Last time: • End of unit on oracles, ckts, poly-time hierarchy:

- Karp-Lipton thm: $NP \subseteq P/poly \Rightarrow$ PH collapses to Σ_2^P.
- Baker-Gill-Solovay: $P^A = NP^A$, $P^B \neq NP^B$ some oracles A, B.

• Started next unit:

Hierarchy thms, relationships among resources

Today:
- "padding arguments" — Papad. 20.1, Sipser 9.1,
- clocked simulation + diagon. — Papad. 7.1, 7.2
- time (+ space) hierarchy theorems

Recall our ex. padding thm:

Thm: If $NTIME(n^2) \subseteq TIME(n^3)$, then

then $NTIME(n^{10}) \subseteq TIME(n^{15})$.

Contrap: if $NTIME(n^{10}) \not\subseteq TIME(n^{15})$,
then
$NTIME(n^2) \not\subseteq TIME(n^3)$.

"Containment (equality) translates ↑" ("inequality (contrap: separation) translates ↓")

Pf: Let $L_1 \in NTIME(n^{10})$. To show: $L_1 \in TIME(n^{15})$.
↳ "padding" — $L_1 \subseteq \Sigma^*$, $\# \notin \Sigma$.
Let M_1 is a NTM running in n^{10} time, deciding L_1.
Define L_2:

$L_2 := \{ \underbrace{x\#^{|x|^5 - |x|}} : x \in L_1 \}$ where $\#$ is a new "dummy" symbol.

L_2 is in $NTIME(n^2)$:

- check that input is of form (linear time)
- ignore $\#$'s, run M_1 on x.

(whole input length is n, so $|x| = n^{1/5}$, so M_1 on x takes time $n^2 = (n^{1/5})^{10}$.)

By assump, $L_2 \in TIME(n^3)$. Let M_* be n^3-time det TM for L_2.

Here's an n^{15} time det alg for L_1:

on input x, $|x| = n$, write "$x\#^{|x|^5 - |x|}$" (length n^5) + call M_* on this.

$(n^5)^3 = n^{15}$. ∎

General technique; space, time, nondet time, etc.

Hierarchy Theorems

Q's :

- is every decidable L in $TIME(2^n)$?
  (NO)
- is $TIME(n^3) = TIME(n^4)$?
  (NO)

Recall: • a fn $g: \mathbb{N} \to \mathbb{N}$ is computable if some TM, on input n, outputs $g(n)$ $\forall n$.

• a lang L is decidable if there's some TM M st $\forall x \in \Sigma^*$,

$x \in L \Rightarrow M$ accepts x
$x \notin L \Rightarrow M$ rejects x. (always halts).

Thm: Given any computable $f: \mathbb{N} \to \mathbb{N}$, there is a decidable L s.t. $L \notin TIME(f(n))$.

Pf: Diagonalize to construct a TM U^* that

• decides a lang L, but
• $\forall$ $f(n)$-time-bounded TM M, U^* disagrees w/ M on some input.

idea:

Our L will be $\subseteq \{0,1\}^*$.
Let $x_1, x_2, \ldots$ enum. of bin. strings.
Each x_i represents TM M_i.

Let U^* be following TM: on input x_i,
1) compute $f(|x_i|)$ (doable since f computable);
2) runs M_i on x_i for $f(|x_i|)$ steps.

U^* accepts if $M_i(x_i)$ rejects within $f(|x_i|)$ steps or hasn't halted within $f(|x_i|)$ steps; U^* rejects if M_i acc. x_i within $f(|x_i|)$ steps.

Let L = lang. dec. by U^*. L decidable.

$L \notin TIME(f(n))$: Suppose $L \in TIME(f(n))$.

This means there's some machine M_K s.t. $\forall x$ M_K acc x iff U^* acc. x, & M_K runs in time $f(n)$.

