D3: Maps M_1 and M_2 are ***isomorphic*** if there is a homeomorphism of the respective surfaces that induces an isomorphism of the respective graphs.
Notation: $M_1 \approx M_2$.

Example 10.10.1: Triangulations and quadrangulations of a surface are maps on the surface.

D4: The ***vertices*** and ***edges of a map*** M are the vertices and edges, respectively, of its graph $G(M)$.

D5: The ***faces of a map*** M are the connected components of $|M| \setminus G(M)$.

D6: The ***0-, 1-, and 2-dimensional faces of a map*** M are its vertices, edges and faces, respectively.

D7: Given a map M on a surface S, the ***dual map*** M^* is a map on the same surface S,

- whose vertex set V^* consists of one point interior to each face of M and
- whose edge set E^* consists, for each edge e of M, of an edge e^* crossing e and joining the vertices of V^* that correspond to the faces incident with e.

D8: A ***polyhedral map*** M, generalizing the notion of a convex polyhedron, is a map

- whose face boundaries are cycles, and
- such that any two distinct face boundaries are either disjoint or meet in either a single edge or vertex.

Example 10.10.2: All three maps in Figure 10.10.1 are non-polyhedral.

Fig 10.10.1 Three non-polyhedral maps.

- The boundary of map A is not a cycle.
- Two 2-cells of map B meet in two edges.
- The boundary of toroidal map C is not a cycle.

D9: The ***face-width of a map*** M, denoted $fw(M)$, is the minimum number of points $|\tau \cap G(M)|$ over all non-contractible simple closed curves τ on the surface.

D10: The ***edge-width of a map*** M, denoted $ew(M)$, is the length of a shortest cycle in $G(M)$ that is noncontractible on the surface.

D11: A ***large-edge-width (LEW)*** map is a map whose edge-width is greater than the number of edges in any face boundary.

Fig 10.10.2 A map with face-width 2 and edge-width 1.

(JG) The intuition is that a plane is "infinitely wide" and that the larger the width, the more a map will tend to share characteristics with a planar map.

(JG) Face-width and edge-width are not defined for planar maps, because there are no non-contractible curves, and the minimum of the empty set is undefined.

EXAMPLES

E3: A map M on the torus and the dual map M^* appear in Figure 10.10.3. (The torus is obtained by identifying like labeled edges on the boundary of the polygon.) Neither M nor M^* is polyhedral. **(JG: show why not!)**

