Chapter 5

Partitions and Permutations

The two *Stirling recursions*, one used to count partitions and the other to count permutations, are both quite similar to Pascal's recursion for combination coefficients. A secondary topic of this chapter is partially ordered sets, known familiarly as *posets*, some of which have sufficient structure to be what are called *lattices*.

5.1 STIRLING SUBSET NUMBERS

Stirling subset numbers count the number of ways that a set can be partitioned.

REVIEW FROM §1.6:

- A ***partition*** of a set S is a family of mutually disjoint non-empty subsets whose union is S.
- The ***Stirling subset number*** $\left\{ n \atop k \right\}$ is the number of ways to partition a set of n distinct objects into k non-empty non-distinct cells.

REVIEW FROM §3.6:

- **Theorem 3.6.4.** Let n and k be a pair of non-negative integers. Then

$$\left\{ n \atop k \right\} k! \;=\; \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$$

Our immediate concern is careful attention to three properties within the definition of a partition: non-distinctness of the cells, non-emptiness of the cells, and distinctness of the objects of the set.

Non-Distinctness of Cells of a Partition

Non-distinctness of the cells means that they are regarded as a set, not as a list. If a given partition has k labeled cells, there are $k!$ ways to list them.

Example 5.1.1: We consider an *ad hoc* calculation of the Stirling number

$$\left\{ {4 \atop 2} \right\}$$

We observe that the set $\{a, b, c, d\}$ can be partitioned into two subsets of two objects each in three ways:

$$1. \{a, b\}, \{c, d\} \qquad 2. \{a, c\}, \{b, d\} \qquad 3. \{a, d\}, \{b, c\}$$

Changing the order in which the subsets are listed in a representation of a *partition* does not change the partition. Thus, the partition

$$1\prime. \{c, d\}, \{a, b\}$$

is identical to partition (1) above. By way of contrast, if the objects were to be distributed into compartments distinguished by pre-assigned names or, equivalently, by their order in the listing, each of the partitions above would correspond to two such distributions, for a total of six ways to distribute four objects into two distinct compartments with a 2-2 distribution.

In addition to partitions (1), (2), and (3), given above, of four objects into two parts of two objects each, the Stirling subset number $\left\{ {4 \atop 2} \right\}$ also counts the partitions of four objects into subsets of sizes one and three, i.e., these four partitions:

$$4. \{a\}, \{b, c, d\} \qquad 5. \{b\}, \{a, c, d\}$$
$$6. \{c\}, \{a, b, d\} \qquad 7. \{d\}, \{a, b, c\}$$

The two compartments within each of these four partitions could be ordered in two ways, if they were distinct. This would give a total of eight distributions into distinct compartments with a 1-3 (or 3-1) distribution.

It follows that, altogether, there are

$$\left\{ {4 \atop 2} \right\} \;=\; 3+4 \;=\; 7$$

distributions into non-distinct calls, and

$$\left\{ {4 \atop 2} \right\} 2! \;=\; 6+8 \;=\; 14$$

distributions into non-distinct calls. These two results are consistent with an application of Theorem 3.6.4.

$$\begin{aligned}
\left\{ {4 \atop 2} \right\} 2! \;&=\; \sum_{j=0}^{2} (-1)^j \binom{2}{j} (2-j)^4 \\
&=\; (-1)^0 \binom{2}{0} (2-0)^4 \;+\; (-1)^1 \binom{2}{1} (2-1)^4 \\
&\qquad\qquad +\; (-1)^2 \binom{2}{2} (2-2)^4 \\
&=\; 1 \cdot 1 \cdot 2^4 \;+\; (-1) \cdot 2 \cdot 1^4 \;+\; 1 \cdot 1 \cdot 0^4 \\
&=\; 16 \;-\; 2 \;+\; 0 \;=\; 14
\end{aligned}$$

The following proposition summarizes this part of the discussion.

Prop 5.1.1. *The number of ways to distribute n distinct objects into k distinct boxes with none left empty is*

$$k!\left\{ {n \atop k} \right\}$$

Proof: After partitioning the objects into

$$\left\{ {n \atop k} \right\}$$

non-distinct non-empty cells, we can assign k distinct labels to the k cells in $k!$ ways. ♢

Every Cell of a Partition is Non-Empty

Specifying *non-emptiness of the cells* of a partition into k cells is consistent with the everyday notion of dividing a set into parts. (For instance, when Julius Caesar wrote in *The Gallic Wars* that all Gaul is divided into three parts, he meant non-empty parts.)

Example 5.1.1, cont.: If one of the two cells of a distribution of the set $\{a, b, c, d\}$ could be left empty, there would be a total of 8 ways to separate the four objects into two parts, which would include the distribution

$$\{a, b, c, d\}\,\{\,\}$$

If the cells were also distinct, there would be twice as many, for a total of 16 ways. Such a distribution is achievable by assigning one of the two compartment names to each of the four objects, for which, of course, there are $2^4 = 16$ ways.

Proposition 5.1.2. *The number of ways to distribute n distinct objects into k distinct boxes with some possibly left empty is*

$$k^n \qquad \diamondsuit$$

Distinctness of Objects

Distinctness of the objects is a critical feature, since two distributions of a multiset of indistinguishable objects would differ only in the numbers of objects in the cells.

Example 5.1.1, cont.: The only two possible partitions of four indistinguishable objects into two non-empty non-distinct cells have the following forms:

$$\{a\}\,\{a,\, a,\, a\} \quad \text{and} \quad \{a,\, a\}\,\{a,\, a\}$$

They are equivalent to the integer partitions

$$4 \;=\; 1 + 3 \quad \text{and} \quad 4 \;=\; 2 + 2$$

In general, partitioning n indistinguishable objects into k indistinguishable cells is equivalent to partitioning the integer n into a sum of k parts, a topic that is explored further in §9.4.

On the other hand, if the two cells are distinct, then the distribution of four non-distinct objects amounts to choosing four cells from a set of two distinct cells, with repetitions allowed. We developed a counting formula for combinations with repetitions in Chapter 0.

Prop 5.1.3. *The number of ways to distribute n non-distinct objects into k distinct boxes with some possibly left empty is*

$$\binom{k+n-1}{n}$$

Proof: This is equivalent to choosing n objects from a set of k with repetitions allowed. The formula was derived in conjunction with Corollary 0.4.5. $\diamondsuit$

The Type of a Partition

Clearly, the sum of the sizes of the cells of a partition of a set of n objects must be equal to n.

DEF: An arrangement of the sizes of the cells into non-increasing order is called the ***type of a partition***.

Example 5.1.1, cont.: The partitions

1. $\{a, b\}, \{c, d\}$ 2. $\{a, c\}, \{b, d\}$ 3. $\{a, d\}, \{b, c\}$

are of type 22, and the partitions

4. $\{a\}, \{b, c, d\}$ 5. $\{b\}, \{a, c, d\}$ 6. $\{c\}, \{a, b, d\}$ 7. $\{d\}, \{a, b, c\}$

are of type 31.

Stirling's Subset Number Recurrence

A recurrence similar to Pascal's recurrence provides a systematic means to calculate a Stirling subset number $\left\{ {n \atop k} \right\}$, without resorting to separate counts for each partition type. Since there is no simple closed formula for a Stirling number, unlike the situation for a binomial coefficient, there is no simple algebraic proof, and we resort to a combinatorial proof.

Prop 5.1.4 [*Stirling subset-# recurrence*]. *Stirling subset numbers satisfy the following recurrence:*

$$\left\{ {0 \atop k} \right\} = (k = 0) \qquad \left\{ {n \atop 0} \right\} = (n = 0)$$

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} \quad \text{for } n \geq 1$$

Combinatorial Proof: The initial conditions are clear.

The recursion is verified by splitting the partitions of the integer interval $[1 : n]$ into two kinds, as per the Method of Distinguished Element, which was used with Pascal's recursion in §1.3. The first kind contains every partition in which the integer n gets a cell to itself. Since the other $n-1$ integers must then be partitioned into $k-1$ non-empty cells, there are

$$\left\{ {n-1 \atop k-1} \right\}$$

cases of the first kind.

In the second kind, the set $[1 : n-1]$ is partitioned into k non-empty cells, and then one of those k cells is selected as the cell for the integer n. There are

$$k \left\{ {n-1 \atop k} \right\}$$

cases of the second kind. The sum of the numbers of cases in these two kinds is the total number of partitions. $\diamondsuit$

Stirling's Triangle for Subset Numbers

Stirling subset #s have a △ similar to Pascal's △.

Table 5.1.1 Stirling's triangle for values of $\left\{ {n \atop k} \right\}$.

n	$\left\{ {n \atop 0} \right\}$	$\left\{ {n \atop 1} \right\}$	$\left\{ {n \atop 2} \right\}$	$\left\{ {n \atop 3} \right\}$	$\left\{ {n \atop 4} \right\}$	$\left\{ {n \atop 5} \right\}$	$\left\{ {n \atop 6} \right\}$	B_n
0	1							1
1	0	1						1
2	0	1	1					2
3	0	1	3	1				5
4	0	1	7	6	1			15
5	0	1	15	25	10	1		52
6	0	1	31	90	65	15	1	203

The rest of this section is devoted to the development of formulas for Stirling's subset triangle that are analogous to the formulas of §4.1 for Pascal's triangle.

