CS E6204 Lecture 1
Computer-Graphics Models for Woven Images on Surfaces

Abstract

A construct from topological graph theory called *graph rotation systems* is extended into a solid mathematical model for the development of an interactive-graphics cyclic-weaving system. It involves a systematic exploration and characterization of dynamic surgery operations on graph rotation systems, such as edge-insertion, edge-deletion, and edge-twisting. This talk explains the underlying mathematics and some high-level aspects of the programming system for the interactive-graphics system.

* This lecture is based on a research paper [ACXG09] presented at SIGGRAPH 2009 in New Orleans.

1 Introduction

The following images were created by a system (TopMod) that is based on the mathematical model presented here.

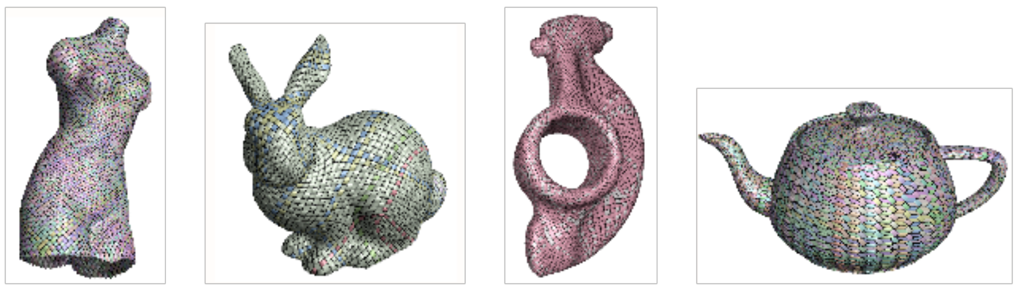


Figure 1: Weaves on 3D-meshes.

Several recent papers [AkCh00, AkChSr00, AkChSr03, ACSE01] on graphics by my co-authors Ergun Akleman and Jianer Chen use classical topological graph theory [GrTu87, Wh01], especially graph rotation systems, as a solid mathematical basis for 3D-mesh modeling and sculpturing systems.

Some advantages of using this formal mathematical model as a basis for software application development:

(A1) **universal**: its techniques can be adopted by any existing modeling software system;

(A2) **robust**: it never generates invalid non-manifold structures;

(A3) **powerful**: it can perform all necessary topological surgery operations;

(A4) it has **simple and intuitive primary operations**;

(A5) many **secondary operations** based on the built-in primary operations can be readily implemented at the user-level.

2 Links, surfaces, and cyclic weaving

Definition. A ***link*** $\sigma : \mathcal{C} \to \mathbb{R}^3$ is a homeomorphism from the union $\mathcal{C} = \cup\{c_1, \ldots, c_k\}$ of a set of disjoint circles into $\mathbb{R}^3$. A ***knot*** is a link with only one component.

Definition. A ***projection of a link*** onto a surface $S \subset \mathbb{R}^3$ is an ***immersion*** $\sigma : \mathcal{C} \to S$, with the following properties:

- $\exists$ finitely many singular points, each called a ***crossing***;
- each crossing is a 2-to-1 singularity;
- the preimages of each ***crossing*** point $y \in S$ are ***ordered***.

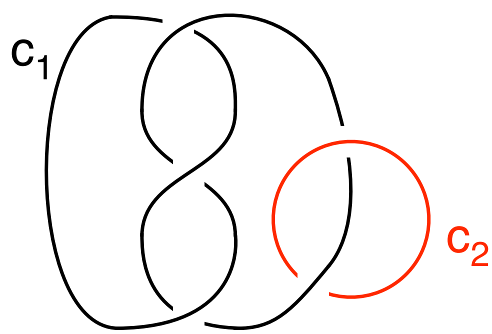
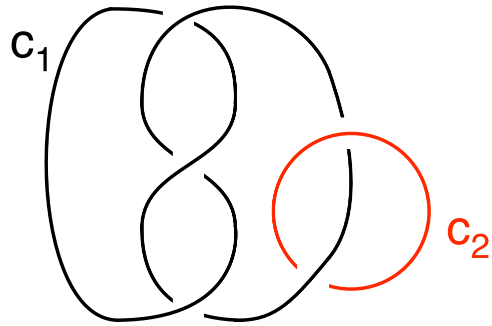


Figure 2: An alternating link projection.

Properties of a projection of a link



We observe the following:

- every intersection is a true crossing (no tangencies).
- the images of two circles may intersect;
- the image of a circle may self-intersect;

Definition. A link projection is ***alternating*** if on a traversal of each of its components, the over-crossings and under-crossings alternate, as on the left of Figure 3. An ***alternating link*** is a link that has an alternating projection.