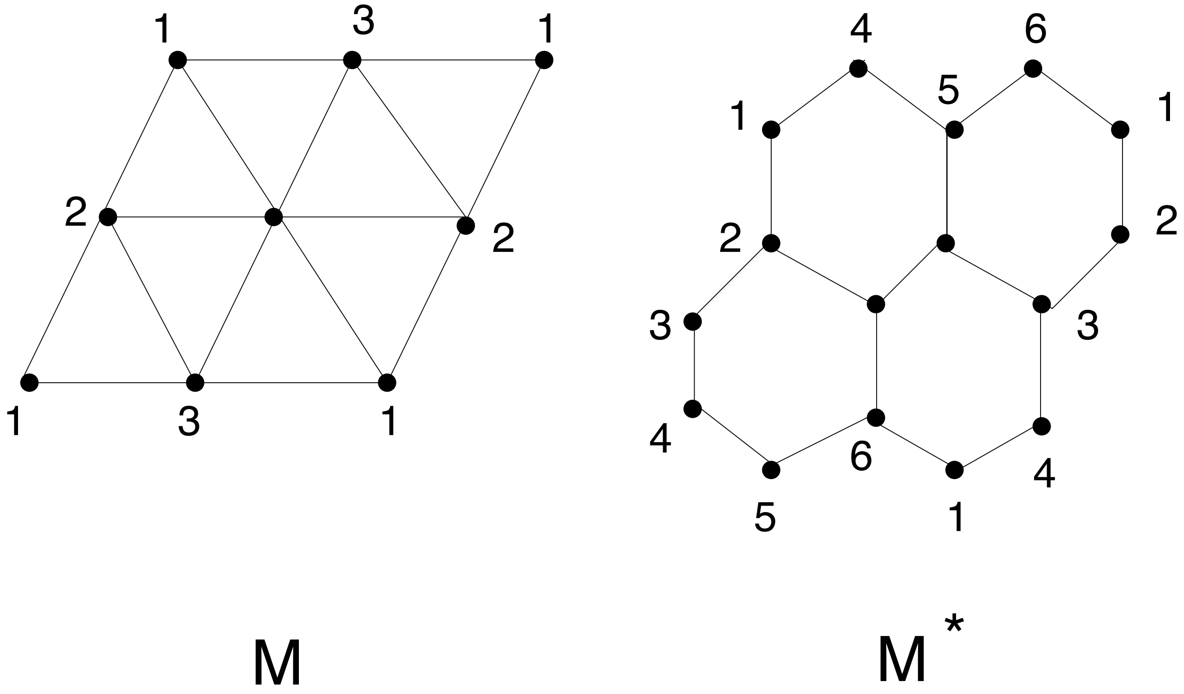


Fig 10.10.3 A torus map and its dual.

(JG) The face-width of map M is 2. A curve that starts at vertex 2 at the left, arcs slightly upward into the interior of a triangle then downward to the unlabeled vertex, then arcs slightly downward into the interior of the opposite triangle then back upward to the other copy of vertex 2, intersects the graph only twice. The 2-cycle from vertex 2 to itself is non-separating, so the edge-width is 2.

Exer 10.10.1 (JG) What are the face-width and edge-width of map $M*$?

E4: Figure 10.10.4 shows two nonisomorphic maps on the sphere that have isomorphic 2-connected 1-skeletons. The 1-skeletons are not 3-connected. The maps are related by a *Whitney flip*.

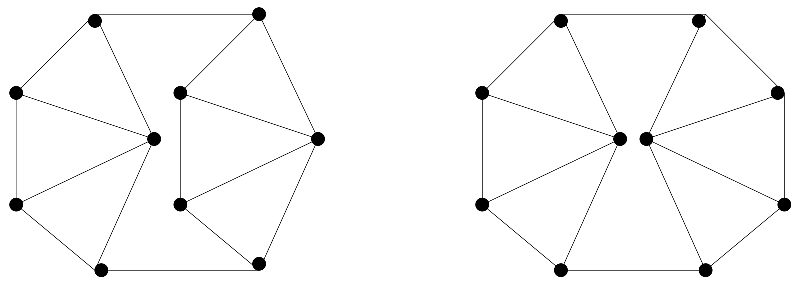


Fig 10.10.4 Maps on the sphere with the same 2-connected graph.

E5: Figure 10.10.5 shows two polyhedral maps on the projective plane with isomorphic 3-connected graphs. (A projective plane is commonly depicted as a disc with antipodal points identified.)

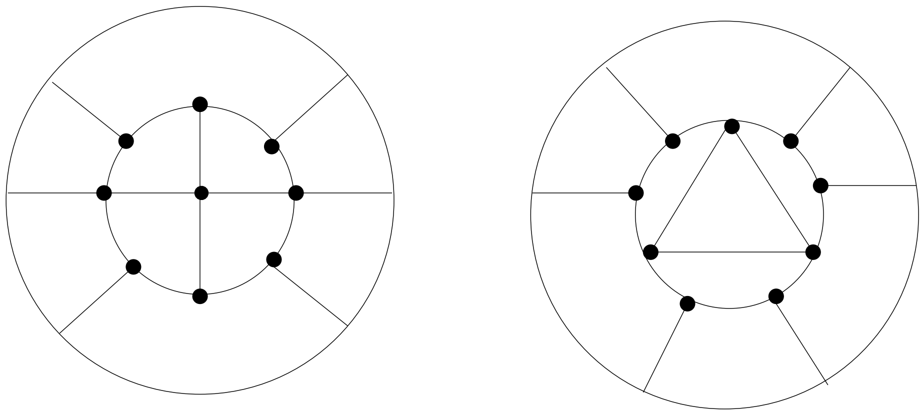


Fig 10.10.5 Maps on the sphere with the same 3-connected graph.

