Admin: End of class: pick up
PS2 sol.
Midterm sol.
Your midterm

Last time: Proved $\Omega(d/e)$
S.o. I.b. for PAC learning.
\[ d = \text{vcoim}(e). \]

\[ \text{started I.b. : CRE using } C \approx \frac{d}{e} + \log(1/e) \]
is a PAC learner.
(like old Occam's
arg, but now $d \leq \ln(19H)$; considering
labelings of sample)

Today: prove AMAZING FACT (Sauer's)

\[ \text{use } \tilde{A} \text{ to give analogue of } A. \]

Applic: eff PAC learn LTFs over $\mathbb{R}^n$

First: Thm (Sauer's Lemma): Let $d = \text{vcoim}(e)$.

Then $T_e(m) \leq \Phi_d(m)$. $\bigstar$

Pf: Induc. on $m+d$.
Consider $m=0$:

$T_e(0) = 1 = \Phi_d(0)$ $\checkmark$

Consider $d=0$:

$T_e(m) = 1 = \Phi_0(m) \checkmark$

\[ \text{vcoim}(e) = 0! \]

Induc. step: assume $\bigstar$ true for all $m', d'$
s.t. $m'+d' < m+d$ + $m' \leq m$, $d' \leq d$. $\blacksquare$
Let $S \subseteq X$ have $|S| = m$. $E$ has $\text{VCOM}(E) = d$.

Want $\Theta$ (bound $|\Pi_E(S)|$).

Fix a distinguished elt $x \in S$.

Some els of $\Pi_E(S)$: Consider $S + S - \{x\}$.

$|\Pi_E(S - \{x\})| \leq \Phi_d(m-1)$ (IH).

Undercounts $|\Pi_E(S)|$ b/c of pairs like $c_1, c_2$ that differ only on $x$. Let $E' := \{ c \in \Pi_E(S) : x \notin c, c \cup \{x\} \in \Pi_E(S) \}$ ($c_i$ - type sets).

So $|\Pi_E(S)| = |E'| + |\Pi_E(S - \{x\})|$.

Sneaky obs: $E' = \Pi_E(S - \{x\})$ (b/c every elt of $E'$ is a subset of $S - \{x\}$).

So $|\Pi_E(S)| = |\Pi_E(S - \{x\})| + |\Pi_E(S - \{x\})| \leq \Phi_d(m-1)$.

If have $\text{VCOM}(E') \leq d-1$: then $V \leq \Phi_{d-1}(m-1)$ by IH, $+ \text{RHS} \vdash \Phi_d(m)$. 
In fact $\text{vc}_0(M(e')) \leq d - 1$; if had $d$ pts in $S - \{x\}$ shatl. by $e'$, then adding in $x$, there'd be $d + 1$ pts in $S$ shatl. by $e$; contrad. assump. that $\text{vc}_0(M(e)) = d$.

Now know $\text{Te}(m) \leq \Phi_d(m) = (\binom{m}{0}) + (\binom{m}{1}) + \ldots + (\binom{m}{d})$.

**Fact: $\Phi_d(m) = 2^m$ if $m \leq d$**

\[
\Phi_d(m) = \left( \binom{m}{0} + \binom{m}{1} + \ldots + \binom{m}{d} \right) \approx 2^m \quad \text{just now!}
\]

- If $m > d$, then $\Phi_d(m) \leq \left( \frac{em}{d} \right)^d = O(m^d)$ for $d$ const.

\\
\[
\left( \frac{d}{m} \right)^d \cdot \Phi_d(m) = \left( \frac{d}{m} \right)^d \cdot \sum_{i=0}^{d} \binom{m}{i} \leq \binom{m}{0} \leq \sum_{i=0}^{m} \left( \frac{d}{m} \right)^i \binom{m}{i} \quad \text{if } \frac{d}{m} < 1
\]

\[
\leq \sum_{i=0}^{m} \left( \frac{d}{m} \right)^i \binom{m}{i} \quad \text{if } m > d
\]
\[
\begin{align*}
\left(1 + \frac{d}{m}\right)^m &\leq \left(e \frac{d}{m}\right)^m = e^d \\
&\leq \left(1 + x \leq e^x\right)
\end{align*}
\]

Now: pf that CHF work. "bad": error \( \geq \varepsilon \) wrt \( c, \delta \).