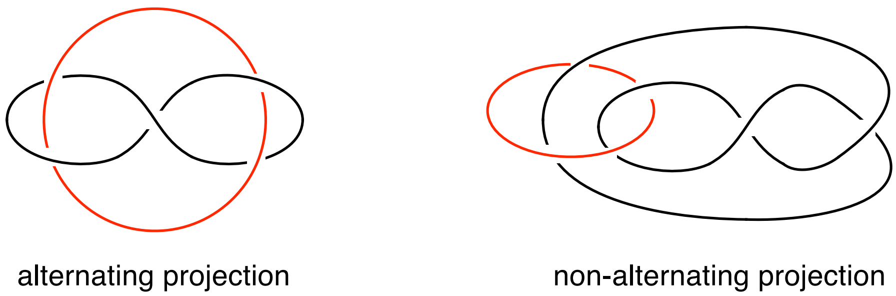


Figure 3: Two projections of the Whitehead link.

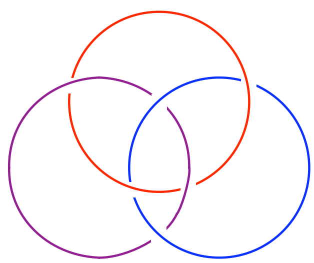


Figure 4: The Borromean link.

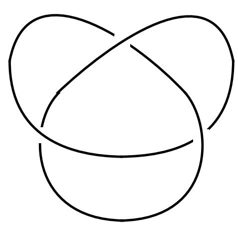


Figure 5: The trefoil knot.

Surfaces

A closed surface in 3-space separates 3-space into two parts, by a 3-dimensional **analogue of the Jordan curve theorem**. The part that goes to infinity is called the *outside* and the other part is called the *inside*.

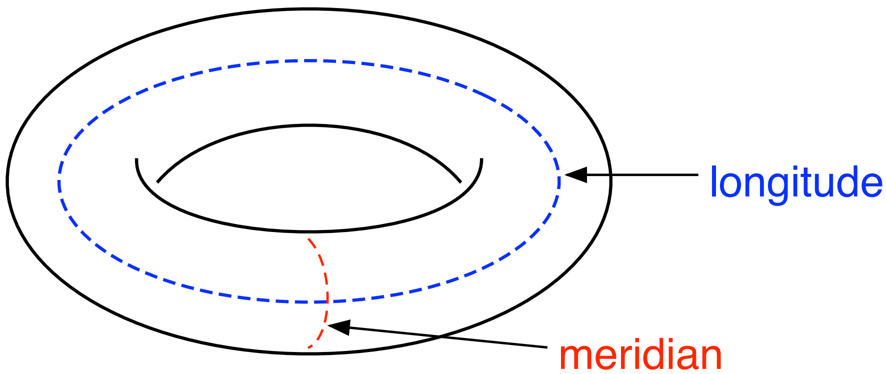


Figure 6: An unknotted torus in $\mathbb{R}^3$.

Definition. *Restoration of a link L* from a projection onto a surface is the result of pulling each crossing apart: a small over-crossing segment is pulled outside the surface and a small undercrossing segment is pushed inside the surface.

Seifert surface for a knot or link

Theorem 2.1 (Seifert, 1934) *Every oriented knot or link in space is the boundary of a connected oriented surface. Each component of the link is a boundary componenet of that surface.*

PROOF. [Sei34] or [Ad04]. □

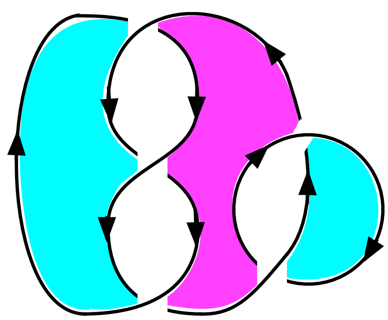


Figure 7: Seifert surface for a trefoil knot linked to an unknot.

An arc that runs from crossing to crossing is called a ***segment*** of the projection. In Figure 7, every segment is oriented. The orientations of the segments are inherited from the orientations of the components of the link itself.

We now need to know two things about Seifert surfaces:

1. how to **draw a Seifert surface**
2. how to **calculate the genus** of the surface

Consider a link projection in which the segments have inherited orientations. We observe that at the head of a segment s, the segment on the other side of the crossing has its tail. Also, one of the other two segments incident on that crossing has its tail there. We call that segment $next(s)$.

A ***Seifert circle*** is a cycle of the permutation $next$.