This example demonstrates that the analogy to the Whitney uniqueness theorem (Fact 6) fails for imbedding 3-connected graphs in the projective plane.

(JG: Prove that the 1-skeletons are isomorphic.)

REMARKS

R1: Face-width, introduced in [RoSe88], is a measure of locally planarity, or of how dense the graph is on the surface, or of how well the graph "represents" the surface.

R2: The concept of map has been extended to cell-complexes whose underlying topological space is a manifold of dimension greater than 2. This includes, in particular, the boundary complex of any polytope. The generalization to higher dimensions, though natural and interesting, is omitted here.

R3: A map may have multiple edges, self-loops, and vertices of degree 1 or 2. A polyhedral map, however, can have none of these.

Moveover, in a polyhedral map, the closure of each face is topologically a closed disc, because the face boundary is a cycle.

FACTS

F1: Euler's formula For any map M with f_0 vertices, f_1 edges, f_2 faces and characteristic $c(M)$,

$$f_0 - f_1 + f_2 \;=\; c(M)$$

F2: If M is a map, then $(M^*)^* = M$.

F3: If M is a map, then $fw(M^*) = fw(M)$.

F4: Map M is polyhedral if and only if its graph $G(M)$ is 3-connected and $fw(M) \geq 3$. Moreover, M is polyhedral if and only if its dual is polyhedral.

F5: Every connected graph G admits a map.

The *rotation scheme* described in §10.10.6 gives a systematic method for obtaining all 2-cell imbeddings of G.

F6: [Wh32] **Whitney Uniqueness Theorem:**
A 3-connected, planar graph has a unique imbedding on the sphere.

F7: [Th90] A uniqueness theorem for general surfaces: if two LEW maps M_1 and M_2 with the same 1-skeleton, then $|M_1| = |M_2|$, i.e., they have the same surface. Moreover, if the graph is 3-connected, then $M_1 \approx M_2$.

REMARKS

R4: According to Fact 5 above, every connected graph has a 2-cell imbedding on a surface.

Whether a graph can be imbedded on a surface such that the face boundaries are (simple) cycles is problematic (see the conjectures below).

R5: [SeTh96] gives a uniqueness result similar to Fact 6 (Whitney unique imbedding thm) for maps with sufficiently large face-width as a function of the genus.

[Ar92] provides an example, for every pair of integers k, b, of two maps M_1, M_2 with the same k-connected graph such that $fw(M_1), fw(M_2) > b$ and $|M_1| \neq |M_2|$. Thus, the [SeTh96]-uniqueness result depends on the face-width exceeding an *increasing* function of the genus.

CONJECTURES

The Cycle Double Cover Conjecture: Every 2-connected graph contains a set $\mathcal{C}$ of cycles such that every edge is contained in exactly two cycles of $\mathcal{C}$.

The Strong Imbedding Conjecture: Every 2-connected graph can be imbedded on a surface so that each face is bounded by a cycle in the graph. The strong imbedding conjecture implies the Cycle Double Cover Conjecture.

Remark: The Cycle Double Cover Conjecture says that every 2-connected graph can be the 1-skeleton of a cellular decomposition of a ***pseudosurface***.

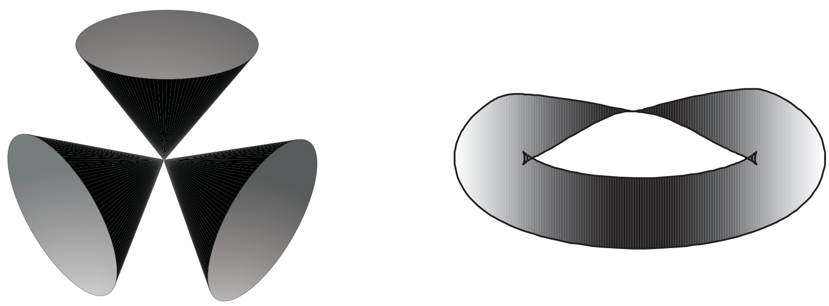


Fig 10.10.6 **Pseudosurfaces: pinched disk and pinched torus.**

10.10.2 Cell Counts, Face Inventory, Degrees

Elementary equations hold among the basic parameters of a map. This section asks,

1. when are these necessary conditions also sufficient for the existence of a map with these parameters, and,
2. when can the map be imbedded in Euclidean space E^3 or E^4 such that the faces are plane convex polygons.

