Last time: start unit on
Hierarchy thms, relationships among resources
- "padding arguments" Papad. 20.1, Sipser 9.1,
- clocked simulation + diagon. Papad. 7.1, 7.2
- time (+ space) hierarchy theorems : rel. between same resource

Today: end of unit on
- Relationships between different resources
  time, space, nondet.;
  nondet space vs time
- start unit on space complexity

Papad. 7.3

Motiv: L vs P ? NL vs P ?
NP vs PSPACE ? NP vs EXPTIME ?

Some easy first observations:

Obs: For f a p.c.f., have
1. • TIME(f(n)) ⊆ SPACE(f(n)) ✓ in f(n) time, can only touch f(n) tape cells.
2. • TIME(f(n)) ⊆ NTIME(f(n)) ✓
3. • SPACE(f(n)) ⊆ NSPACE(f(n)) ✓

Slightly stronger ver. of 1 + 2:

Thm (Hopcroft, Paul, Valiant):
If $f(n) \geq n$ a p.c.f., then
$$TIME(f(n) \cdot \log f(n)) \subseteq SPACE(f(n)).$$

Thm (Paul, Pippinger, Szemeredi, Trotter):
If $f(n) \geq n$ a p.c.f., then

$\log^* N$ = # k s.t. $\underbrace{\log \log \cdots \log}_{k} N \leq 1$.

$$TIME(f(n)\cdot(\log^* f(n))^{\frac{1}{4}}) \subseteq NTIME(f(n))$$

More:

<u>Thm:</u> for $f(n)\geq n$ $\overbrace{\text{a p.c.f.}}^{\text{(don't need)}}$, have

$$NTIME(f(n)) \subseteq \bigcup_{K>1} TIME(K^{f(n)}).$$

<u>Pf:</u> Fix $L\in NTIME(f(n))$.

Let M be NTM running in time $f(n)$ + deciding L.

We'll describe a det M' running in time

$\leq K^{f(n)}$ for some fixed K (dep. on M)

that decides L.

Overall nondet. comp. of M on x: a tree of depth $f(|x|)$, branching factor at each node $\leq C_M$ (const. dep. on M.)

M acc x iff some leaf reaches q_{acc}.

Any partic. outcome of M on x $\Longleftrightarrow$ fixed path thru tree $\Longleftrightarrow$ a fixed elt of $\{1,\dots,c_M\}^{f(n)}$.

j^{th} entry is 7: make choice #7 at step j of comput.

$(C_M)^{f(n)}$ such lists.

$|x|=n$

Det M' • computes $f(|x|)$ (p.c.f.)

• runs thru all; for each list, M' sim. M's comput. for that seq. of nondet. choices.

If some seq. causes M to accept, M' accepts;

if no " " " " ", M' rejects.

Runtime $\le (c_M)^{f(n)} \cdot \underline{poly(f(n))}$

$\le K^{f(n)}$ for some K. ∎

Sps f not a p.c.f. Then can't

Workaround: try all lists in $\{1, \dots, c_M\}^t$ for $t = 1, 2, \dots$ successively, until reach a value t^* s.t. all comput. of M on x halt within t^* steps.

(if all rej, M' rej; if one accepts, M' acc.)

This will happen with $t^* \le f(n)$.

So runtime now is $\le \sum_{t=1}^{f(n)} (c_M)^t \cdot poly(f(n))$

$\le K^{f(n)}$ some K. ∎

Cor: $NP \subseteq EXP$.

Thm: For f a p.c.f. with $f(n) \geq n$, have
$NTIME(f(n)) \subseteq SPACE(f(n))$.

Pf: M' as before can run in $O(f(n))$: reuse space.

Tape 1 hold curr. seq. in $\{1, \dots, c_M\}^{f(n)}$ being tried; $\to O(f(n))$

other tapes do the simul. of M on this seq.
$\hookrightarrow O(f(n))$ space

When trying next seq. in $\{1, \dots, c_M\}^{f(n)}$, reuse space on other tapes.

Cor: $NP \subseteq PSPACE$.

Recall BGS thm & the 10U from it:

There is a lang A s.t. $P^A = NP^A$.