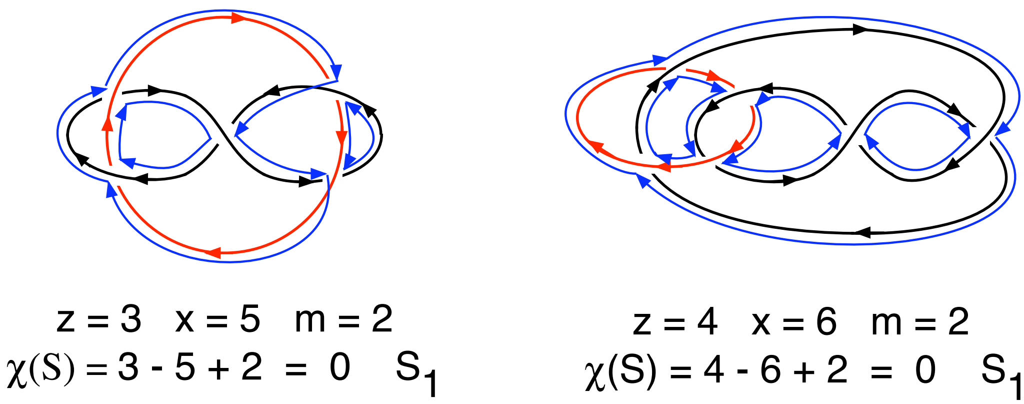


Figure 8: Seifert circles for two link projections.

The Euler characteristic of the Seifert surface equals $z - x$, where

- z is the number of Seifert circles
- x is the number of crossings in the projection

If m is the number of components of the link, then $z - x + m$ is the Euler characteristic $\chi(S)$ of the closed surface of the same genus. And the equation

$$\chi(S) \;=\; 2 - 2\gamma(S)$$

is used to calculate the genus of the surface.

A ***torus knot*** is a knot that lies on an unknotted torus in 3-space.

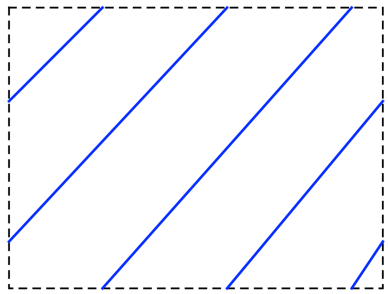


Figure 9: A torus knot.

Proposition 2.2 *For any link L in $\mathbb{R}^3$ with $m(L)$ components, there is a closed orientable surface S of genus $m(L)$ in $\mathbb{R}^3$ on which L is imbedded.*

PROOF. Thicken each component C_j into a solid torus, so that S_j lies on the surface of that solid torus, and so that the solid tori are mutually disjoint. Nest discard the interiors of the solid tori, so that each component of the link lies on a torus. Then connect the $m(L)$ tori with $m(L) - 1$ tubes, to obtain a copy S of the surface $S_{m(L)}$ of genus $m(L)$. □

Corollary 2.3 *Every link L in 3-space has an alternating projection onto some closed surface in $\mathbb{R}^3$.*

PROOF. By Proposition 2.2, there is a closed orientable surface S in $\mathbb{R}^3$ such that L is imbedded on S. An imbedding is an alternating projection with zero crossings. □

Cyclic plain-weaving

Definition. A ***cyclic plain-weaving*** is an alternating projection of a link onto a surface in $\mathbb{R}^3$.

Remark A ***cyclic weaving*** is like a cyclic plain weaving, except that

- the projection need not be alternating;
- crossings on the surface S may have pre-images in the link L with more than two points.

The ***thickness*** of a weaving is the maximum number of points in a preimage.

3 Graph imbeddings and rotation systems

This presentation of topological graph theory is consistent with more detailed discussions of these issues to be found in [GrTu87].

Topological graphs may have multi-edges and self-loops.

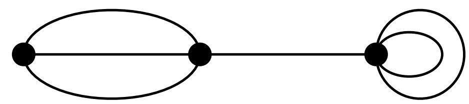


Figure 10: A graph.

An edge ALWAYS has two ***edge-ends***, which are small neighborhoods of the limit points 0 and 1 of a parametrization of the edge, even when there is only one ***endpoint***.

Figure 11: A proper edge and a self-loop.

Each edge e induces two ***oriented edges***, each running from one edge-end of edge e to the other edge-end of e.

Surfaces and imbeddings

- ***surface***: a closed, compact 2-dimensional manifold;
- ***imbedding***: a homeomorphism $G \to S$ of a graph G onto a topological subspace of the surface S;
- ***cellular***: every connected component of $S - G$ is homeomorphic to an open disk.
- ***rotation*** at a vertex v of G: a cyclic ordering of the *oriented edges* originating at v;
- ***(pure) rotation system*** of graph G: a set of n rotations, one for each vertex of G.