The classic results for maps on the sphere are Eberhard's theorem of 1891 and Steinitz's theorem of 1922.

DEFINITIONS

D12: A ***map*** is ***of type***

$$\{p, q\}$$

if each face has p edge incidences (JG: ***size*** p) and each vertex has q edge incidences (JG: ***degree*** q).

Remark: No global symmetry is implied; in fact, the automorphism group of the map, as defined in §10.10.5, may be trivial.

D13: The ***cell-distribution vector (f-vector)*** of a map M is the 3-tuple

$$(f_0,\, f_1,\, f_2)$$

where f_0, f_1, f_2 are the numbers of vertices, edges, and faces of M, respectively.

D14: The ***face-size sequence (p-sequence)*** of a polyhedral map M is the sequence

$$\{p_i\}_{i\geq 3}$$

where p_i is the number of i-gonal faces in M.

D15: The ***vertex-degree sequence (v-sequence)*** of a polyhedral map M is the sequence

$$\{v_i\}_{i\geq 3}$$

where v_i is the number of vertices of degree i in M.

EXAMPLES

E6: The map M from Figure 10.10.3 is of type $\{3, 6\}$ with face vector $(4, 12, 8)$. Its dual M^* is of type $\{6, 3\}$ with face vector $(8, 12, 4)$.

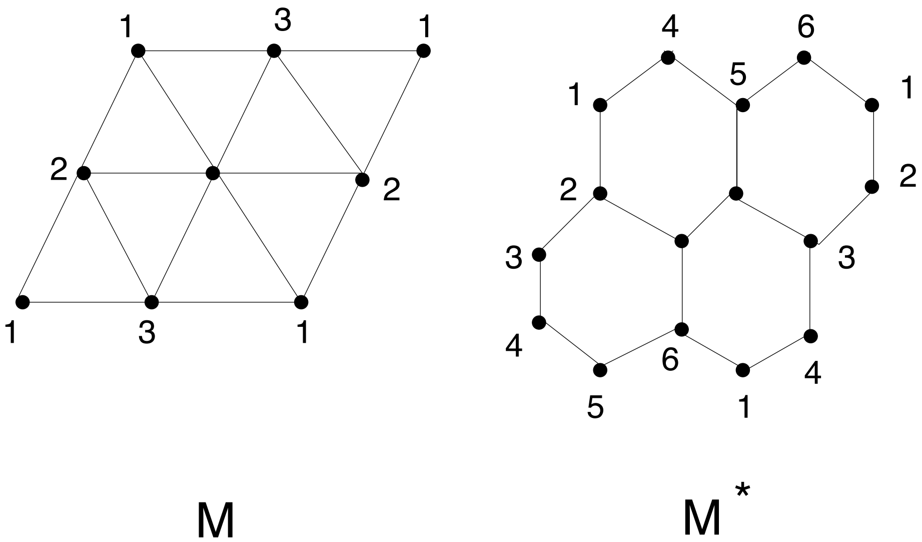


Fig 10.10.1 A torus map and its dual.

E7: The maps from Figure 10.10.6 both have v-sequence $(6, 3)$, but the first has p-sequence $(0, 6, 0, 1)$ while the second has p-sequence $(1, 3, 3)$.

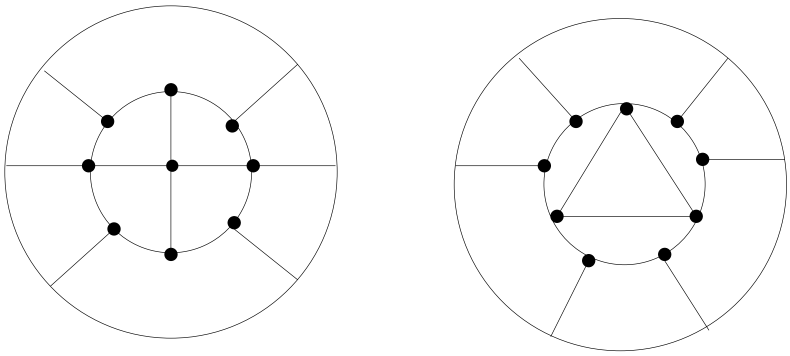


Fig 10.10.3 Maps on the sphere with the same 3-connected graph.