Rows Are Log-Concave

REVIEW FROM §1.5:

- A sequence $\langle x_n \rangle$ is a *log-concave sequence* if

$$x_{n-1}\, x_{n+1} \le x_n^2, \quad \text{for all } n \ge 1$$

- A log-concave sequence is unimodal.

Example 5.1.2: Figure 5.1.1 illustrates the unimodality of row 6. In fact, every row of Stirling's triangle for subset numbers (see Table 5.1.1) is unimodal.

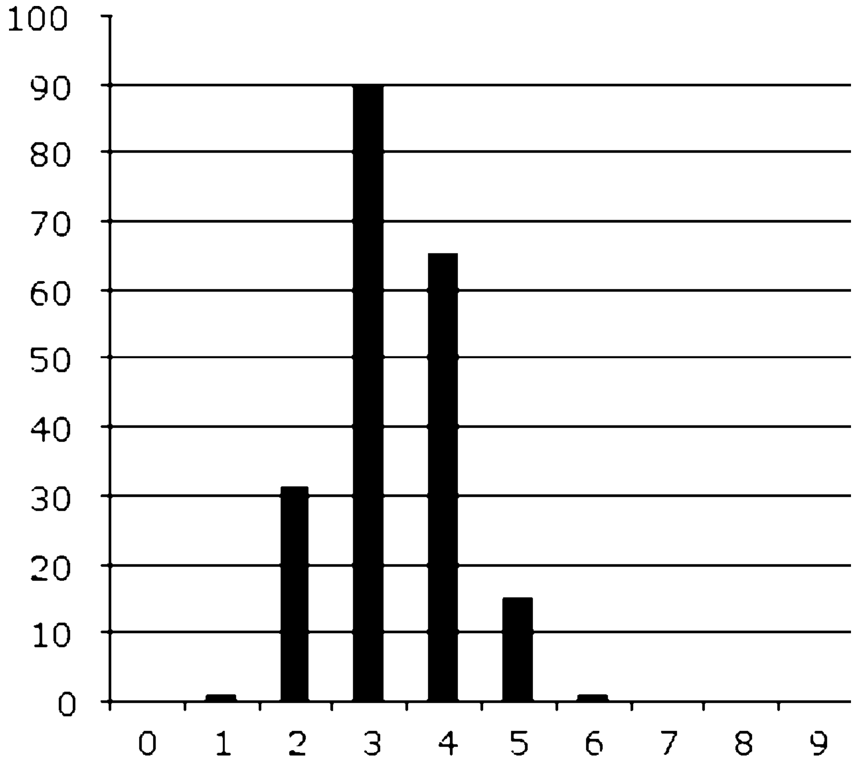

Fig 5.1.1 Graph of the values of $\left\{ {6 \atop k} \right\}$.

Lemma 5.1.5. *Let $\langle x_n \rangle$ be a log-concave sequence. Then*

$$x_{n-2}\, x_{n+1} \;\leq\; x_{n-1}\, x_n$$

Proof: The log-concavity inequality is applied twice.

$$\begin{aligned}
x_{n-2}\, x_{n+1} &\leq x_{n-2} \cdot \frac{{x_n}^2}{x_{n-1}} \quad \text{since } x_{n-1}\, x_{n+1} \leq x_n^2 \\
&= \frac{x_{n-2}}{x_{n-1}} \cdot x_n^2 \\
&\leq \frac{x_{n-1}}{x_n} \cdot x_n^2 \quad \text{since } x_{n-2}\, x_n \leq x_{n-1}^2 \\
&= x_{n-1}\, x_n
\end{aligned}$$

◇

Prop 5.1.6. *For all $n \geq 0$, the sequence of Stirling subset numbers*

$$\left\{ {n \atop 0} \right\},\ \left\{ {n \atop 1} \right\},\ \ldots\ ,\ \left\{ {n \atop n} \right\}$$

is log-concave. That is,

$$\left\{ {n \atop k-1} \right\} \left\{ {n \atop k+1} \right\} \;\leq\; \left\{ {n \atop k} \right\} \left\{ {n \atop k} \right\}$$

Proof: This is an algebraic proof by induction on n.

BASIS: Rows 0 and 1 are surely log-concave.

IND HYP: Assume that row $n-1$ is log-concave.

IND STEP: Under Stirling's subset recurrence, the product

$$\left\{ {n \atop k-1} \right\} \left\{ {n \atop k+1} \right\}$$

has the expansion

$$\left(\left\{ n-1 \atop k-2 \right\} + (k-1)\left\{ n-1 \atop k-1 \right\}\right) \times \left(\left\{ n-1 \atop k \right\} + (k+1)\left\{ n-1 \atop k+1 \right\}\right)$$

$$= \left\{ n-1 \atop k-2 \right\}\left\{ n-1 \atop k \right\} + (k^2-1)\left\{ n-1 \atop k-1 \right\}\left\{ n-1 \atop k+1 \right\}$$

$$+ (k-1)\left\{ n-1 \atop k-1 \right\}\left\{ n-1 \atop k \right\} + (k+1)\left\{ n-1 \atop k-2 \right\}\left\{ n-1 \atop k+1 \right\}$$

to which log-concavity and Lemma 5.1.5 are applied, under the induction hypothesis

$$\leq \left\{ n-1 \atop k-1 \right\}^2 + (k^2-1)\left\{ n-1 \atop k \right\}^2$$

$$+ (k-1)\left\{ n-1 \atop k-1 \right\}\left\{ n-1 \atop k \right\} + (k+1)\left\{ n-1 \atop k-1 \right\}\left\{ n-1 \atop k \right\}$$

$$\leq \left\{ n-1 \atop k-1 \right\}^2 + k^2\left\{ n-1 \atop k \right\}^2 + 2k\left\{ n-1 \atop k-1 \right\}\left\{ n-1 \atop k \right\}$$

$$= \left(\left\{ n-1 \atop k-1 \right\} + k\left\{ n-1 \atop k \right\}\right)^2$$

$$= \left\{ n \atop k \right\}^2 \qquad \diamondsuit$$

Bell Numbers

DEF: The ***Bell number*** B_n is the number of partitions of a set of n distinct objects.

Thus, the n^{th} Bell number is the sum

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

of row n of Stirling's triangle for subset numbers.

Theorem 5.1.7. *The Bell numbers satisfy the recurrence*

$$\begin{aligned} B_0 &= 1 \\ B_n &= \sum_{k=0}^{n-1} \binom{n-1}{k} B_k \quad \text{for } n \geq 1 \end{aligned}$$

Proof: This proof has combinatorial steps and algebraic steps. The initial condition

$$B_0 = \left\{ {n \atop 0} \right\} = 1$$

is clearly satisfied.

For $n \geq 1$, consider the case in which there are k other objects in the cell of a partition of $[1 : n]$ that contains the number n. There are

$$\binom{n-1}{k}$$

ways to select these k numbers and then B_{n-k-1} ways to partition the remaining $n-k-1$ numbers. Thus, the total number of partitions of n objects is

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-k-1}$$

which is transformable, by symmetry of binomial coefficients, to

$$= \sum_{k=0}^{n-1} \binom{n-1}{n-k-1} B_{n-k-1}$$

Reversing the order of summation yields the conclusion

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k \qquad \diamondsuit$$

Example 5.1.3: Table 5.1.1 provides the Bell numbers

B_0	B_1	B_2	B_3	B_4	B_5	B_6
1	1	2	5	15	52	203

We observe, for instance, that

$$\binom{3}{0}B_0 + \binom{3}{1}B_1 + \binom{3}{2}B_2 + \binom{3}{3}B_3$$
$$= 1 \cdot 1 + 3 \cdot 1 + 3 \cdot 2 + 1 \cdot 5$$
$$= 15 = B_4$$

Column-Sum Formulas

There are two readily accessible summation formulas for column c of the triangle for Stirling subset numbers. They both assert that a weighted partial sum of the entries in column c can be found in column $c+1$. In the two formulas, the weightings differ.