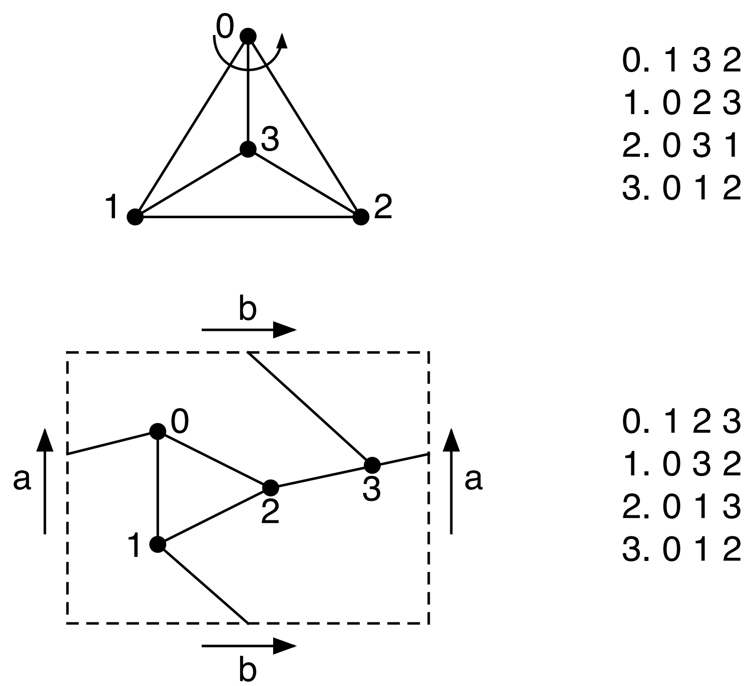
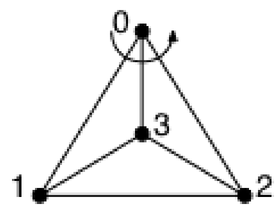


Figure 12: Two inequivalent rotation systems for K_4.

A rotation system is a permutation $\rho : \overrightarrow{E} \to \overrightarrow{E}$ on the oriented edges of a graph. It has one cycle for each vertex of the graph.



0. 1 3 2
1. 0 2 3
2. 0 3 1
3. 0 1 2

$$\begin{array}{c} (\ [0\ 1]\ \ [0\ 3]\ \ [0\ 2]\ ) \\ (\ [1\ 0]\ \ [1\ 2]\ \ [1\ 3]\ ) \\ (\ [2\ 0]\ \ [2\ 3]\ \ [2\ 1]\ ) \\ (\ [3\ 0]\ \ [3\ 1]\ \ [3\ 2]\ ) \end{array}$$

Let $\iota : \overrightarrow{E} \to \overrightarrow{E}$ be the permutation that reverses every oriented edge. That is,

$$\iota([u, v]) \ = \ [v, u]$$

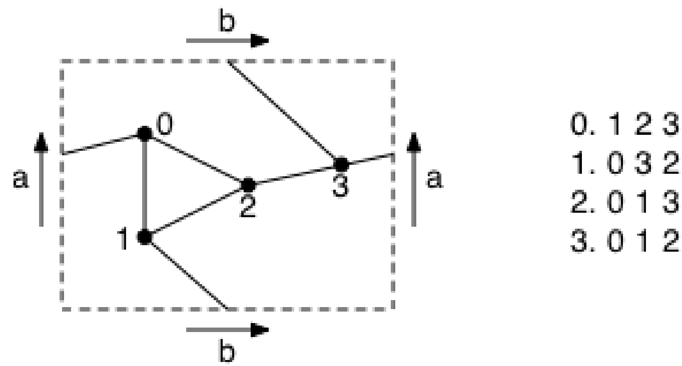
The composition permutation $\iota \circ \rho : \overrightarrow{E} \to \overrightarrow{E}$ is often called the dual of ρ. For instance,

$$[0\ 1] \overbrace{\longrightarrow}^{\iota} [1\ 0] \overbrace{\longrightarrow}^{\rho} [1\ 2]$$

Its cycles are the fb-walks of the graph imbedding.

$$\begin{array}{c} (\ [0\ 1]\ \ [1\ 2]\ \ [2\ 0]\ ) \\ (\ [0\ 2]\ \ [2\ 3]\ \ [3\ 0]\ ) \\ (\ [0\ 3]\ \ [3\ 1]\ \ [1\ 0]\ ) \\ (\ [1\ 3]\ \ [3\ 2]\ \ [2\ 1]\ ) \end{array}$$

Similarly,



([0 1] [0 2] [0 3])
([1 0] [1 3] [1 2])
([2 0] [2 1] [2 3])
([3 0] [3 1] [3 2])

Has as its dual permutation

([0 1] [1 3] [3 2] [2 0] [0 3] [3 1] [1 2] [2 3] [3 0])
([0 2] [2 1] [1 0])

We observe that the 9-cycle and the 3-cycle are the fb-walks of the given toroidal graph imbedding of K_4.