D16: A polyhedral map M is ***simplicial*** (or a ***triangulation***) if the boundary of each face is a 3-cycle.

D17: A polyhedral map M is ***simple*** if its graph is 3-regular.

D18: A ***geometric realization (realization)*** of a polyhedral map M is an imbedding of M into Euclidean space E^d (no self intersection) such that each face is a plane convex polygon and that adjacent faces are not coplanar.

E8: Five maps on the sphere and their corresponding 3-dimensional realizations appear in Figure 10.10.7.

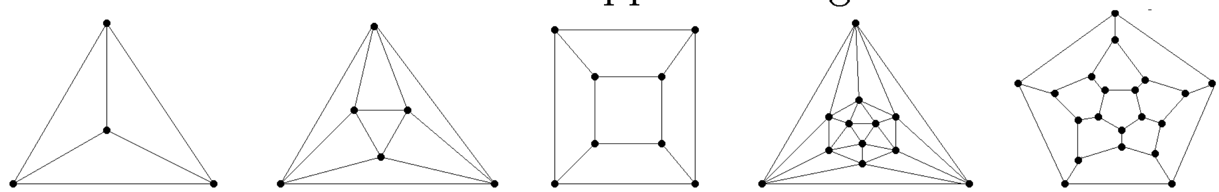


Fig 10.10.7 The Platonic solids as realizations of maps.

FACTS

F8: The f-vector, the p-sequence and the v-sequence satisfy the following elementary equations:

- Sum the number of vertices of each degree.

  $\sum v_i = f_0$

- Sum the number of faces of each size.

  $\sum p_i \;=\; f_2$

- Euler's thm on degree sum and its dual.

  $\sum ip_i \;=\; 2f_1 \;=\; \sum iv_i$

F9: For an orientable map M of genus g, with and choice of real numbers $\alpha, \beta \geq 0$, such that $\alpha + \beta = 1$, Euler's formula implies that

$$\sum(\alpha i - 2)v_i + \sum(\beta i - 2)p_i \;=\; 4(g-1) \qquad (1)$$

Example 10.10.9: For M a simple map on S_0, taking $\alpha = 1/3$, we have

$$\sum(6-i)p_i \;=\; 12 \qquad (2)$$

Exer 10.10.2 (JG) Prove Fact F9.

F10: [Eb1891] **Eberhard's Theorem:** Condition (2) above is sufficient for the existence of a sphere map, in the following sense: if a face-size sequence

$$\{p_i \mid i \geq 3,\ i \neq 6\} \quad \text{satisfies} \quad \sum_{k \neq 6} (6-k)p_k \;=\; 12$$

then $\exists$ a value of p_6 such that $\{p_i \mid i \geq 3\}$ is the face-size-sequence of a 3-regular polyhedral map on the sphere.

Remark: There is no known generalization of Eberhard's theorem to arbitrary surfaces.

Example 10.10.10: We are given 1 3-gon, 4 4-gons, and 1 5-gon, we calculate:

$$F = 6 + p_6$$

$$E = \frac{1}{2}(1 \cdot 3 + 4 \cdot 4 + 1 \cdot 5 + 6 \cdot p_6) \;=\; 12 + 3p_6$$

Using $V - E + F = 2$, we calculate

$$V = 2 + (12 + 3p_6) - (6 + p_6) \;=\; 8 + 2p_6$$

for $p = 0$, we draw the following sphere map:

F11: [St22] **Steinitz's Theorem:** Every polyhedral map on the sphere is isomorphic to the boundary complex of a 3-dimensional polytope. Thus, any polyhedral map on the sphere has a realization in E^3.