Prop 5.1.8. *Let n and c be non-negative integers. Then*

$$\left\{ {n+1 \atop c+1} \right\} = \sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop c} \right\}$$

Proof: In partitioning the $n+1$ numbers of the integer interval $[1 : n+1]$ into $c+1$ cells, there are

$$\binom{n}{k}$$

ways to select $n-k$ other numbers to be in the same cell as the number $n+1$ and then

$$\left\{ {k \atop c} \right\}$$

ways to partition the remaining k numbers into c additional cells. ♢

Example 5.1.4: In column $c = 1$ of the triangle for Stirling subset numbers, all the non-zero entries are 1's. Thus, Proposition 5.1.8 takes the form

$$\left\{ {n+1 \atop 2} \right\} = \sum_{k=1}^{n} \binom{n}{k} \left\{ {k \atop 1} \right\} = \sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$$

Example 5.1.5: In column $c = 2$ of the Stirling triangle for subset numbers, there appear the consecutive entries

$$\left\{ {2 \atop 2} \right\} = 1 \qquad \left\{ {3 \atop 2} \right\} = 3 \qquad \left\{ {4 \atop 2} \right\} = 7$$

In row 4 of Pascal's triangle, there are the consecutive entries

$$\binom{4}{2} = 6 \qquad \binom{4}{3} = 4 \qquad \binom{4}{4} = 1$$

Proposition 5.1.8 asserts for this case that

$$\begin{aligned} \left\{ {5 \atop 3} \right\} &= \sum_{k=2}^{4} \binom{4}{k} \left\{ {k \atop 2} \right\} \\ &= \binom{4}{2}\left\{ {2 \atop 2} \right\} + \binom{4}{3}\left\{ {3 \atop 2} \right\} + \binom{4}{4}\left\{ {4 \atop 2} \right\} \\ &= \ 6 \cdot 1 \ + \ 4 \cdot 3 \ + \ 1 \cdot 7 \ = \ 25 \end{aligned}$$

The sum in Proposition 5.1.8 can be visualized as a dot product of a row of Pascal's triangle with a column of Stirling's triangle.

$r \rightarrow$	2	3	4
$\binom{4}{r}$	6	4	1

$$6 \cdot 1 + 4 \cdot 3 + 1 \cdot 7 = 25$$

$n \downarrow$	$\left\{ {n \atop 2} \right\}$	$\left\{ {n \atop 3} \right\}$
2	1	
3	3	
4	7	
5		25

Prop 5.1.9. *Let n and c be non-negative integers. Then*

$$\left\{ {n+1 \atop c+1} \right\} = \sum_{k=0}^{n} (c+1)^{n-k} \left\{ {k \atop c} \right\}$$

Proof: By induction.

BASIS: The equation is clearly true when $n = 0$.

IND HYP: Assume, for inductive purpose, that

$$\left\{ {n \atop c+1} \right\} = \sum_{k=0}^{n-1} (c+1)^{n-k-1} \left\{ {k \atop c} \right\}$$

IND STEP: Then

$$\begin{aligned}
\left\{ {n+1 \atop c+1} \right\} &= \left\{ {n \atop c} \right\} + (c+1) \left\{ {n \atop c+1} \right\} && \text{(Stirling recursion)} \\
&= \left\{ {n \atop c} \right\} + (c+1) \sum_{k=0}^{n-1} (c+1)^{n-k-1} \left\{ {k \atop c} \right\} && \text{(ind hyp)} \\
&= \left\{ {n \atop c} \right\} + \sum_{k=0}^{n-1} (c+1)^{n-k} \left\{ {k \atop c} \right\} \\
&= \sum_{k=0}^{n} (c+1)^{n-k} \left\{ {k \atop c} \right\} && \diamondsuit
\end{aligned}$$

Example 5.1.6: Proposition 5.1.9 implies that

$$\begin{aligned}
\left\{ {5 \atop 3} \right\} &= \sum_{k=2}^{4} 3^{4-k} \left\{ {k \atop 2} \right\} \\
&= 3^{4-2} \left\{ {2 \atop 2} \right\} + 3^{4-3} \left\{ {3 \atop 2} \right\} + 3^{4-4} \left\{ {4 \atop 2} \right\} \\
&= 3^2 \cdot 1 + 3^1 \cdot 3 + 3^0 \cdot 7 \;=\; 25
\end{aligned}$$

Southeast Diagonal Sum

Along a southeast diagonal from column 0 to column c, multiply each entry by its column number and take the sum. This equals the number immediately below the last entry in that diagonal.

Proposition 5.1.10. *Let n and c be non-negative integers. Then*

$$\left\{ {n+c+1 \atop c} \right\} = \sum_{k=0}^{c} k \left\{ {n+k \atop k} \right\}$$

Proof: Again by induction.

BASIS: The equation is clearly true for all $n \geq 0$ when $c = 0$.

IND HYP: Assume for all $n \geq 0$ that

$$\left\{ {n+c \atop c-1} \right\} = \sum_{k=0}^{c-1} k \left\{ {n+k \atop k} \right\}$$

IND STEP: Then

$$\left\{ {n+c+1 \atop c} \right\} = \left\{ {n+c \atop c-1} \right\} + c \left\{ {n+c \atop c} \right\} \qquad \text{(Stirling's recursion)}$$

$$= \sum_{k=0}^{c-1} k \left\{ {n+k \atop k} \right\} + c \left\{ {n+c \atop c} \right\} \qquad \text{(ind hyp)}$$

$$= \sum_{k=0}^{c} k \left\{ {n+k \atop k} \right\} \qquad \diamondsuit$$

Example 5.1.7: The sum in Proposition 5.1.10 can be visualized as a dot product of a southeast diagonal of Stirling's triangle with a vector of column numbers.

$n \downarrow$	$\left\{ {n \atop 1} \right\}$	$\left\{ {n \atop 2} \right\}$	$\left\{ {n \atop 3} \right\}$
3	1		
4		7	
5			25
6			$\boxed{90}$

$$1 \cdot 1 + 2 \cdot 7 + 3 \cdot 25 \;=\; 90$$

Stirling Numbers of the Second Kind

REVIEW FROM §1.6: The *Stirling numbers of the second kind* were defined as the coefficients $S_{n,k}$ in the sum

$$x^n \;=\; \sum_{k=0}^{n} S_{n,k} x^{\underline{k}}$$

Prop 5.1.11. *For all non-negative integers n and k,*

$$S_{n,k} \;=\; \left\{ {n \atop k} \right\} \tag{5.1.1}$$

Proof: We use the Stirling subset-number recurrence, as verified in Proposition 5.1.4.

$$\left\{ {0 \atop k} \right\} \;=\; (k=0) \qquad\qquad \left\{ {n \atop 0} \right\} \;=\; (n=0)$$

$$\left\{ {n \atop k} \right\} \;=\; \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} \qquad \text{for } n \geq 1$$

It is sufficient to show that the Stirling numbers of the second kind satisfy the same recurrence. The initial conditions

$$S_{0,k} \;=\; (k=0) \qquad \text{and} \qquad S_{n,0} \;=\; (n=0)$$

hold, because

$$x^0 = 1x^{\underline{0}}$$

and because the constant term of the expansion

$$x^n \;=\; \sum_{k=0}^{n} S_{n,k} x^{\underline{k}} \tag{5.1.2}$$

is 0, unless $k = 0$.

The Stirling numbers $S_{n-1,k}$ of the second kind are defined with the specification

$$x^{n-1} \;=\; \sum_{k=0}^{n-1} S_{n-1,k} x^{\underline{k}} \tag{5.1.3}$$

Accordingly,

$$\begin{aligned} x^n &= x \cdot x^{n-1} \\ &= x \cdot \sum_{k=0}^{n-1} S_{n-1,k}\, x^{\underline{k}} \qquad \text{(by (5.1.2))} \\ &= \sum_{k=0}^{n-1} S_{n-1,k}\, x^{\underline{k}} \cdot x \\ &= \sum_{k=0}^{n-1} S_{n-1,k}\, x^{\underline{k}} \cdot (x-k) + \sum_{k=0}^{n-1} S_{n-1,k}\, x^{\underline{k}} \cdot k \\ &= \sum_{k=0}^{n-1} S_{n-1,k}\, x^{\underline{k+1}} + \sum_{k=0}^{n-1} k S_{n-1,k}\, x^{\underline{k}} \\ &= \sum_{k=1}^{n} S_{n-1,k-1}\, x^{\underline{k}} + \sum_{k=0}^{n-1} k S_{n-1,k}\, x^{\underline{k}} \\ &= \sum_{k=0}^{n} (S_{n-1,k-1} + k S_{n-1,k})\, x^{\underline{k}} \end{aligned} \tag{5.1.4}$$

Since $x^{\underline{k}}$ must have the same coefficient in the two expansions (5.1.2) and (5.1.4) of x^n, it follows that

$$S_{n,k} = S_{n-1,k-1} + k S_{n-1,k}$$

Thus, the Stirling numbers of the second kind have the same recurrence as the Stirling subset numbers, which implies that they have the same values. ♢

Table 5.1.2 Basic Formulas for Stirling Subset #s

Stirling's recurrence:

$$\left\{ {0 \atop k} \right\} = (k = 0) \qquad \left\{ {n \atop 0} \right\} = (n = 0)$$

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} \quad \text{for } n \geq 1 \tag{5.1.5}$$

Special values:

$$\left\{ {n \atop 1} \right\} = (n > 0) \quad \left\{ {n \atop 2} \right\} = 2^{n-1} - 1 \text{ (for } n \geq 1) \quad \left\{ {n \atop n} \right\} = 1$$

Converting ordinary powers to falling powers:

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} \tag{5.1.6}$$

Using binomial coefficients to calculate Stirling subset #s:

$$\left\{ {n \atop k} \right\} k! = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (m-j)^n = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \tag{5.1.7}$$

Bell numbers:

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \tag{5.1.8}$$

$$B_0 = 1; \qquad B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k \quad \text{for } n \geq 1 \tag{5.1.9}$$

