Last time:

- simplifying assumptions for small-depth ckts
- PAR : $\approx 2^n$ u.b. l.b. for depth-2

  $\approx 2^{n^{\frac{1}{d-1}}}$ u.b. for depth d

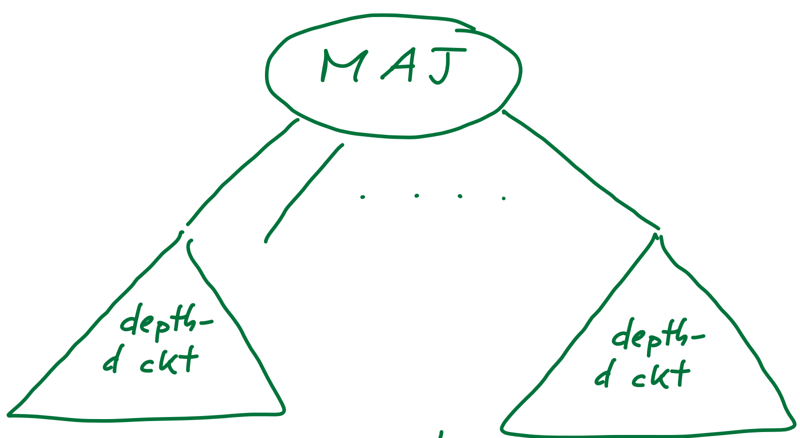
  ⊛ $\approx 2^{n^{\frac{1}{d-1}}}$ l.b. " " " :

based on random restrictions, Håstad Switching Lemma,

(iterative depth reduction) (didn't prove)

Today: • using HSL for $2^{\Omega(n^{\frac{1}{d-1}})}$ l.b.

• start proof that



needs size $2^{\Omega(n^{\frac{1}{4d}})}$ to compute PAR

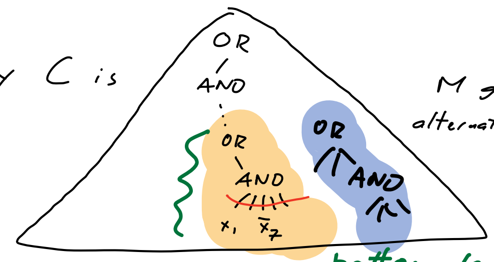
using polynomial threshold functions (PTFs)

"PTF degree"

"weak PTF degree"

Questions?

Recall setup: Say C is



depth d,
M gates,
alternating formula
computes PAR

bottom level

We did: Stage 0: "initial trim of fan-in"
Hit C with $\rho \leftarrow R_p$, $p = \frac{1}{100}$. What happens?

We got: some outcome of ρ makes C "collapse" to C_0:

- C_0 has size $\le M$
  bott fanin $\le 10 \log M$
- C_0 computes PAR (or $\overline{\text{PAR}}$) over $\ge \frac{n}{200}$ vars.

Stage 1: Hit C_0 w/ r.r. $\rho \leftarrow R_{\frac{1}{100 \log M}}$. $\frac{1}{100 \log M} = p$

Each depth-2 from-bottom form. in C_0 is a width-$(10 \log M)$ DNF.

– Apply HSL w/ $w = 10 \log M$,
$t = 10 \log M$:

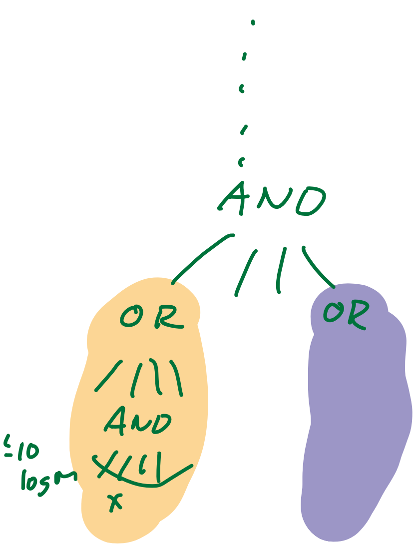
get that this whp $\left(> 1 - \frac{1}{M^{10}}\right)$ becomes a CNF of width $\le t = 10 \log M$.

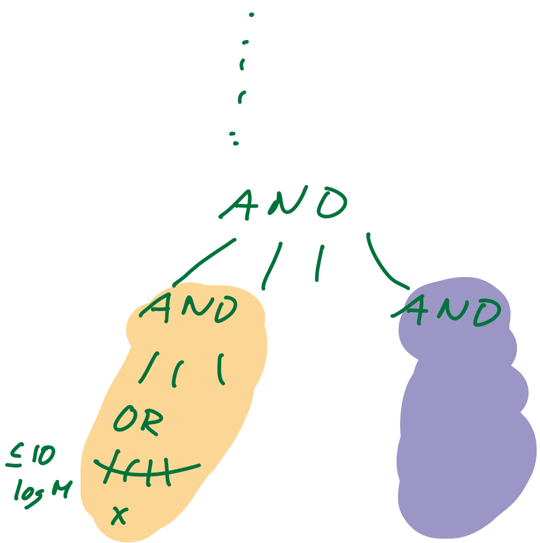
– UB over all depth-2-from-bottom formulas:
w.p. $> \frac{1}{2}$ all these become CNFs of width $\le 10 \log M$.