10.10.3 Map Coloring

The most famous results on map coloring are the Four Color Theorem for the sphere and the Heawood Map Coloring Theorem, which is the generalization of the Four Color Theorem to surfaces of higher genus. Also in this section are a few results on coloring densely imbedded graphs.

DEFINITION

D19: The **chromatic number** $\chi(S)$ of a surface S is the least number of colors sufficient to properly color the faces of any map on S. By duality, it is also the least number of colors sufficient to properly color the vertices of any map on S. In this section, coloring will mean vertex coloring.

FACTS

F14: [ApHa76] **Four Color Theorem:** $\chi(S_0) = 4$.

F15: [Fr34] $\chi(N_2) = 6$.

F16: [RiYo68] **Heawood Map Coloring Theorem:** For every surface S except the Klein bottle N_2,

$$\chi(S) = \left\lfloor \frac{7 + \sqrt{49 - 24c}}{2} \right\rfloor$$

where c is the Euler characteristic of S. The right-hand side of the equation is called the ***Heawood formula***.

F17: A graph G imbeddable on the torus with

$$ew(M) \geq 4$$

is 5-colorable. It is not known whether this same statement holds for surfaces of higher genus.

Example 10.10.11: The essentially unique imbedding $K_7 \to S_1$ has a non-contractible 3-cycle. Of course, its chromatic number is 7.

F18: [Th93] Any graph imbeddable on S_g with

$$ew(M) \geq 2^{14g+6}$$

is 5-colorable.

F19: [Th97] For a fixed surface S, there is a polynomial-time algorithm to decide if a map on S can be 5-colored.

F20: Even on the sphere, the problem of deciding whether a map can be 3-colored is NP-complete.

F21: [RSST96] On the sphere, a 4-coloring can be found in $O(n^2)$ steps.

REMARKS

R6: The problem of determining the chromatic number of the sphere appeared in a 1852 letter from Augustus de Morgan to Sir William Hamilton, and was likely due to Francis Guthrie, the brother of a student of de Morgan.

R7: The computer dependent proof of Appel and Haken [ApHa76] that four colors suffice was simplified considerably [RSST97] (but is still computer dependent).

R8: That the formula in the Heawood Map Coloring Theorem gives an upper bound on $\chi(S)$ was proved by Heawood [He1890].

R9: That there exist graphs that actually require the number of colors given by that formula is a consequence of the formula for the genus of complete graphs due to Ringel and Youngs [RiYo68].

R10: Whether there is a polynomial-time algorithm for deciding whether a map on an arbitrary surface can be 4-colored is unknown.

EXAMPLES

E12: Figure 10.10.8a (preview) is a map on the projective plane that requires 6 colors for a proper coloring. This shows that $\chi(N_1) \geq 6$.

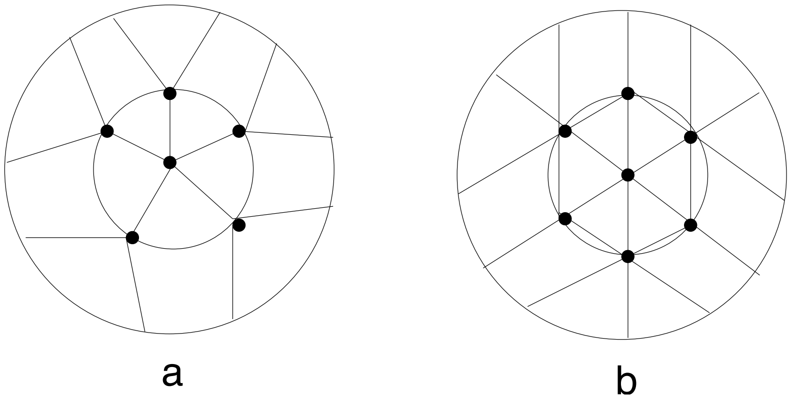


Fig 10.10.9 **Minimal triangulations of the projective plane.**

E13: Figure 10.10.8 is map on the torus that requires 7 colors. This shows that $\chi(S_1) \geq 7$. In fact, $\chi(N_1) = 6$ and $\chi(S_1) = 7$, in accordance with Fact 23.