Column-sum formulas:

$$\left\{ {n+1 \atop c+1} \right\} = \sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop c} \right\} \tag{5.1.10}$$

$$\left\{ {n+1 \atop c+1} \right\} = \sum_{k=0}^{n} (c+1)^{n-k} \left\{ {k \atop c} \right\} \tag{5.1.11}$$

SE diagonal-sum formula:

$$\left\{ {n+c+1 \atop c} \right\} = \sum_{k=0}^{c} k \left\{ {n+k \atop k} \right\} \tag{5.1.12}$$

Example 5.1.8: The following recursive calculation of the Stirling number $S_{n,k}$ of the 2^{nd} kind illustrates how these numbers conform to Stirling's subset recursion.

$$\begin{aligned}
x &= x^{\underline{1}} \\
\Rightarrow x^2 &= x \cdot x^{\underline{1}} = x^{\underline{1}}(x-1) + x^{\underline{1}} = x^{\underline{2}} + x^{\underline{1}} \\
\Rightarrow x^3 &= x \cdot x^{\underline{2}} + x \cdot x^{\underline{1}} = x^{\underline{2}}(x-2) + 2x^{\underline{2}} + x^{\underline{2}} + x^{\underline{1}} \\
&= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \\
\Rightarrow x^4 &= x \cdot x^{\underline{3}} + 3x \cdot x^{\underline{2}} + x \cdot x^{\underline{1}} \\
&= [x^{\underline{3}}(x-3) + 3x^{\underline{3}}] + 3\,[x^{\underline{2}}(x-2) + 2x^{\underline{2}}] \\
&\qquad + [x^{\underline{1}}(x-1) + x^{\underline{1}}] \\
&= [x^{\underline{4}} + 3x^{\underline{3}}] + 3\,[x^{\underline{3}} + 2x^{\underline{2}}] + [x^{\underline{2}} + x^{\underline{1}}] \\
&= x^{\underline{4}} + [3x^{\underline{3}} + 3x^{\underline{3}}] + [6x^{\underline{2}} + x^{\underline{2}}] + x^{\underline{1}} \\
&= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}}
\end{aligned}$$

5.2 STIRLING CYCLE NUMBERS

Stirling cycle numbers count the possible partitions of a set into cycles, in effect, the number of the permutations of the set. They satisfy a recurrence similar to Pascal's recurrence, and their non-zero entries form a triangle.

REVIEW FROM §1.6:

- The ***Stirling cycle number*** $\left[{n \atop k}\right]$ is the number of ways to partition n distinct objects into k non-empty non-distinct cycles.
- Since every permutation of a set of n objects can be represented as a composition of disjoint cycles, it follows that the ***Stirling cycle number*** $\left[{n \atop k}\right]$ is the number of permutations with exactly k cycles.

In general,

$$\left[{n \atop k}\right] \geq \left\{{n \atop k}\right\}$$

since the number of ways to form a cycle from s objects already in a cell of a partition is $(s-1)!$. Thus, to calculate $\left[{n \atop k}\right]$, one could multiply the number of partitions of a given partition type $t_1\, t_2\, \cdots\, t_r$ by

$$(t_1 - 1)!(t_2 - 1)! \cdots (t_r - 1)!$$

and sum over all such partitions.

Example 5.2.1: The set $\{a,\ b,\ c,\ d\}$ can be partitioned into two cycles in 11 ways, which correspond to the 11 permutations of the set $\{1,2,3,4\}$ with two cycles:

$$\begin{array}{llll} (a)(b\,c\,d) & (a)(b\,d\,c) & (b)(a\,c\,d) & (b)(a\,d\,c) \\ (c)(a\,b\,d) & (c)(a\,d\,b) & (d)(a\,b\,c) & (d)(a\,c\,b) \\ (a\,b)(c\,d) & (a\,c)(b\,d) & (a\,d)(b\,c) & \end{array} \qquad \left[\begin{matrix}4\\2\end{matrix}\right] = 11$$

We observe that each of the four partitions of type 31 into cells corresponds to $(3-1)!(1-1)! = 2$ partitions into cycles. For instance, the partition

$$\{a\}\,\{b,\ c,\ d\}$$

corresponds to the two permutations

$$(a)(b\,c\,d) \qquad \text{and} \qquad (a)(b\,d\,c)$$

Each of the three partitions of type 22 into cells yields only $(2-1)!(2-1)! = 1$ partition into cycles. For instance, the partition

$$\{a\,b\}\{c\,d\}$$

yields only the permutation

$$(a\,b)(c\,d)$$

We observe, moreover, that

$$4\cdot 2 + 3\cdot 1 = 11$$

Non-Distinctness of the Cycles

Changing the order in which its cycles are written does not change a permutation.

Example 5.2.2: For instance,

$$(a\,b)(c\,d) \qquad \text{and} \qquad (c\,d)(a\,b)$$

are representations of the same permutation.

Stirling's Cycle Number Recurrence

As with Stirling subset numbers, a recurrence similar to Pascal's recurrence provides a systematic means to calculate a Stirling cycle number $\left[{n \atop k}\right]$, without resorting to separate counts for each partition type. Moreover, here too there is no simple algebraic proof, and we resort again to a combinatorial proof.

Prop 5.2.1 [*Stirling cycle-#recurrence*]. *Stirling cycle numbers satisfy the following recurrence:*

$$\begin{bmatrix} 0 \\ k \end{bmatrix} = (k = 0) \qquad\qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = (n = 0)$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} \quad \text{for } n \geq 1$$

Combinatorial Proof: The initial conditions are clear. As with Stirling subset numbers, the recursion is verified by splitting the permutations of the integer interval $[1:n]$ that have k cycles into two kinds. The first kind contains every permutation in which the number n gets a cycle to itself, and the other $n-1$ numbers are partitioned into $k-1$ non-empty cycles, so there are

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

cases of the first kind. In the second kind, in which the number n does not have a cycle to itself, the other $n-1$ numbers are partitioned into k non-empty cycles, and then the number n is inserted immediately after some number j in one of those k cycles. There are

$$(n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}$$

cases of the second kind, because there are, in total, $n-1$ other numbers after which the number n could be inserted. The sum of the numbers of cases in these two types is the total number of partitions of $[1:n]$ into k cycles. $\diamondsuit$

Stirling's Triangle for Cycle Numbers

There is a triangle for the Stirling cycle numbers, like Pascal's triangle and the triangle for Stirling subset numbers. It appears as Table 5.2.1.

Table 5.2.1 Stirling's triangle for values of $\left[{n \atop k}\right]$.

n	$\left[{n \atop 0}\right]$	$\left[{n \atop 1}\right]$	$\left[{n \atop 2}\right]$	$\left[{n \atop 3}\right]$	$\left[{n \atop 4}\right]$	$\left[{n \atop 5}\right]$	$\left[{n \atop 6}\right]$	Σ
0	1							1
1	0	1						1
2	0	1	1					2
3	0	2	3	1				6
4	0	6	11	6	1			24
5	0	24	50	35	10	1		120
6	0	120	274	225	85	15	1	720

We observe that Column 1 of Stirling's triangle for cycle numbers is the sequence $(n-1)!$.

Proposition 5.2.2. *Let n be a positive integer. Then*

$$\left[{n \atop 1}\right] = (n-1)!$$

Proof: The number of ways to arrange n objects in a cycle with a designated starting point is $n!$. Two cycles may be regarded as equivalent if they differ only in the choice of starting point. There are n possible starting points. Thus, by the Rule of Quotient,

$$\left[{n \atop 1}\right] = (n-1)! \qquad \diamondsuit$$

It is less apparent, but not hard to prove, that Column 2 also has a tractable closed formula.

Proposition 5.2.3. *Let n be a positive integer. Then*

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!\, H_{n-1}$$

Proof: Once again, by induction on n.

BASIS: $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1 = (2-1)!\, H_1.$

IND HYP: Assume for some $n \geq 2$ that

$$\begin{bmatrix} n-1 \\ 2 \end{bmatrix} = (n-2)!\, H_{n-2}$$

IND STEP: Then, by Stirling's recursion,

$$\begin{aligned}
\begin{bmatrix} n \\ 2 \end{bmatrix} &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ 2 \end{bmatrix} \\
&= (n-2)! + (n-1) \begin{bmatrix} n-1 \\ 2 \end{bmatrix} && \text{(Prop 5.2.2)} \\
&= (n-2)! + (n-1)(n-2)!\, H_{n-2} && \text{(ind hyp)} \\
&= \frac{(n-1)!}{n-1} + (n-1)!\, H_{n-2} \\
&= (n-1)! \left(\frac{1}{n-1} + H_{n-2} \right) = (n-1)!\, H_{n-1} \quad \diamondsuit
\end{aligned}$$

Example 5.2.3: The following table helps to illustrate Proposition 5.2.3. In each column, the product of the entries in the row labeled H_{n-1} and the row labeled $(n-1)!$ is the entry in the row labeled $\left[{n \atop 2} \right]$.

n	1	2	3	4	5	$\cdots$
H_{n-1}	0	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\cdots$
$(n-1)!$	1	1	2	6	24	$\cdots$
$\left[{n \atop 2}\right]$	0	1	3	11	50	$\cdots$

Rows are Log-Concave

As with the rows of the Stirling triangle for subset numbers, the rows of the Stirling triangle for cycle numbers are log-concave and, thus, unimodal.