Also, w.p. $> \frac{1}{2}$, # $*$s $\ge$ $p/2$ frac. $= \frac{1}{200 \log M}$ frac.

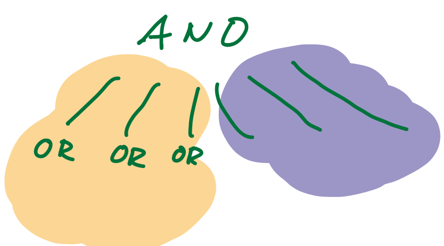
* Collapse adj AND gates at depths 3 & 2 from bottom! Call Result C_1.

Pre-restric: $\overset{\rho}{\Longrightarrow}$ Post-restric





collapse to a CNF:



DEPTH REDUCTION.

Repeat:

l.b. 07

Current ckt:	Depth	Bott. fan-in	# vars "alive"
C	d	n	n
C_0	d	$10 \log M$	$\frac{n}{200}$
C_1	$d-1$	$10 \log M$	$\frac{n}{200} \cdot \frac{1}{200 \log M}$
C_2	$d-2$	$10 \log M$	$\frac{n}{200} \cdot \left(\frac{1}{200 \log M}\right)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_{d-2}	2	$10 \log M$	$\frac{n}{200} \cdot \left(\frac{1}{200 \log M}\right)^{d-2}$

Note: any depth-2 ckt w/ bott. fan-in r can compute PAR only on $\leq r$ vars.

So

$$\frac{n}{200 \cdot (200 \log M)^{d-2}} \leq 10 \log M.$$

$$\frac{n}{10 \cdot 200^{d-1}} \leq (\log M)^{d-1}$$

$$2^{\Omega(n^{\frac{1}{d-1}})} \leq M. \quad \blacksquare$$

STRONGER LB: AC^0 augmented w/ MAJ.

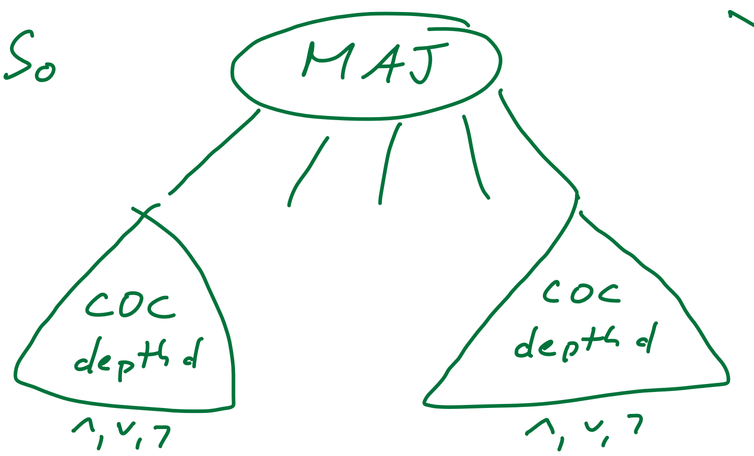
$$\left(\begin{array}{l} MAJ = \operatorname{sign}(x_1 + \dots + x_r) \\ \{-1,1\}^r \to \{-1,1\} \end{array} \right)$$

$MAJ(x_1, \dots, x_r) : \{0,1\}^r \to \{0,1\}$

$$MAJ(x_1, \dots, x_r) = \begin{cases} 1 & \text{if } \sum_{i=1}^{r} x_i \geq \frac{r}{2} \\ 0 & \text{o/w.} \end{cases}$$

FACT: COC ($\wedge, \vee, \neg$ gates) can't compute $MAJ_n(x_1, \dots, x_n)$ in size less than $2^{\Theta(n^{\frac{1}{d-1}})}$.

So



stronger than just COC.

We'll do a l.b. for these:

<u>Thm</u> [ABFR94]: Let C be any size-s depth-d ckt for PAR of above form:

MAJ

COC ... COC of $\wedge/\vee, \neg$.

Then $s \geq 2^{\Omega(n^{\frac{1}{4d}})}$.

Pf: uses <u>polynomial threshold functions</u> (PTFs).

<u>Def</u>: Let $f: \{0,1\}^n \to \{-1,1\}$ be a Bool. fn.

A (real) polynomial $p(x_1, \ldots, x_n): \mathbb{R}^n \to \mathbb{R}$ is
$\{0,1\}^n \to \mathbb{R}$

a <u>PTF for f</u> if

- $p(x) \neq 0$ for all $x \in \{0,1\}^n$
- for all $x \in \{0,1\}^n$, have $\text{sign}(p(x)) = f(x)$.

A Bool. $f: \{0,1\}^n \to \{-1,1\}$ has a <u>degree-d PTF</u> if there's such a p with $\deg(p) \leq d$.

$\text{PTFdeg}(f) = \min d$ s.t. f has a deg-d PTF.