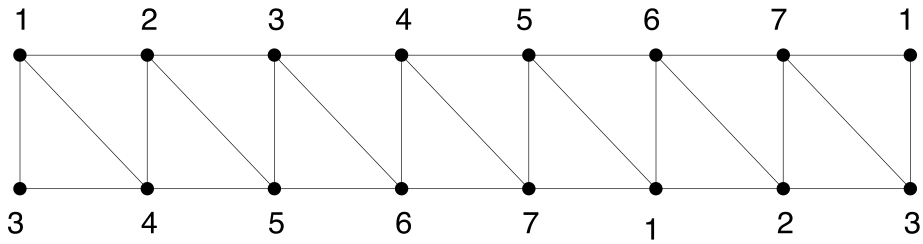


Fig 10.10.8 **A map on the torus whose graph is K_7.**

E14: An example of Fisk [Fi78] shows that no 4-color analogue of Thomassen's result (Fact 18 above) can hold. See Figure 10.10.9, where the torus is obtained by identifying opposite sides of the square.

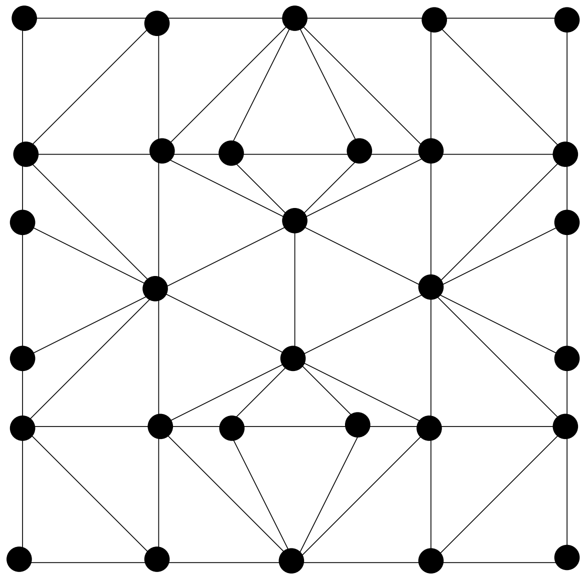


Fig 10.10.9 A graph on the torus with exactly two odd-degree vertices that is not 4-colorable.

Remark: It is not easy to show that the graph above requires more than 4 colors. It is even harder to show that a graph that is $n \times n$ in this pattern would also require more than 4 colors.

10.10.4 Minimal Maps

A map can be quite "degenerate", for example, the map on the sphere with 2 vertices, 1 edge, and 1 face. Polyhedral maps (and maps with edge-width or face-width bounded from below) cannot be this small. This section concerns maps that are in some sense minimal — either with respect to the number of vertices, or with respect to being polyhedral, or with respect to having edge-width k. Also covered in this section are weakly neighborly polyhedral maps.

DEFINITIONS

D20: A polyhedral map is ***neighborly*** if every pair of distinct vertices is joined by an edge.

D21: A polyhedral map is ***weakly neighborly*** (abbr. a ***wnp-map***) if every two vertices are contained on a face.

D22: The operation of ***edge-contraction*** for a triangulation, and its inverse operation ***vertex-splitting***, are depicted in Figure 10.10.10. After contracting an edge in a triangulation, the map may no longer be a triangulation, i.e., no longer polyhedral; this occurs if the edge is contained in a 3-cycle that is not a face boundary or if the map is the tetrahedral map.

D23: A ***minimal triangulation*** of a surface S is a triangulation such that the contraction of any edge results in a map that is no longer polyhedral.

D24: A ***k-minimal triangulation*** is a triangulation with edge-width k, such that each edge is contained in a noncontractible k-cycle. (Except on the sphere, minimal and 3-minimal are equivalent.)