Proposition 5.2.4. *For all $n \geq 0$, the sequence of Stirling cycle numbers*

$$\left[{n \atop 0}\right], \left[{n \atop 1}\right], \ldots, \left[{n \atop n}\right]$$

is log-concave. That is,

$$\left[{n \atop k-1}\right]\left[{n \atop k+1}\right] \leq \left[{n \atop k}\right]\left[{n \atop k}\right]$$

Proof: Rows 0 and 1 are vacuously log-concave. Assume that row $n-1$ is log-concave, and consider row n. Under Stirling's recurrence for cycle numbers, the product

$$\left[{n \atop k-1}\right]\left[{n \atop k+1}\right]$$

has the expansion

$$\left(\begin{bmatrix} n-1 \\ k-2 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right) \times \left(\begin{bmatrix} n-1 \\ k \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k+1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} n-1 \\ k-2 \end{bmatrix}\begin{bmatrix} n-1 \\ k \end{bmatrix} + (n-1)^2\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ k+1 \end{bmatrix}$$

$$+ (n-1)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ k \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}\begin{bmatrix} n-1 \\ k+1 \end{bmatrix}$$

to which log-concavity and Lemma 5.1.5 are applied under the induction hypothesis.

$$\leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^2 + (n-1)^2\begin{bmatrix} n-1 \\ k \end{bmatrix}^2$$

$$+ (n-1)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ k \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ k \end{bmatrix}$$

$$\leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^2 + (n-1)^2\begin{bmatrix} n-1 \\ k \end{bmatrix}^2 + 2(n-1)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\begin{bmatrix} n-1 \\ k \end{bmatrix}$$

$$= \left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}\right)^2 = \begin{bmatrix} n \\ k \end{bmatrix}^2 \qquad \diamondsuit$$

Figure 5.2.1 illustrates the unimodality of row 6.

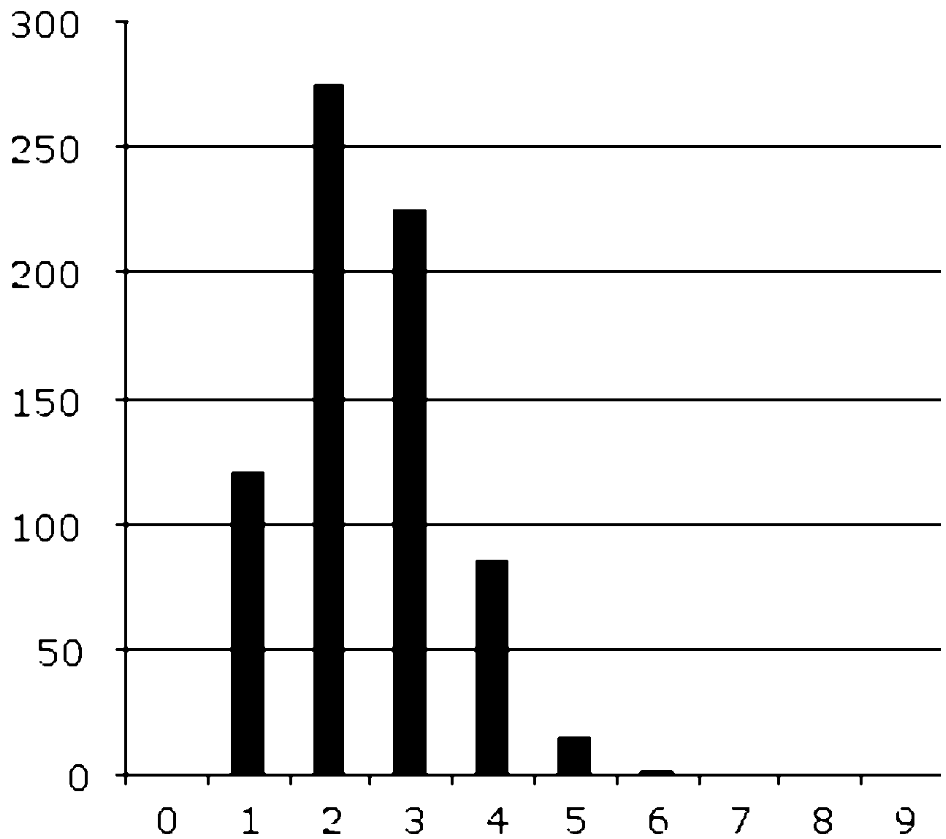

Fig 5.2.1 **Unimodality of the sequence** $\langle \left[{6 \atop k}\right] \mid k = 0, \ldots, 6 \rangle$.

Row Sums

The rows of Stirling's triangle for cycle numbers have several other interesting properties. The following property is apparent in Table 5.2.1.

Proposition 5.2.5. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n} \left[{n \atop k}\right] = n!$$

Proof: The simplest proof is that each row sum of Stirling's triangle for cycle numbers is the total number of permutations of a set of n objects. An alternative proof proceeds inductively on the row number, n. $\diamondsuit$

A subtler property is how each entry in the second column is related to the row immediately above that entry, by a weighted row sum.

Proposition 5.2.6. *Let n be a positive integer. Then*

$$\sum_{j=0}^{n} j \begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}$$

Proof: By induction.

BASIS: If $n = 1$, then

$$0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 + 1 = 1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

IND HYP: For some $n \geq 2$, assume that

$$\sum_{j=0}^{n-1} j \begin{bmatrix} n-1 \\ j \end{bmatrix} = \begin{bmatrix} n \\ 2 \end{bmatrix}$$

IND STEP: Then, by Stirling's recursion,

$$\sum_{j=0}^{n} j \begin{bmatrix} n \\ j \end{bmatrix} = \sum_{j=0}^{n} j \left(\begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ j \end{bmatrix} \right)$$

$$= \sum_{j=0}^{n} j \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (n-1) \sum_{j=0}^{n} j \begin{bmatrix} n-1 \\ j \end{bmatrix}$$

Now split the first sum.

$$= \sum_{j=0}^{n} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + \sum_{j=0}^{n} (j-1) \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (n-1) \sum_{j=0}^{n} j \begin{bmatrix} n-1 \\ j \end{bmatrix}$$

Apply Proposition 5.2.5 to the first sum.

$$= (n-1)! + \sum_{j=0}^{n} (j-1) \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (n-1) \sum_{j=0}^{n} j \begin{bmatrix} n-1 \\ j \end{bmatrix}$$

Next apply the ind hyp to the other two sums.

$$= (n-1)! + \begin{bmatrix} n \\ 2 \end{bmatrix} + (n-1) \begin{bmatrix} n \\ 2 \end{bmatrix}$$

$$= (n-1)! + n \begin{bmatrix} n \\ 2 \end{bmatrix}$$

Then apply Proposition 5.2.2

$$= \begin{bmatrix} n \\ 1 \end{bmatrix} + n \begin{bmatrix} n \\ 2 \end{bmatrix}$$

and conclude by applying Stirling's recursion.

$$= \begin{bmatrix} n+1 \\ 2 \end{bmatrix} \qquad \diamondsuit$$

Example 5.2.4: With data from Table 5.2.1, we now illustrate Proposition 5.2.6.

$$1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 1 \cdot 6 + 2 \cdot 11 + 3 \cdot 6 + 4 \cdot 1$$

$$= 50$$
$$= \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Proposition 5.2.6 has Theorem 5.2.7 as a fascinating consequence.

Theorem 5.2.7. *The average number of cycles in a random permutation of n objects is H_n.*

Proof: Let the random variable X be the number of cycles in a permutation on n objects. Then

$$\Pr(k \text{ cycles}) = \frac{1}{n!} \begin{bmatrix} n \\ k \end{bmatrix}$$

Therefore, the expected number of cycles is

$$\begin{aligned}
\mu &= \sum_{k=0}^{n} k \cdot \frac{1}{n!} \begin{bmatrix} n \\ k \end{bmatrix} \\
&= \frac{1}{n!} \sum_{k=0}^{n} k \cdot \begin{bmatrix} n \\ k \end{bmatrix} \\
&= \frac{1}{n!} \cdot \begin{bmatrix} n+1 \\ 2 \end{bmatrix} && \text{(Proposition 5.2.6)} \\
&= \frac{1}{n!} \cdot n!\, H_n && \text{(Proposition 5.2.3)} \\
&= H_n
\end{aligned}$$

◊

Proposition 5.2.8 concerns a generalization of Proposition 5.2.6. It asserts that every entry, not just the entries in column 2, is a weighted sum of the elements of the row just above.

Proposition 5.2.8. *Let n and c be non-negative integers. Then*

$$\sum_{j=0}^{n} \binom{j}{c} \begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n+1 \\ c+1 \end{bmatrix}$$

Proof: For $c > n$, both sides of the equation are 0. Thus, in what follows, it is assumed that $c \leq n$.