Pf: Let A be any PSPACE-complete.
($A \in PSPACE$, & $\forall B \in PSPACE$, have $B \leq_p A$).
We'll exhibit PSPACE-complete A soon.

Then
$$NP^A \underset{①}{\subseteq} PSPACE^A \underset{②}{\subseteq} PSPACE \underset{③}{\subseteq} P^A \subseteq NP^A$$

①: we just saw can sim. any NP machine in PSPACE (try all poss seq. of guesses, reuse space).

②: Since A is in PSPACE, don't need A-oracle: PSPACE alg. can answer all queries to A in PSPACE by itself.

③: since A is PSPACE-hard, any lang in PSPACE $\leq_p A$.

So all of are $=$, so $P^A = NP^A$.

What about nondet space vs det time?

Next:

Thm: Let $f(n)$ be a p.c.f.

Have $NSPACE(f(n)) \subseteq \bigcup_{k>1} TIME(k^{f(n)+\log n})$.

---

if $f(n) \geq \log(n)$, then

$$NSPACE(f(n)) \subseteq \bigcup_{k>1} TIME(k^{f(n)}).$$

Def: Let M be a c-tape TM, i.e. (Det. or nondet)

$M = (K, \Sigma, \delta, s)$

set of K_M states — alphabet — transition rule (function if M is det.) — start state

A configuration of M on input x is:

- input head position (n choices);
- current state of M (K_M ");
- current tape contents + tape head position for each of the c worktapes (for an $f(n)$-space machine,

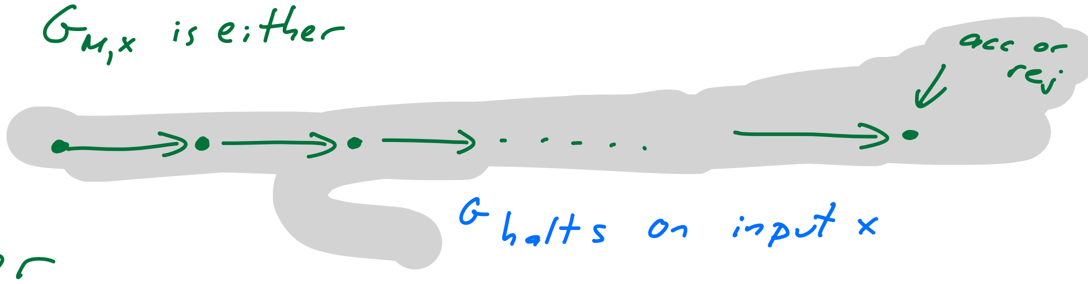
$$\left(f(n) \cdot |\Sigma|^{f(n)}\right)^c.$$

head pos.   # poss. tape contents

Given an input x, a <u>config. graph of M on x</u>

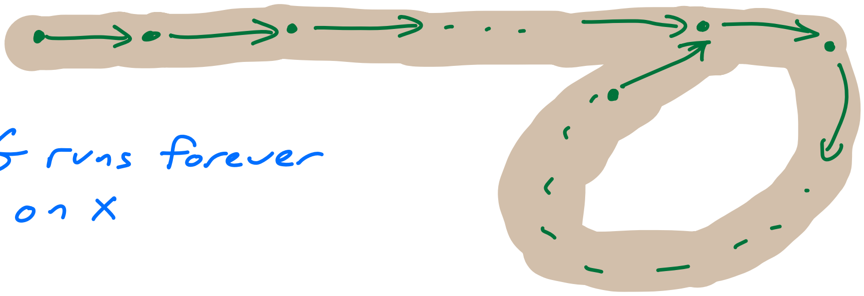
$G_{M,x}$ is a directed graph whose nodes are configs of M on x; directed edge (C_1, C_2) is in $G_{M,x}$ if C_1 can follow from C_2 in one step of M's transition rule.

• If M is det., $G_{M,x}$ has outdeg 1:
if $G_{M,x}$ has finite #nodes (G uses finite space on x), then $G_{M,x}$ is either



G halts on input x

or



G runs forever on x

If M is nondet. + has $\le c_M$ choices at each nondet step, $G_{M,x}$ has outdegree $\le c_M$.