Recall old thm:

**Thm:** Let \( E \) be finite. \( \forall c \in E, \forall \delta > 0 \), given \( m \) ex from \( \text{EX}(c, \delta) \), with
\[
m > \frac{1}{\varepsilon} \left( \ln |E| + \ln \frac{1}{\delta} \right)
\]
 w.p. 1-\( \delta \) all bad concepts in \( E \) inconsistent w/ the ex.

**New thm:** Let \( E \) be any c.c. \( d = \text{vc}(E) \).
\( \forall c \in E, \forall \delta > 0 \), given \( m \) ex. from \( \text{EX}(c, \delta) \) with now
\[
m > \frac{2}{\varepsilon} \left( \ln \pi_c(2m) + \ln \frac{2}{\delta} \right)
\]

Looking ahead: Fact that
\[
\pi_c(2m) \leq \left(\frac{2em}{d}\right)^d
\]
what makes this useful.

Key to Occam pf: u.b. over \( \mathcal{H} \).

"new thm" u.b. over labelings of \( 2^n \)-elt set.
Pf of new thm: Consider foll. prob. experiment:

First run: 2n ex from \( \chi(c, \theta) \), draw \( \xi_j \). 

Second run: 

Event \( A \): Some \( \xi_j \) is bad (\( \Pr (\xi_j \neq c(\xi_j, \theta) > \epsilon) \)), but is cons. \( \forall S_1 \). (To prove thm: must show) \( \Pr (A) \leq \delta \).

Event \( B \): Some \( \xi_j \) is cons. \( \forall S_1 \), but wrong on \( > \frac{\epsilon}{2} \) ex. in \( S_2 \).

Claim: \( \Pr (B) \leq \frac{\delta}{2} \). Then \( \Pr (A) \leq 2 \cdot \Pr (B) \).

Pf: \( \Pr (B) \leq \Pr (A + B) = \Pr (A) \cdot \Pr (B/A) \).

Suff to show \( \Pr (B/A) \leq \frac{\delta}{2} \).

Suppose \( A \) occurs. So \( \exists \xi_j \) bad, \( \xi_j \) cons. \( \forall S_1 \). True error of \( \xi_j \) is \( > \epsilon \), so by mult. CB. (\( r = \frac{1}{2} \))

\[ \Pr (\xi_j \text{ err. then } \epsilon/2 \text{ in ex wrong in } S_2) \leq e^{-\epsilon m/2} = e^{-\epsilon m/8} \]

\[ \frac{\epsilon}{2} : \quad \frac{1}{8} < \frac{1}{2} \quad \text{(claim)} \]

Need only show \( \Pr (B) \leq \frac{\delta}{2} \).

Equiv. way to view \( B \): Draw \( S \) set of \( \xi_j \) ex.

Randomly split \( S \) into \( S_1 \) \& \( S_2 \) (1st \& 2nd halves),

\( B = \) event that some lab. of \( S \) results in 0 mist on \( S_1 \), 

\( > \frac{\epsilon}{2} \) mist on \( S_2 \).

Fix some partic. lab. of \( S \). Assume it makes \( > \frac{\epsilon}{2} \) mistakes in total.
Barrel $2m$ balls: $k$ black, $2m-k$ white.

Rand. part. into $S_1$ & $S_2$.

$$\Pr[\text{all } k \text{ end up in } S_2] = \frac{k^k}{\binom{2m}{k}}$$

$$= \frac{k!}{(k-1)! m!} \cdot \frac{m(m-1)\ldots(m-k+1)}{2m(2m-1)\ldots(2m-k+1)} \leq \frac{1}{2^k}.$$  

UB over all $\prod_e(2m)$ lab. of $S$:

$$\Pr[C_B] \leq \prod_e(2m) \cdot \frac{1}{2^k} \leq \frac{\prod_e(2m)}{2^{en/k}}.$$  

So all done if

$$\frac{e}{k} \leq \frac{5}{2}.$$  

Algebra

$$m \geq \frac{2}{e} \left( \ln \prod_e(2m) + \ln \frac{3}{5} \right)$$

suff.