BASIS: If $n = 0$, then for $c = 0$,

$$\binom{0}{0} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \cdot 1 = 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

IND HYP: For some $n \geq 1$, assume for all k that

$$\sum_{j=0}^{n-1} \binom{j}{k} \begin{bmatrix} n-1 \\ j \end{bmatrix} = \begin{bmatrix} n \\ c \end{bmatrix}$$

IND STEP: Then for any $c \leq n$, Stirling's recursion implies

$$\sum_{j=0}^{n} \binom{j}{c} \begin{bmatrix} n \\ j \end{bmatrix} = \sum_{j=0}^{n} \binom{j}{c} \left(\begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ j \end{bmatrix} \right)$$

which splits like this:

$$= \sum_{j=0}^{n} \binom{j}{c} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (n-1) \sum_{j=0}^{n} \binom{j}{c} \begin{bmatrix} n-1 \\ j \end{bmatrix}$$

which reduces, by the induction hypothesis, to

$$= \sum_{j=0}^{n} \binom{j}{c} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (n-1) \begin{bmatrix} n \\ c+1 \end{bmatrix}$$

Applying Pascal's recursion, we continue

$$= \sum_{j=0}^{n} \left(\binom{j-1}{c-1} + \binom{j-1}{c} \right) \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (n-1) \begin{bmatrix} n \\ c+1 \end{bmatrix}$$

$$= \sum_{j=0}^{n} \binom{j-1}{c-1} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + \sum_{j=0}^{n} \binom{j-1}{c} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + (n-1) \begin{bmatrix} n \\ c+1 \end{bmatrix}$$

which reduces, by the induction hypothesis, to

$$= \begin{bmatrix} n \\ c \end{bmatrix} + \begin{bmatrix} n \\ c+1 \end{bmatrix} + (n-1) \begin{bmatrix} n \\ c+1 \end{bmatrix}$$

$$= \begin{bmatrix} n \\ c \end{bmatrix} + n \begin{bmatrix} n \\ c+1 \end{bmatrix}$$

and we finish, by applying Stirling's recursion.

$$= \begin{bmatrix} n+1 \\ c+1 \end{bmatrix} \qquad \diamondsuit$$

Example 5.2.5: Some data from Table 5.2.1 helps us to illustrate Proposition 5.2.8.

$$\begin{aligned}\binom{1}{2}\begin{bmatrix}4\\1\end{bmatrix}+\binom{2}{2}\begin{bmatrix}4\\2\end{bmatrix}+\binom{3}{2}\begin{bmatrix}4\\3\end{bmatrix}+\binom{4}{2}\begin{bmatrix}4\\4\end{bmatrix}& \\ &= 0\cdot 6 + 1\cdot 11 + 3\cdot 6 + 6\cdot 1 \\ &= 35 \\ &= \begin{bmatrix}5\\3\end{bmatrix}\end{aligned}$$

Columns

Proposition 5.2.9 asserts that a weighted partial sum of the entries in column c can be found in column $c+1$. It is analogous to Proposition 5.1.9 for Stirling subset numbers.

Proposition 5.2.9. *Let n and c be non-negative integers. Then*

$$\begin{bmatrix}n+1\\c+1\end{bmatrix} = \sum_{k=0}^{n} n^{\underline{n-k}}\begin{bmatrix}k\\c\end{bmatrix}$$

Proof: The equation is clearly true when $n = 0$. Assume, for inductive purpose, that it is true for $n-1$. After starting with Stirling's recursion,

$$\begin{bmatrix}n+1\\c+1\end{bmatrix} = \begin{bmatrix}n\\c\end{bmatrix} + n\begin{bmatrix}n\\c+1\end{bmatrix}$$

we apply the inductive hypothesis.

$$= \begin{bmatrix} n \\ c \end{bmatrix} + n \sum_{k=0}^{n-1} (n-1)^{\underline{n-k-1}} \begin{bmatrix} k \\ c \end{bmatrix}$$

$$= \begin{bmatrix} n \\ c \end{bmatrix} + \sum_{k=0}^{n-1} n^{\underline{n-k}} \begin{bmatrix} k \\ c \end{bmatrix}$$

$$= \sum_{k=0}^{n} n^{\underline{n-k}} \begin{bmatrix} k \\ c \end{bmatrix} \qquad \diamondsuit$$

The sum in Proposition 5.2.9 can be visualized as a dot product of a row of falling powers of a fixed base with a column of Stirling's triangle.

Example 5.2.6: Consider column 2.

$k \rightarrow$	2	3	4	5
$5^{\underline{5-k}}$	60	20	5	1

$$60 \cdot 1 + 20 \cdot 3 + 5 \cdot 11 + 1 \cdot 50 = 225$$

$n \downarrow$	$\begin{bmatrix} n \\ 2 \end{bmatrix}$	$\begin{bmatrix} n \\ 3 \end{bmatrix}$
2	1	
3	3	
4	11	
5	50	
6		225

That is,

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot 5^{\underline{3}} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot 5^{\underline{2}} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot 5^{\underline{1}} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} \cdot 5^{\underline{0}}$$

$$= 1 \cdot 60 + 3 \cdot 20 + 11 \cdot 5 + 50 \cdot 1 = 225 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

Southeast Diagonal

The entries along each SE diagonal from column 0 to column c satisfy a summation formula.

Proposition 5.2.10. *Let n and c be non-negative integers. Then*

$$\begin{bmatrix} n+c+1 \\ c \end{bmatrix} = \sum_{k=0}^{c} (n+k) \begin{bmatrix} n+k \\ k \end{bmatrix}$$

Proof: The equation is clearly true for all $n \geq 0$ when $c = 0$. Assume, for inductive purpose, that it is true for $c - 1$. Then, by Stirling's recursion,

$$\begin{bmatrix} n+c+1 \\ c \end{bmatrix} = \begin{bmatrix} n+c \\ c-1 \end{bmatrix} + (n+c) \begin{bmatrix} n+c \\ c \end{bmatrix}$$

Now apply the induction hypothesis.

$$= \sum_{k=0}^{c-1} (n+k) \begin{bmatrix} n+k \\ k \end{bmatrix} + (n+c) \begin{bmatrix} n+c \\ c \end{bmatrix}$$

$$= \sum_{k=0}^{c} (n+k) \begin{bmatrix} n+k \\ k \end{bmatrix} \qquad \diamondsuit$$

Example 5.2.7: The sum in Proposition 5.2.10 is a dot product of a southeast diagonal of Stirling's triangle with a vector of row numbers.

$n \downarrow$	$\left[{n \atop 1}\right]$	$\left[{n \atop 2}\right]$	$\left[{n \atop 3}\right]$
3	2		
4		11	
5			35
6			$\boxed{225}$

$$3 \cdot 2 + 4 \cdot 11 + 5 \cdot 35 \;=\; 225$$

Stirling Numbers of the First Kind

REVIEW FROM §1.6: The *Stirling numbers of the first kind* were defined as the coefficients $s_{n,c}$ in the sum

$$x^{\underline{n}} \;=\; \sum_{c=0}^{n} s_{n,c}\, x^c$$

Prop 5.2.11. *Let n and c be any non-negative integers. Then*

$$s_{n,c} \;=\; (-1)^{n+c} \left[{n \atop c}\right] \tag{5.2.1}$$

Proof: We recall the Stirling cycle recurrence of Prop 5.2.1.

$$\begin{bmatrix} 0 \\ k \end{bmatrix} = (k = 0) \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = (n = 0)$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} \quad \text{for } n \geq 1$$

BASIS: The initial conditions

$$s_{0,c} = (-1)^{0+c}\,(c = 0) \quad \text{and} \quad s_{n,0} = (-1)^{n+0}\,(n = 0)$$

hold, because

$$x^{\underline{0}} = 1x^0$$

and because the constant term of the expansion

$$x^{\underline{n}} = \sum_{c=0}^{n} s_{n,c}\, x^c \tag{5.2.2}$$

is 0, unless $n = 0$.