<u>Fact:</u> if M is an NTM, can "standardize" it to M' s.t. on every x, M' has a unique C_{acc}

halting "accept" configuration: "cleanup" by erasing all worktapes, moving all tape heads back to initial position.

M' doesn't use any more space than M.

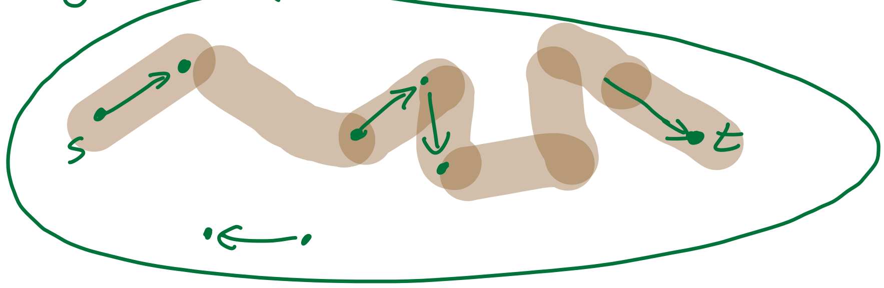
For a standardized NTM M, have $\forall x$,

$$M \text{ accepts } x \iff C_{acc} \text{ is reachable from } C_{init} \text{ in } G_{M,x}.$$

This is an instance of REACHABILITY on config graph $G_{M,x}$.

REACH $= \{ (G, s, t) :$ G a digraph; s, t nodes in G; t reachable from s via path of directed edges.

G



Pf of thm:

> Thm: Let $f(n)$ be a p.c.f.
> Have $NSPACE(f(n)) \subseteq \bigcup_{k>1} TIME(k^{f(n)+\log n})$

Let $L \in NSPACE(f(n))$, let M be NTM deciding L in space $f(n)$.

Given $|x| = n$, # nodes in $G_{M,x}$ is # configs

$$\le n \cdot k_M \cdot \left(f(n) \cdot |\Sigma|^{f(n)} \right)^c = N$$

$\leq a^{\log n + f(n)}$ some const. a.

Since REACH $\in P$,

can solve REACH on $G_{M,x}$ in poly(N) time,
so determining whether M acc. x can be done by
a det TM M' in

poly( ) time. So $L \in \bigcup_{k>1} \text{TIME}(k^{f(n)+\log n})$.

(• Think of M' as "knowing" M; its finite control "knows" (depends on) M's.

• On input x, M' could write down whole $G_{M,x}$ graph & then run REACH alg. on it.
That's fine.
But M' doesn't need to explicitly write down $G_{M,x}$ to run the REACH alg.

Whenever M' needs to read part of $G_{M,x}$ (i.e. check whether C' is reachable from C in one step, ie) / (C, C') edge is in $G_{M,x}$),
M' can do it by writing C & C' (say on an extra tape) & checking that $C \to C'$ in 1 step under M's transit. rule.

max{$f(n)$, $\log n$}

Takes space $\le O(f(n) + \log n)$ because
log n bits for input head location, $O(f(n))$ bits to write rest of config.

$a, b \ge 0$:

$\max\{a,b\} \le a + b \le 2 \cdot \max\{a,b\}$

So... now know $\quad \left(k^{\log n} = 2^{(\log k)(\log n)} = n^{\log k} \right)$

$NL = NSPACE(\log n) \underset{\text{(just did)}}{\subseteq} \bigcup_{k>1} TIME(k^{\log n}) = P.$

So...

We think:

next time

$$L \overset{\neq}{\subseteq} NL \overset{\neq}{\subseteq} P \overset{\neq}{\subseteq} NP \overset{\neq}{\subseteq} PSPACE \overset{=}{\subseteq} NPSPACE \overset{\neq}{\subseteq} EXPTIME \overset{\neq}{\subseteq} EXPSPACE$$

Know

- $L \subsetneq PSPACE$: (space hierarchy thm)

  $L \subsetneq SPACE(\log^2 n) \subsetneq SPACE(n) \subseteq PSPACE$

- $P \subsetneq EXPTIME$: (time hierarchy)

  $P \subseteq TIME(n^{\log n}) \subsetneq TIME(2^n) \subsetneq EXPTIME$

Next time: start unit on space complexity.