Consider running M_K on x_K:

- $x_K \in L \Rightarrow U^*(x_K)$ acc $\Rightarrow M_K$ acc x_K in $f(|x_K|)$ time $\Rightarrow U^*$ rej x_K. CONTRAD.
- $x_K \notin L \Rightarrow U^*(x_K)$ rej. $x_K \Rightarrow M_K$ rej $x_K \Rightarrow U^*$ acc x_K. CONTRAD.

So $L \notin TIME(f(n))$. ■

Can conclude from this that there is an ∞ hierarchy of time classes:

starting w/ $f = f_1$, saw $\exists$ dec $L \notin TIME(f_1)$, $L \in TIME(f_1')$. Let $f_2(n) = \max\{f_1(n), f_1'(n)\}$. $L \in TIME(f_2)$.

$$TIME(f_1) \subsetneq TIME(f_2) \subsetneq TIME(f_3) \subsetneq \ldots$$

Can we be more quant. precise?

$TIME(n^2) = TIME(n^8)$?

Look closely at time needed to decide L in prev pf... the machine U^* dec L essentially does following:

• given (M, x), runs M on x for $f(|x|)$ steps.

How eff. can we do this?

Pot. concern: need to compute $f(|x|)$.

what if computing $f(|x|)$ on input x takes $\gg f(|x|)$ time steps?

Ex: $f(n) = \begin{cases} n & \text{if } 2^{2^{2^{2^{2^n}}}} \text{th digit of } e^{\pi^e} \text{ is } 3 \\ n^2 & \text{otherwise} \end{cases}$

Legislate weird f's away: consider only "nice" time/space bounds.

Def: A fn $f: \mathbb{N} \to \mathbb{N}$ is a proper complexity fn (p.c.f.) if

(a) $f(n+1) \geq f(n) \ \forall n$;

(b) there's a TM which, for every input x, outputs a string of length $f(|x|)$ & runs in time $O(|x| + f(|x|))$ & space $O(f(|x|))$

Fact: $\lceil \log n \rceil$, $n \cdot \lceil \log n \rceil$, $\lceil \sqrt{n} \rceil$, 2^n, n^2, etc all p.c.f.'s.

How eff. can we run a UTM for f steps?

p.c.f.

$\underline{\text{Thm}}$: (Clocked simulation.)

Fix any p.c.f. $f(n) \geq n$.

There's a TM U_f s.t. on input $(\underline{M}, \underline{x})$,

TM

input

U_f runs in time $O(f(|(M,x)|)^3)$ +

acc iff M acc x within $\underline{\underline{f(|x|)}}$ steps.

$\underline{\text{Pf}}$: Suppose M is a c-tape TM.

U_f has 4 worktapes, used as follows:

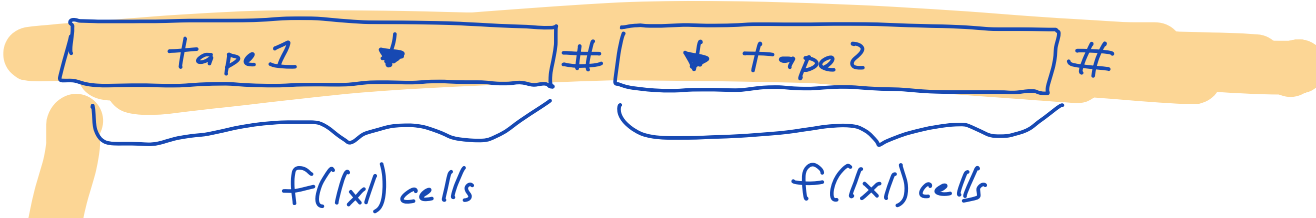
$\underline{\text{Tape 1}}$: U_f starts off by measuring off $f(|x|)$ cells; it'll use this as "clock."

p.c.f.

↪ takes $O(|M| + |x| + f(|x|))$ time.

(neglig.)

$\underline{\text{Tape 2}}$: U_f stores all c of M's tape contents, head positions, sequentially:



$f(|x|)$ cells per simulated tape.