IND HYP: Now assume that

$$x^{\underline{n-1}} = \sum_{c=0}^{n-1} s_{n-1,c}\, x^c$$

IND STEP: Then

$$\begin{aligned} x^{\underline{n}} &= (x - n + 1) \cdot x^{\underline{n-1}} \\ &= x \cdot \sum_{c=0}^{n-1} s_{n-1,c}\, x^c \;-\; (n-1) \sum_{c=0}^{n-1} s_{n-1,c}\, x^c \end{aligned}$$

$$
\begin{aligned}
&= \sum_{c=0}^{n-1} s_{n-1,c} x^{c+1} \;-\; (n-1) \sum_{c=0}^{n-1} s_{n-1,c}\, x^c \\
&= \sum_{c=1}^{n} s_{n-1,c-1} x^{c} \;-\; (n-1) \sum_{c=0}^{n-1} s_{n-1,c}\, x^c \\
&= \sum_{c=0}^{n} \big(s_{n-1,c-1} - (n-1) s_{n-1,c}\big)\, x^c \qquad (5.2.3)
\end{aligned}
$$

Since x^c must have the same coefficient in both expansions, (5.2.2) and (5.2.3), of $x^{\underline{n}}$, it follows that

$$s_{n,c} \;=\; s_{n-1,c-1} - (n-1) s_{n-1,c}$$

Thus, the absolute values of the Stirling numbers of the first kind satisfy the same recurrence as the Stirling cycle numbers. That is,

$$|s_{n,c}| \;=\; \left[{n \atop c} \right]$$

This implies, by an induction, that

$$|s_{n,c}| \;=\; (-1)^{n+c} \left[{n \atop c} \right]$$

$\diamondsuit$

Example 5.2.8: The values of $s_{n,c}$ are calculated recursively, as in the proof of Proposition 5.2.11.

$$\begin{aligned}
x &= x^{\underline{1}} \\
\Rightarrow x^2 &= x \cdot x^{\underline{1}} \\
&= x^{\underline{1}}(x-1) + x^{\underline{1}} \\
&= x^{\underline{2}} + x^{\underline{1}} \\
\Rightarrow x^3 &= x \cdot x^{\underline{2}} + x \cdot x^{\underline{1}} \\
&= x^{\underline{2}}(x-2) + 2x^{\underline{2}} + x^{\underline{2}} + x^{\underline{1}} \\
&= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \\
\Rightarrow x^4 &= x \cdot x^{\underline{3}} + 3x \cdot x^{\underline{2}} + x \cdot x^{\underline{1}} \\
&= [x^{\underline{4}} + 3x^{\underline{3}}] + [3x^{\underline{3}} + 6x^{\underline{2}}] + [x^{\underline{2}} + x^{\underline{1}}] \\
&= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}}
\end{aligned}$$

Table 5.2.2 Basic Formulas for Stirling Cycle #s

Stirling's recurrence:

$$\begin{bmatrix} 0 \\ k \end{bmatrix} = (k=0) \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = (n=0)$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} \quad \text{for } n \geq 1 \qquad (5.2.4)$$

Special values for $n \geq 1$:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)! \qquad \begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!\, H_{n-1} \qquad \begin{bmatrix} n \\ n \end{bmatrix} = 1$$

Converting falling powers to ordinary powers:

$$x^{\underline{n}} = \sum_{k=0}^{n} \left[{n \atop k} \right] (-1)^{n-k} x^k \tag{5.2.5}$$

Row sum formulas:

$$n! = \sum_{k=0}^{n} \left[{n \atop k} \right] \tag{5.2.6}$$

$$\left[{n+1 \atop c+1} \right] = \sum_{j=0}^{n} \binom{j}{c} \left[{n \atop j} \right] \tag{5.2.7}$$

Column-sum formula:

$$\left[{n+1 \atop c+1} \right] = \sum_{k=0}^{n} n^{\underline{n-k}} \left[{k \atop c} \right] \tag{5.2.8}$$

SE diagonal-sum formula:

$$\left[{n+c+1 \atop c} \right] = \sum_{k=0}^{c} (n+k) \left[{n+k \atop k} \right] \tag{5.2.9}$$

5.3 INVERSIONS AND ASCENTS

Stirling cycle numbers inventory the set of all $n!$ permutations of the integer interval $[1:n]$, according to the number of cycles. In particular, the Stirling cycle number

$$\begin{bmatrix} n \\ k \end{bmatrix}$$

is the number of partitions with k cycles. This section is concerned with two other ways of partitioning those $n!$ permutations, one according to their number of *inversions* and the other according to their number of *ascents*.

NOTATION: Specifying a permutation

$$\pi \;=\; \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

of the integer interval $[1:n]$ by its lower line

$$a_1\, a_2\, \cdots\, a_n$$

is called the ***one-line representation*** of π.

Inversions

DEF: In a permutation π of the integer interval $[1:n]$, an ***inversion*** is a pair of integers $i < j$ with $\pi(j) < \pi(i)$.

In any permutation π of the integer interval $[1 : n]$, each instance of an inversion corresponds to some larger integer preceding an integer j in the one-line representation of π, so they would appear to be inverted in that line. There are $\binom{n}{2}$ pairs of integers in $[1 : n]$, each of which could possibly be inverted. At the low end, the identity permutation of $[1 : n]$ has no inversions. At the high end, the permutation that reverses the order of $[1 : n]$ has $\binom{n}{2}$ inversions.

DEF: The ***inversion vector*** of a permutation π is the vector

$$b_1\, b_2\, \cdots\, b_n$$

such that b_j equals the number of larger integers preceding j in the one-line representation of π.

Example 5.3.1: The permutation

$$\pi = 3\,5\,1\,6\,2\,4$$

has the inversion vector

$$2\,3\,0\,2\,0\,0$$

We observe that the coordinate b_j of the inversion vector $b_1\, b_2\, \cdots\, b_n$ is an integer in the range $[0 : n - j]$. Moreover, the total number of inversions of a permutation is the sum of the coordinates of its inversion vector.

Example 5.3.1, cont.: The perm $\pi = 3\,5\,1\,6\,2\,4$ has a total of 7 inversions, the sum of the coordinates of its inversion vector $2\,3\,0\,2\,0\,0$.

DEF: The ***inversion coefficient*** $I_n(k)$ is the number of permutations of the integer interval $[1:n]$ with exactly k inversions.

Table 5.3.1 gives the values of some inversion coefficients.

Table 5.3.1 **Inversion coefficients.**

n	$I_n(0)$	$I_n(1)$	$I_n(2)$	$I_n(3)$	$I_n(4)$	$I_n(5)$	$I_n(6)$	$I_n(7)$	$I_n(8)$	
0	1									
1	1									
2	1	1								
3	1	2	2	1						
4	1	3	5	6	5	3	1			
5	1	4	9	15	20	22	20	15	9	
6	1	5	14	29	49	71	90	101	101	$\cdots$

The table of inversion coefficients can be constructed using the following proposition. We take $I_n(c)$ to be 0 if $c < 0$.

Proposition 5.3.1. *The inversion coefficients satisfy the following recurrence.*

$$\begin{aligned} I_0(0) &= 1 \\ I_n(c) &= \sum_{j=0}^{n-1} I_{n-1}(c-j) \quad \textit{for } n \geq 1 \end{aligned}$$

Proof: The initial condition is true, since the null permutation on the empty set has no inversions.

To affirm the recursion inductively, assume that the recursion holds for the permutations of $[1:n-1]$. Now consider the one-line representation of a permutation π on $[1:n]$ with c inversions

$$\pi: \quad \pi_1 \pi_2 \cdots \pi_n$$

Then the number of inversions contributed by the placement of the integer n within this line equals the number j of integers that follow n on that line. Thus, if n is erased from that line, then the number of inversions in the permutation corresponding to the resulting line equals $c-j$. There are exactly $I_{n-1}(c-j)$ such permutations of $[1 : n-1]$. Thus, $I_n(c)$ is the sum of the numbers $I_{n-1}(c-j)$ over the possible values of j. ♢

Example 5.3.2: We observe in Table 5.3.1 that

$$\begin{aligned} I_4(3) &= I_3(3) + I_3(2) + I_3(1) + I_3(0) \\ &= 1 + 2 + 2 + 1 \;=\; 6 \end{aligned}$$

$$\begin{aligned} I_4(4) &= I_3(4) + I_3(3) + I_3(2) + I_3(1) \\ &= 0 + 1 + 2 + 2 \;=\; 5 \\ I_4(5) &= I_3(5) + I_3(4) + I_3(3) + I_3(2) \\ &= 0 + 0 + 1 + 2 \;=\; 3 \end{aligned}$$

Donald Knuth (see [Knut1973], p.12) regards the following observation of Marshall Hall as the most important single fact about inversions.

Theorem 5.3.2 [Hall1956]. *A permutation π on the integer interval $[1:n]$ is reconstructible from its inversion vector*

$$b_1\, b_2\, \cdots\, b_n$$

Proof: To reconstruct a one-line representation of the permutation π, begin by writing the number n. After the integers

$$k\,,\,\ldots\,,\,n$$

have been written as directed here, insert the integer $k-1$ so that it immediately follows the first b_{k-1} integers. ♢

Corollary 5.3.3. *There is a bijective correspondence between permutations on $[1:n]$ and inversion vectors $b_1\, b_2\, \cdots\, b_n$ with $b_j \in [0:n-j]$ for $j = 1, \ldots, n$.*

Proof: The number of permutations of $[1:n]$ and the number of such inversion vectors are both equal to $n!$. By Theorem 5.3.2, the correspondence of permutations to inversion vectors is one-to-one. It follows by the pigeonhole principle that it is onto. ♢

Example 5.3.3: The one-line representation of the permutation of the integer interval $[1 : 7]$ corresponding to the inversion vector

$$4\,5\,1\,2\,0\,1\,0$$

is reconstructed as follows:

$$\begin{array}{ccccccc} 7 \\ 7 & 6 \\ 5 & 7 & 6 \\ 5 & 7 & 4 & 6 \\ 5 & 3 & 7 & 4 & 6 \\ 5 & 3 & 7 & 4 & 6 & 2 \\ 5 & 3 & 7 & 4 & 1 & 6 & 2 \end{array}$$

Ascents

DEF: An index j of a permutation

$$\pi = a_1\, a_2 \cdots a_n$$

is an ***ascent*** if $a_j < a_{j+1}$ and a ***descent*** if $a_j > a_{j+1}$.

