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Computer Science 4236: Introduction to Computational Complexity
Final exam, Spring 2023

Due 11:59pm Friday, May 5, 2023

Allowed materials include any of the readings for this course (linked to from the course web page); Rocco's notes from lecture; the responses that Rocco and the other instructors have provided on Ed Discussion; and any notes that you produced (typed or handwritten) yourself for the course. **Disallowed materials** include everything else.

You are **not allowed** to collaborate or communicate with anyone else while taking the exam; it must reflect entirely your own work.

Be *clear, precise, and succinct* in your answers. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.

If you have questions during the exam, please make a **private post** on Ed Discussion.

Problem 0 (2 points) Summarize the most important aspects of the Computer Science Department's academic honesty policy, as relevant to this exam, in your own words, in one sentence. As a reminder, some key points of the policy are reproduced below:

> *The Department enforces the general university and school policies on academic honesty, as described in the Bulletins of the School of Engineering and Applied Science, of Columbia College, and of the School of General Studies; in the Honor Code booklet of Barnard College; and in other related publications ...*
>
> *Unless specifically authorized by the instructor, no external aids or electronic devices are allowed in exams ...*
>
> *In general, the academic penalty for a first offense of academic dishonesty within the Department is a grade of zero on the assignment, project, or exam, or reduction of the course grade at the discretion of the instructor ...*
>
> *SEVERE ACADEMIC DISHONESTY: ... A student planning or executing with another student a cooperative subterfuge during an exam ... A student making use of unauthorized material during an exam.*

Problem 1 (20 points) Recall that an *independent set* in a graph G is a subset S of vertices such that no edge is present between any pair of vertices in S.

Show that the problem of counting the number of satisfying assignments of a monotone 2CNF formula (i.e. a CNF in which each clause has exactly 2 distinct unnegated variables x_i and x_j) is equivalent to the problem of counting the number of independent sets in a graph (i.e. a polynomial time algorithm for one problem yields a polynomial time algorithm for the other).

Problem 2 (20 points) Give a probabilistic polynomial-time algorithm which generates a uniform random satisfying assignment for an input DNF formula. In more detail, your algorithm should

- take as input an m-term n-variable DNF formula ϕ and a value $\delta > 0$;
- run in $\text{poly}(n, m, \log(1/\delta))$ time; and
- with probability at most δ output $\perp$ (meaning "failure") and with the remaining (at least $1-\delta$) probability, output a uniform random satisfying assignment, i.e. a random $\boldsymbol{x} \in \{0,1\}^n$ such that for each satisfying assignment z of ϕ, we have $\mathbf{Pr}[\boldsymbol{x} = z] = \frac{1}{\#\text{ sat. assignments of }\phi}$.

(You may assume that the input DNF formula has at least one satisfying assignment.)

Problem 3 (20 points) Show that all but a $1/2^{2^n}$ fraction of Boolean functions $f(x,y) : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ have deterministic communication complexity at least $n - c$ for some absolute constant c independent of n.

Problem 4 (20 points) The *majority* function on n binary inputs, $MAJ_n : \{0,1\}^n \to \{0,1\}$, is defined as

$$MAJ_n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } x_1 + \cdots + x_n \geq n/2 \\ 0 & \text{otherwise.} \end{cases}$$

Show that any depth-d unbounded fan-in circuit for MAJ_n must have at least $s(n,d)$ gates for some function $s(n,d)$. You may use any results proved or stated in class, and similar to what we did in class, you may think of d as a constant (and you don't need to worry about constants in the exponent).

Problem 5

Recall that a polynomial $p(x_1, \dots, x_n)$ is *multilinear* if in each monomial no variable has degree greater than 1.

(a) (10 points) Let $f : \{0,1\}^n \to \mathbf{R}$ be any real-valued function on $\{0,1\}^n$. Show that there is a unique multilinear polynomial $p(x)$ such that $p(x) = f(x)$ for all $x \in \{0,1\}^n$.

(b) (10 points) Consider the following computational problem:

BOOLEAN-POLYNOMIAL
Input: A polynomial $p(x_1, \ldots, x_n)$ (you may assume all coefficients are given as integers and that the polynomial is given in standard form)
Output: Does p compute a Boolean function over $\{0,1\}^n$? (In other words, does there exist any input $z \in \{0,1\}^n$ such that $p(z) \notin \{0,1\}$?)

Show that BOOLEAN-POLYNOMIAL is in P.