Total length needed:

$$\underset{\substack{\nearrow \\ \#\text{tapes}}}{c} \cdot \underset{\substack{\uparrow \\ \text{cells per tape}}}{f(|x|)} \cdot (\text{overhead})$$

$\leftarrow$ needed b/c M's alphabet may be > U_f's alphabet.

$\underline{c \cdot \text{overhead}} < \underline{|M|}$. ($c$ is < $\log|M|$, etc.)

So overall, tot. length of this tape < $\underbrace{c \cdot \text{overhead}}_{\leq M} \cdot f(|x|)$

$\leq f(|(M,x)|)^2$.

Tape 3: holds M's program, state of finite control, etc; length $\leq |M| \cdot \text{overhead} \ll f(|(M,x)|)^2$;

Tape 4: worktape for U_f.

To sim. one step of M on x, U_f does:

- scans tape 2, copies to tape 4 the c symb. M's tape heads are reading. (U_f can't use finite control: M's alphabet may be too big for U_f.)
  Time needed $\leq f(|(M,x)|)^2$.

- scans tape 3, figures out M's next move given c cells being read + M's current state; writes next move to tape 4. Time needed $\leq f(|(M,x)|)^2$.

- Updates tape 2, advances tape 1 clock by 1 tick.

Time needed $\leq f(|\langle M,x\rangle|)^2$.

So, sim. each step of M on x can be done in $O(f(|\langle M,x\rangle|)^2)$ time.

If clock runs out or if M rej x within $f(|x|)$ steps, U_f rej.
If M acc x within $f(|x|)$ steps, U_f accepts.

Whole thing takes time

$$O(f(|\langle M,x\rangle|)^2) \cdot f(|x|)$$

$$\leq O(f(|\langle M,x\rangle|)^3) \text{ time.} \quad \blacksquare$$

<u>Cleverer sim.</u> (we won't prove):

<u>Thm:</u> (Strong clocked sim.)
Let f,g be any p.c.f. s.t. $f(n) \geq n$
$g(n) = \omega(f(n) \cdot \log f(n))$.
There's a U_f TM s.t. on input $(\overset{\text{TM}}{\underline{M}}, \overset{\text{input.}}{\underline{x}})$,
U_f runs in time $g(|\langle M,x\rangle|)$ &
acc iff M acc x within $\underline{\underline{f(|x|)}}$ steps.

This gives us a <u>quantitative</u> time hierarchy thm:

<u>Thm</u>: (Time Hier. Thm):

Fix any p.c.f.'s f, g s.t. $f(n) \geq n$, $g(n) = \omega(f(n) \cdot \log f(n))$.

Have $TIME(f(n)) \subsetneq TIME(g(n))$.

<u>Pf sketch</u>: Like earlier pf. Let D be det TM which, on input x_i, runs U_f on (M_i, x_i) + accepts iff U_f rejects.

Lang L acc. by D is in $TIME(g(n))$.

Same arg. as before tells us that $L \notin TIME(f(n))$.

Intuitively, log factor in strong clocked sim. is b/c U_f has fixed # (4) tapes, + M (being sim.) can have $k > 4$ tapes; have to store mult. M-tapes of M on 1 tape; entails moving back + forth.

<u>Space</u>? Space version of clocked sim: don't mind moving back + forth; only factors in play are c + "overhead".

Recall that every TM occurs ∞ly many times in enum: so c, "overhead" are, for some $\equiv$ machine M, arbitrarily small compared to M. Working this out, get:

<u>Thm</u>: (Space Hierarchy Thm): Let $f(n), g(n)$ be p.c.f. s.t. $f(n) \geq \log n$ + $g(n) = \omega(f(n))$.

Then $SPACE(f(n)) \subsetneq SPACE(g(n))$.

Properness is essential...

Old hier. thms :) : rel. between $TIME(f)$ & $TIME(g)$
$SPACE(f)$ & $SPACE(g)$

Next time:

rel. between diff resources:

TIME vs SPACE

TIME vs NTIME etc.