Two ***equivalent orientable imbeddings*** of a graph G have the same rotation at every vertex of G.

Example 3.1 Imbeddings of the complete graph K_4.

- 2 in S_0 with four 3-gons, like top drawing in Fig. 12.
- 8 in S_1 with 3-gon and 9-gon, like bottom drawing in Fig. 12.
- 6 in S_1 with a 4-gon and an 8-gon

Thus, the genus distribution of K_4 is

$$g_0(K_4) = 2 \qquad g_1(K_4) = 14$$

Example 3.2 Two more genus distributions.

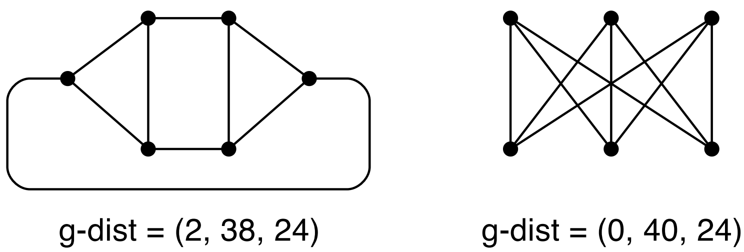


Figure 13: Two non-isomorphic graphs.

Two basic artifacts of context

Proposition 3.3 *For any graph G,*

$$\sum_{i\geq 0} g_i(G) \;=\; \prod_{v\in V(G)} ((deg(v)-1)!)$$

Theorem 3.4 (Thomassen) *The minimum-genus problem is NP-complete.*

Theorem 3.5 (Mohar) *For every closed surface S, there is a linear-time algorithm to decide whether a given graph can be imbedded in S.*

The catch is that the multiplicative constant grows rapidly with increasing genus of the surface.

Face-tracing

The ***face-tracing algorithm*** [Ed60, GrTu87] constructs the fb-walks. Matching the perimeter of each s-sided polygon to each fb-walk of length s reconstructs the surface S.

A ***face corner*** is a triple (v, e, e') comprising a vertex v and two oriented edges e and e', both oriented out of v, where the v-edge-end of e' immediately follows the v-edge-end of e' in the rotation at v. If neither e nor e' is a self-loop, we say that e' is 0-*next* to e at v and that e is 1-*next* to e' at v. For a self-loop, we must say which orientation is 0-next or 1-next.

Subroutine FaceTrace$(t_0, \langle u_0, w_0 \rangle)$
Input: $\langle u_0, w_0 \rangle$ is an oriented edge, and $t_0 \in \{0, 1\}$ is its "trace type".
Output: the sequence of oriented edges in the fb-walk containing $\langle u_0, w_0 \rangle$.

1. trace and print $\langle u_0, w_0 \rangle$;
2. $t = t_0 + type([u_0, w_0])\ (mod\ 2)$;
3. $\langle u, w \rangle =$ the t-next to $\langle w_0, u_0 \rangle$ at w_0; \\$u = w_0$
4. **while** $(\langle u, w \rangle \neq \langle u_0, w_0 \rangle)$ or $(t \neq t_0)$ **do**
   trace and print $\langle u, w \rangle$;
   $t = t + type([u, w])\ (mod\ 2)$;
   $\langle w', u' \rangle = \langle w, u \rangle$;
   $\langle u, w \rangle =$ the t-next to $\langle w', u' \rangle$ at w'. \\$u = w'$

Algorithm FbWalks($\rho(G)$**)**
Input: $\rho(G)$ is a general graph rotation system.
Output: the collection of all fb-walks in $\rho(G)$.

while there is an untraced face corner (u, e, e') in $\rho(G)$ **do**
  suppose that $e' = \langle u, w \rangle$; call FaceTrace$(0, \langle u, w \rangle)$.

4 Rotation systems and surgery

Surgery operations on pure graph rotation systems (thus, on graph imbeddings) have been extensively studied [Ch90, GrTu87]. They are relatively easy to understand.

Pure edge-insertion and edge-deletion surgery

Edge-Insert-0

(a) If both ends of a new edge e are inserted into corners of the same face, then e splits that face into two faces, and the two oriented edges corresponding to e belong to the different fb-walks in the new imbedding.

(b) If the two ends of e are inserted into corners of two different faces, then e merges those two faces into a single face, and the two oriented edges corresponding to e belong to the fb-walk of that single face in the new imbedding.