<u>Ex</u>: f is $\overset{\text{DNF}}{x_2 \bar{x}_3 x_4 x_5 \vee x_1 x_4 \vee \bar{x}_2 \bar{x}_3 x_6}$:

$\text{PTFdeg}(f) \leq 4$:

$$\underbrace{x_2 \cdot (1-x_3) \cdot x_4 \cdot x_5}_{0/1} + \underbrace{x_1 \cdot x_4}_{0/1} + \underbrace{(1-x_2) \cdot (1-x_3) \cdot x_6}_{0/1} - \frac{1}{2} \; : \; + \text{ iff DNF is true}$$

Ex: $f = x_1 \wedge x_2 \wedge \dots \wedge x_\ell$:

$PTFdeg(f) = 1$: $x_1 + \dots + x_\ell - (\ell - \frac{1}{2})$

Some obs. :

- $PTFdeg(f)$ same whether we think of $f: \{-1,1\}^n \to$ or $f: \{0,1\}^n \to$.

If p is a deg-d PTF for $f: \{0,1\}^n \to \{-1,1\}$, then

$$q(x_1, \dots, x_n) = p\left(\frac{x_1+1}{2}, \frac{x_2+1}{2}, \dots, \frac{x_n+1}{2}\right)$$

is a deg-d PTF for $f: \{-1,1\}^n \to \{-1,1\}$

$2x-1$ for other direc.

Can show that every $f: \{-1,1\}^n \to \{-1,1\}$ is computed exactly by a multilinear poly of degree $\le n$.

↳ max deg of each var ≤ 1.

↳ So every n-var fn f has $PTFdeg(f) \le n$.

- WLOG, any poly p is multilinear:
PTF for f
inputs 0/1 : $0^2=0$, $1^2=1$.

inputs −1/1 : $(-1)^2=1$.

Def: Let $f:\{0,1\}^n \to \{-1,1\}$ be a Bool. fn.
A poly $p(x_1,\ldots,x_n)$ is a weak PTF for f if

i) $p(x)\neq 0$ for at least one $x\in\{0,1\}^n$; &
ii) $\forall x\in\{0,1\}^n$, if $p(x)\neq 0$ then $\mathrm{sign}(p(x))=f(x)$.

f has weak PTF deg d if there's such a p of deg $\le d$.

WeakPTFdeg(f) = min such d.

For every $f:\{0,1\}^n \to \{-1,1\}$, have

$$\mathrm{WeakPTFdeg}(f) \le \mathrm{PTFdeg}(f) \le n.$$

Do any f's have WeakPTF(f) = n ? YES:

Lemma: Weak PTF deg $(PAR_n) = n$.

Pf: View inputs as ± 1:

$PAR_n(x_1, \dots, x_n) = x_1 x_2 \dots x_n$.

Suppose $p(x_1, \dots, x_n)$ is a weak PTF for PAR_n + $\deg(p) = d < n$.

Write $p(x) = \sum_{S \subset [n]} c_S X_S$.

$X_S = \prod_{j \in S} x_j$

(since $d < n$, each S is proper subset: $|S| < n$.)

Consider $\circledast = \sum_{x \in \{-1,1\}^n} p(x) \cdot PAR(x)$: Since p is a weak PTF for PAR,

For every x have $p(x) \cdot PAR(x) \geq 0$, +

for some x, have $p(x) \neq 0$; so $\circledast > 0$.

But,

$$\circledast = \sum_x p(x) \cdot x_1 \cdots x_n = \sum_x \left(\sum_{S \subset [n]} c_S X_S \right) \cdot X_{[n]}$$

$$= \sum_x \sum_{S \subset [n]} c_S X_{[n] \setminus S}$$

$(-1)^2 = 1^2 = 1$

$$= \sum_{x} \sum_{\substack{T=[n]\setminus S \\ T \neq \emptyset \text{ b/c } d<n}} c_{[n]\setminus T}\, x_T$$

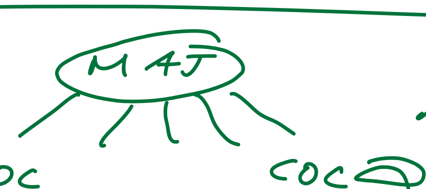
$$= \sum_{\substack{T=[n]\setminus S \\ T\neq\emptyset}} c_{[n]\setminus T} \left(\sum_{x} x_T \right)$$

But for any T in sum, since $T \neq \emptyset$,

$$\sum_{x} x_T = 0. \quad \text{So } (*) = 0.$$

Contrad! So $d = n$. $\blacksquare$

Let's use this for



lb's!

High level proof strategy:

1) (The ckts we're interested in) have "PTF approximators": Sps f has size-s depth-d ckt like:
Then there's a "low-deg" poly p ($\deg \approx (\log s)^{2d}$) that's "almost" a PTF for f:

$\rightarrow$ $\operatorname{sign}(p(x)) \neq f(x)$ only for a few x's.

2) (PTF approx. $\Rightarrow$ weak PTF)

If f is a Bool fn + p is an "almost PTF" for f, we can modify p to get new poly $q(x)$ which is a weak PTF for f, where $\deg(q)$ only a bit larger than $\deg(p)$.

3) PAR has weak deg n. ☺ Did this!

Remains to do 2) + 1). Next time...