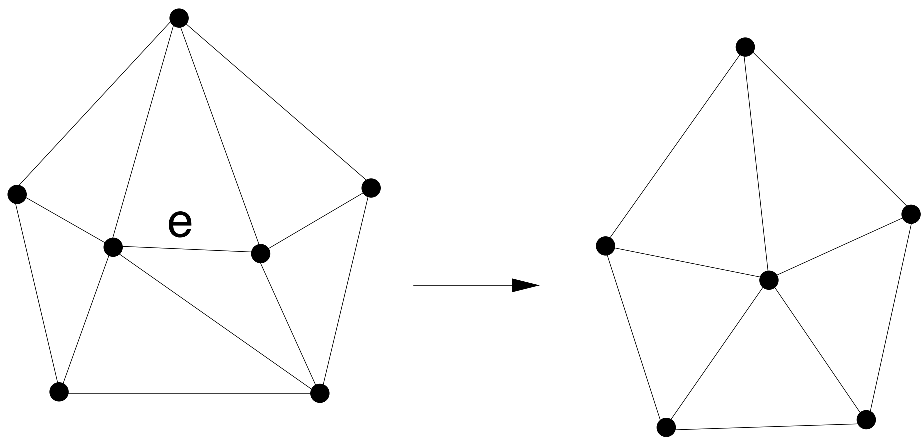


Fig 10.10.10 Edge contraction and vertex-splitting in a triangulation.

EXAMPLES

E15: The only wnp-maps on the sphere are the boundary complexes of the pyramids and triangular prism.

E16: There are 5 wnp-maps on S_1 and none on S_2.

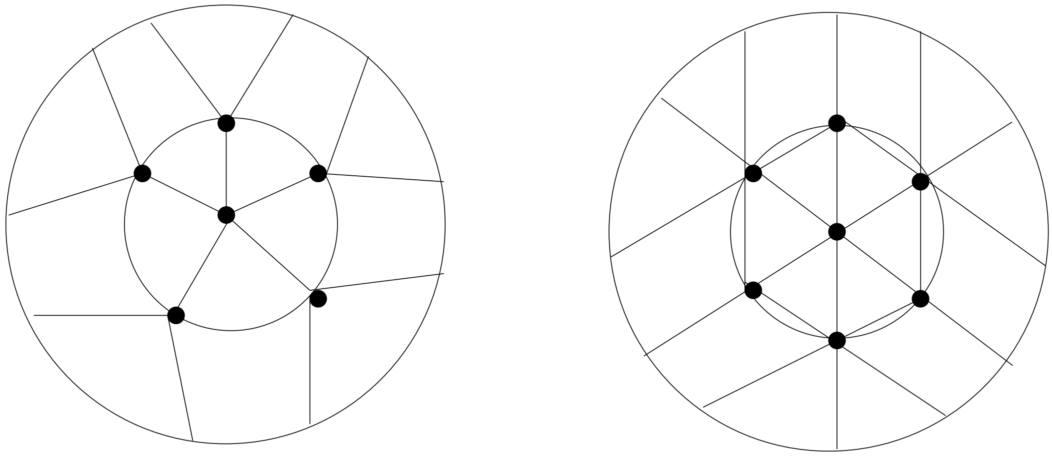
E17: The number of minimal triangulations for various surfaces are as follows:

S_0 has 1 (the tetrahedral map)

N_1 has 2 (see Figure 10.10.11)

S_1 has 21

N_2 has 25.



a b

Fig 10.10.11 The minimal triangulations of the projective plane.

FACTS

F22: If the map M with f_0 vertices and Euler characteristic c is polyhedral, then

$$f_0 \geq \left\lceil \frac{7 + \sqrt{49 - 24c}}{2} \right\rceil,$$

and this lower bound is attained for all surfaces except S_2, N_2, and N_3. By duality the same bound holds for f_2.

F23: The neighborly polyhedral maps attain the bound in Fact 10.10.22.

F24: [AlBr86] Each surface admits at most finitely many wnp-maps. (See Example 8.)

F25: [BaEd89] The set of minimal triangulations is finite for every fixed surface. (See Example 11.) In other words, for each surface, there is a finite set of triangulations from which any triangulation on that surface can be generated by vertex splittings.

F26: For any $k \geq 3$, the set of k-minimal graphs on a fixed surface is finite. ([MoTh01] provides a proof.)

REMARK

R11: [Br90] has provided a (non-tight) **lower bound** for the number f_1 of edges for a polyhedral map of Euler characteristic c.

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