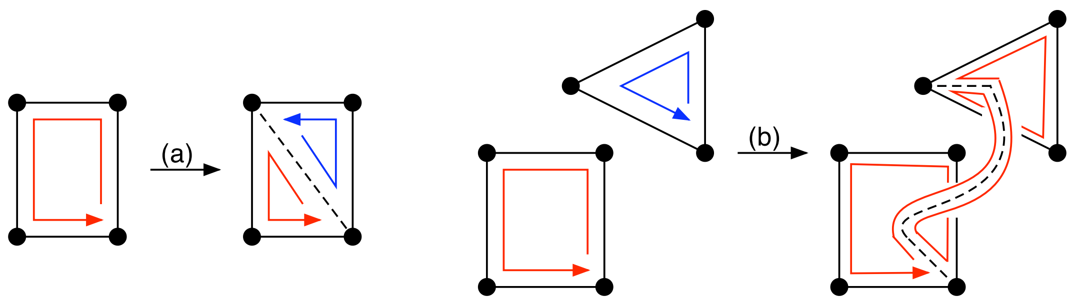


Figure 14: Adding an edge to an orientable imbedding.

The operation of edge deletion on a pure rotation system "reverses" edge insertion.

Edge-Delete-0

If the two oriented edges corresponding to an edge e appear in the boundary walks of two different faces, then deleting the edge e merges the two faces into a single face.

If the two oriented edges corresponding to an edge e belong to the boundary walk of a single face, then deleting the edge e splits that face into two faces.

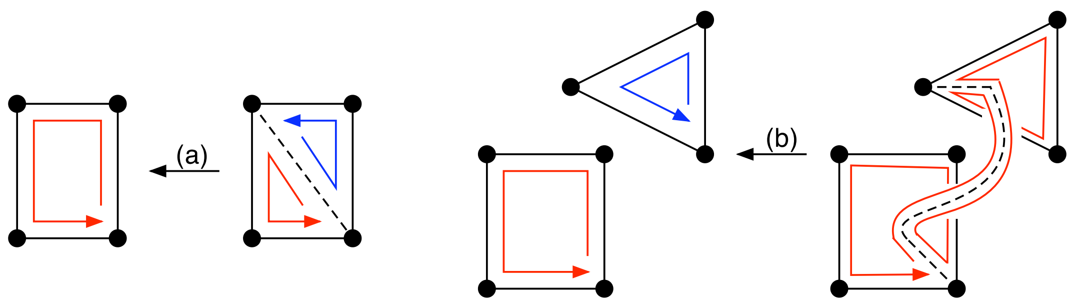


Figure 15: Edge-delete from orientable imbedding (R to L).

*** REMARK ***

The fb-walks for an imbedding of a graph on an orientable surface represent a trivial weave, i.e., a link whose components are completely unlinked and individually unknotted.

We now turn our focus to non-trivial weaves.

5 General rotation systems

A ***general rotation system*** of a graph $G = (V, E)$ consists of a pure rotation system of G plus a function $t : E \to \{0, 1\}$ that assigns to each edge of G an ***edge-type***.

This augmentation of graph rotation systems is sufficient to represent imbeddings on non-orientable surfaces. For this, we regard type-0 edges as ***flat*** and type-1 edges as ***twisted***.

The following figure represents ***band-decompositions*** for imbeddings $K_4 \to S_0$ and $K_4 \to N_1$.

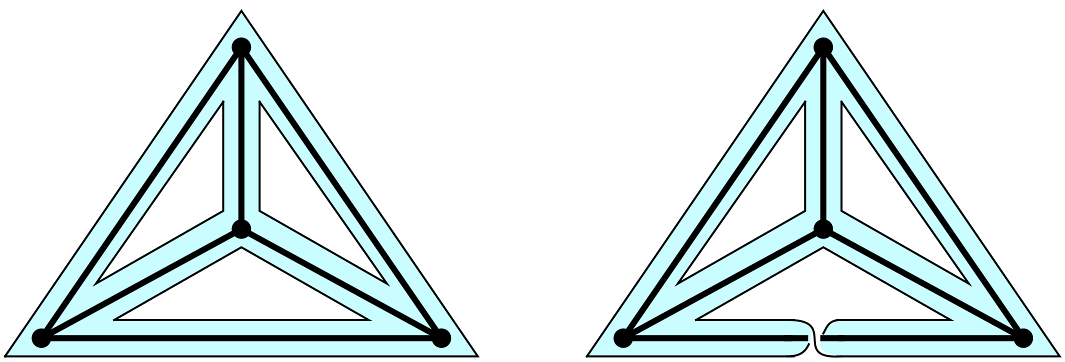


Figure 16: Band-decompositions for $K_4 \to S_0$ and $K_4 \to N_1$.

Proposition 5.1 *A twist that joins two fb-walks decreases the Euler characteristic by 1.* □

Proposition 5.2 *The imbedding surface specified by a general rotation system is orientable if and only if the following condition holds for every pair of vertices u and v:*

> *The parity of the number of twisted edges is the same along every path between u and v.* □

The ***induced weaving*** of a general rotation system is the projection of the boundary of its band-decomposition onto the surface specified by its underlying pure rotation system.

Example 5.3 The second band-decomposition also gives us a non-trivial weaving on S_0 of a link with three components, which are the fb-walks for the imbedding $K_4 \to N_1$ projected onto S_0.

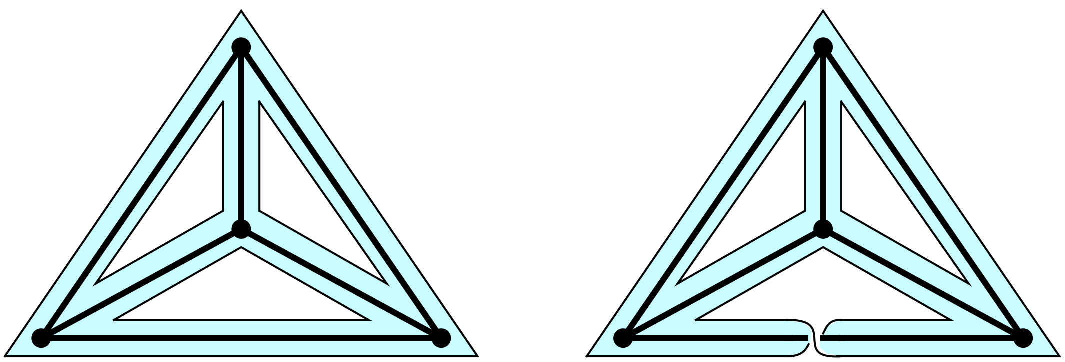


Figure 17: Band-decomps for $K_4 \to S_0$ and $K_4 \to N_1$ again.

6 On edge-twisting surgery

Twisting an edge in a general rotation system means changing its type, either from 0 to 1, or from 1 to 0.

REMARK When traversing a twisted edge during face-tracing, the cyclic direction at the terminating vertex (at which one selects the next oriented edge) is taken to be opposite from the direction at the originating vertex.

Theorem 6.1 *Twisting an edge e in a general rotation system $\rho(G)$ satisfies the following rules:*

(A) *Suppose that the two trace-pairs induced by e belong to the boundaries of two different faces in the imbedding. Then twisting e merges the two faces into a single face;*

(B) *Suppose that the two trace-pairs induced by e belong to the boundary of the same face F in the imbedding.*

- (B1) *If the two trace-pairs induced by e use the same oriented edge, then twisting e splits the face F into two faces;*
- (B2) *If the two trace-pairs induced by e use different oriented edges, then twisting e converts the face F into a new single face.*

The results of twisting an edge both of whose induced trace-pairs belong to the boundary walk of the same face, in particular case (B2) of Theorem 6.1, seem to have been absent from the existing literature in topological graph theory. **It takes several pages of technical detail to close this gap**.

7 General edge-inserts and edge-deletes

One might expect that most results for pure graph rotation systems would extend naturally to general graph rotation systems. However, there seem to be some subtle issues that are quite different, which, to our knowledge, have not been thoroughly studied in the literature.

Example 7.1 Figure 18(1) corresponds to a 1-face imbedding of the bouquet B_1 (one vertex with one self-loop) on the projective plane. In particular, face corners c_1 and c_2 in Figure 18(1) belong to the same face. Now suppose that we insert a new type-0 edge e_2 between these two face corners, as depicted in Figure 18(2).

Figure 18: Inserting an edge into a general rotation system

The rules in §2 for pure graph rotation systems say that an edge insertion (necessarily type-0 for pure rotation systems) between two corners of the same face would split that face into two faces. However, by applying the general face-tracing algorithm to Figure 18(2), we find out that the resulting rotation system corresponds to a 1-face imbedding of the bouquet B_2 (on the Klein bottle)!

Rules for edge-insertion surgery

Theorem 7.2 *Suppose that we insert the ends of a type-*0 *edge* e *into two face corners* c_1 *and* c_2 *in a general rotation system* $\rho(G)$*. Then the following rules hold:*

(A) *Suppose that corners* c_1 *and* c_2 *belong to two different faces. Then inserting edge* e *between* c_1 *and* c_2 *merges the two faces into a single face.*

(B) *Suppose that corners* c_1 *and* c_2 *belong to the same face. Then*

(B1) *if* c_1 *and* c_2 *have the same corner-type, then inserting edge* e *between* c_1 *and* c_2 *splits the face into two faces;*

(B2) *if* c_1 *and* c_2 *have different corner-types, then inserting* e *between* c_1 *and* c_2 *results in a new face.*

Rules for edge-deletion surgery

Now we turn to edge-deletion on general rotation systems. Since deleting a type-1 edge e can be implemented by first twisting e then deleting the twisted e that is of type-0, it is sufficient to focus on deleting a type-0 edge.

Theorem 7.3 *Deleting a type-0 edge e from a general graph rotation system $\rho(G)$ satisfies the following rules:*

(A) *Suppose that the two trace-pairs induced by e belong to the boundary walks of two different faces of the imbedding. Then deleting edge e merges the two faces into a single face.*

(B) *Suppose that the two trace-pairs induced by e both belong to the boundary walk of the same face F in the imbedding.*

- (B1) *If the two trace-pairs induced by edge e use different oriented edges, then deleting edge e splits the boundary walk of the face F into two closed walks, each the boundary of a new face of the resulting imbedding.*
- (B2) *If the two trace-pairs induced by edge $e = [u, w]$ use the same oriented edge, then deleting edge e changes the boundary walk of face F into the boundary walk of a single new face.*

8 Extended graph rotation systems

Topologically, tracing a twisted edge "reverses" the local orientation of the rotation system. Accordingly, retwisting an edge is equivalent to untwisting. Here are the differences in our model for cyclic weaving:

- We record which segment goes ***over*** and which segment goes ***under*** at the crossing point.
- We record ***by how many turns*** an edge is twisted.

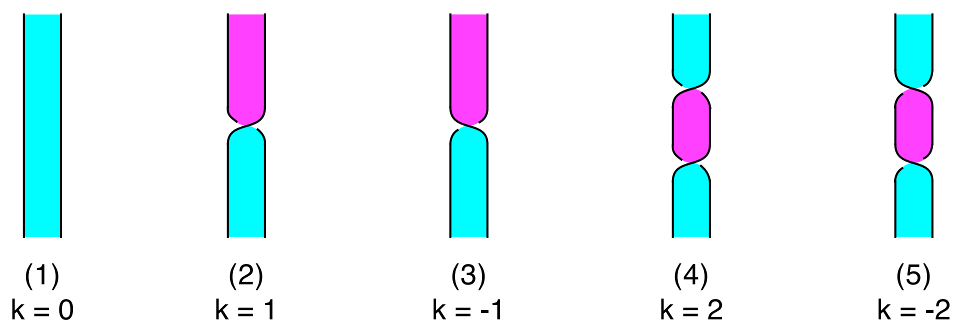


Figure 19: (1) an untwisted edge. (2) a clockwise twisted edge. (3) a counterclockwise twisted edge. (4) a double clockwise twisted edge. (5) a double counterclockwise twisted edge.

An ***extended rotation system*** for a graph G is obtained from a pure rotation system by assigning to every edge e, a number $k(e)$ of twists, with $k(e) \in \mathbb{Z}$.

9 Cyclic plain-weaving on surfaces

Theorem 9.1 *Let $\rho_0(G)$ be a pure rotation system for an imbedding $\pi_0 : G \to S$ of a graph on an orientable surface. Let A be an arbitrary subset of edges of G. If we twist all edges in A positively, or if we twist all edges in A negatively, then the resulting extended rotation system induces a cyclic plain weaving on S.*

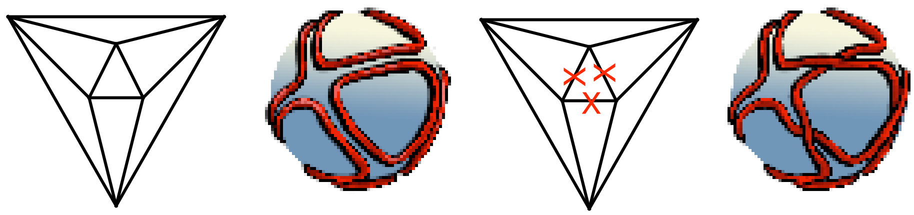


Figure 20: Close-up view of very small ERS-weaves.

Figure 21: Some easily implemented ERS weaves.

Theorem 9.2 *Every cyclic plain-weaving on the sphere can be specified by an ERS.*

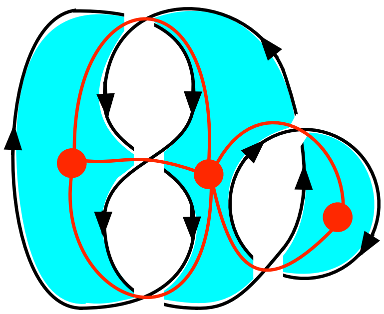


Figure 22: Constructing the graph and the ERS for a link.

Theorem 9.3 *Every CELLULAR cyclic plain-weaving on an orientable surface can be specified by an ERS.*

References

[Ad04] C. C. Adams, *The Knot Book*, Amer. Math. Soc., 2004.

[AkCh00] E. Akleman and J. Chen, Guaranteeing the 2-manifold property for meshes with doubly linked face list, *International J. Shape Modeling* **5** (2000), 149–177.

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