Remark: An ascent is a special kind of non-inversion.

Example 5.3.4: The ascents of the permutation

$$\pi = 3\,5\,1\,6\,2\,4$$

are as follows:

$$\begin{array}{ll} 1: & 3 < 5 \\ 3: & 1 < 6 \\ 5: & 2 < 4 \end{array}$$

Example 5.3.5: The partition of the permutations of $[1:4]$ according to number of ascents is as follows:

$$\begin{array}{lllllllll} 3: & 1234 \\ 2: & 1243 & 1423 & 1324 & 1342 & 2134 & 2314 & 2341 & 2413 \\ & 3124 & 3412 & 4123 \\ 1: & 3421 & 3241 & 4231 & 2431 & 4312 & 4132 & 1432 & 3142 \\ & 4213 & 2143 & 3214 \\ 0: & 4321 \end{array}$$

Eulerian Numbers

DEF: The ***Eulerian number***

$$\left\langle {n \atop k} \right\rangle$$

is the number of permutations of $[1 : n]$ with exactly k ascents.

Prop 5.3.4. *The Eulerian numbers satisfy the recurrence*

$$\left\langle {0 \atop k} \right\rangle = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}$$

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle \quad \text{for } n > 0$$

Combinatorial Proof: The basis for the recurrence is clear. The first summand in the right side of the recursion follows from the fact that a permutation of $[1 : n]$ with k ascents is obtained from a permutation of $[1 : n-1]$ with k ascents by prepending the integer n at the start of the one-line representation or inserting it between the integers of an ascending pair. The second summand corresponds to the $n-k$ ways to increase the number of ascents by 1 in a permutation of $[1 : n-1]$ with $k-1$ ascents either by interposing n between any of the $n-k-1$ descending pairs or by appending n at the end of the line. ♢

As with Pascal's recursion and the Stirling recursions, the Euler recursion leads to a triangular table.

Table 5.3.2 Euler's triangle for values of $\left\langle {n \atop r} \right\rangle$.

n	$\left\langle {n \atop 0} \right\rangle$	$\left\langle {n \atop 1} \right\rangle$	$\left\langle {n \atop 2} \right\rangle$	$\left\langle {n \atop 3} \right\rangle$	$\left\langle {n \atop 4} \right\rangle$	$\left\langle {n \atop 5} \right\rangle$	$\left\langle {n \atop 6} \right\rangle$	B_n
0	1							1
1	1	0						1
2	1	1	0					2
3	1	4	1	0				6
4	1	11	11	1	0			24
5	1	26	66	26	1	0		120
6	1	57	302	302	57	1	0	720

We observe that each row of Euler's triangle is symmetric. This observation is confirmed for all n as follows.

Prop 5.3.5 ***Symmetry for Eulerian Numbers.***

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle$$

Proof: A permutation π of $[1 : n]$ with k ascents has $n-1-k$ descents. Accordingly, the permutation whose one-line representation is the reverse of the representation for π has $n-1-k$ ascents. $\diamondsuit$

5.4 DERANGEMENTS

We recall that a *derangement* is a permutation in which none of the objects is fixed. The ***derangement recurrence*** (from §2.1)

$$\begin{aligned} D_0 &= 1; \quad D_1 = 0; \\ D_n &= (n-1)D_{n-1} + (n-1)D_{n-2} \quad \text{for } n \geq 2 \end{aligned} \tag{5.4.1}$$

is second-degree linear with variable coefficients. From it, a first degree recurrence can be derived.

Proposition 5.4.1. *The derangement sequence satisfies the recurrence*

$$\begin{aligned} D_0 &= 1; \\ D_n &= nD_{n-1} + (-1)^n \quad \textit{for } n \geq 1 \end{aligned} \tag{5.4.2}$$

Proof: Recursion (5.4.1) above implies that

$$D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}] \quad \text{for } n \geq 2 \tag{5.4.3}$$

We now apply recursion (5.4.3) recursively.

$$\begin{aligned} D_n - nD_{n-1} &= (-1)[D_{n-1} - (n-1)D_{n-2}] \\ &= (-1)^2[D_{n-2} - (n-2)D_{n-3}] \\ &= \cdots \\ &= (-1)^{n-1}[D_1 - D_0] = (-1)^{n-1}[0-1] \\ &= (-1)^n \\ \Rightarrow D_n &= nD_{n-1} + (-1)^n \end{aligned}$$

◊

Using either of the derangement recurrences, (5.4.1) or (5.4.2), we can calculate the *derangement number* D_n. The ratio $D_n/n!$ is the proportion of permutations that are derangements. Some values for the ratios $D_n/n!$ and $n!/D_n$ appear in Table 5.4.1.

Table 5.4.1 Ratios of derangements to perms.

n	$n!$	D_n	$D_n/n!$	$n!/D_n$
0	1	1	1	
1	1	0	0	
2	2	1	0.5	2.0
3	6	2	0.333333	3.0
4	24	9	0.375	2.666667
5	120	44	0.366667	2.727273
6	720	265	0.368055	2.716981
7	5040	1854	0.367857	2.718447
8	40320	14833	0.367881	2.718263
9	362880	133496	0.367879	2.718284

Seemingly, the ratios $D_n/n!$ and $n!/D_n$ converge rapidly to e^{-1} and e, respectively. The following proposition and its corollary confirm this reasonable suspicion. This is an application of the familiar technique of guessing the solution to a recurrence and proving the correctness by induction.

Theorem 5.4.2. *For every non-negative integer n,*

$$D_n = n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right] \quad (5.4.4)$$

Proof: For $n = 0$, both sides of equation (5.4.4) have the value 1. We asssume inductively that equation (5.4.4) holds for $n - 1$. Then

$$\begin{aligned} D_n &= nD_{n-1} + (-1)^n \qquad \text{(by (5.4.2))} \\ &= n(n-1)!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right] + (-1)^n \\ &= n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^{n-1}\frac{1}{(n-1)!}\right] + \frac{n!(-1)^n}{n!} \\ &= n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \cdots + (-1)^n\frac{1}{n!}\right] \qquad \diamondsuit \end{aligned}$$

In §3.6, the derangement numbers were calculated by incl-excl. In the proof of Thm 5.4.2, we verified the solution as a "guessed solution" to a recursion. In the next section, the derangement recurrence is solved by generating functions, without resort to guessing.

Corollary 5.4.3. $\lim_{n\to\infty} \frac{D_n}{n!} = e^{-1}$. $\diamondsuit$

Remark: By running a Monte Carlo experiment on a computer, we could use Corollary 5.4.3 to approximate the value of e.

Every permutation of n objects may be regarded as a choice of j objects to fix and a derangement of the other $n - j$ objects. This leads immediately to the following assertion, which was previously noted with Example 4.2.4.

Prop 5.4.4. *Let n be a non-negative integer. Then*

$$n! \ = \ \sum_{j=0}^{n} \binom{n}{j} D_{n-j} \qquad \diamondsuit$$

Example 5.4.1: For $n = 4$, Proposition 5.4.4 corresponds to the equation

$$\begin{aligned} 24 \ &= \ \binom{4}{0} D_4 + \binom{4}{1} D_3 + \binom{4}{2} D_2 + \binom{4}{3} D_1 + \binom{4}{4} D_0 \\ &= \ 1 \cdot D_4 \ + \ 4 \cdot D_3 \ + \ 6 \cdot D_2 \ + \ 4 \cdot D_1 \ + \ 1 \cdot D_0 \\ &= \ 1 \cdot 9 \ + \ 4 \cdot 2 \ + \ 6 \cdot 1 \ + \ 4 \cdot 0 \ + \ 1 \cdot 1 \end{aligned}$$

5.5 EXPONENTIAL GEN FUNCTIONS

OGFs are well-adapted to problems about counting unordered selections. This section develops the other main variety of generating function, called an *exponential generating function*, which is especially useful in counting ordered selections. We will see also how EGFs can be used in solving certain recurrences with variable coefficients.

REVIEW FROM §1.7:

- The ***ordinary generating function*** (abbr. ***OGF***) for a sequence $\langle g_n \rangle$ is any closed form $G(z)$ corresponding to the infinite polynomial

$$\sum_{n=0}^{\infty} g_n z^n$$

  or sometimes, the polynomial itself.

- The ***exponential generating fn*** (abbr. ***EGF***) for a sequence $\langle g_n \rangle$ is any closed form $\hat{G}(z)$ corresponding to the infinite polynomial

$$\sum_{n=0}^{\infty} g_n \frac{z^n}{n!}$$

  or sometimes, the polynomial itself.

- **Prop 1.7.1.** Let $G(z)$ and $H(z)$ be the OGFs for counting unordered selections from two disjoint multisets S and T. Then $G(z)H(z)$ is the OGF for counting unordered selections from the union $S \cup T$.
- The ***convolution of the sequences*** $\langle a_n \rangle$ and $\langle b_n \rangle$ is the sequence

$$a_0b_0, \quad a_0b_1 + a_1b_0, \quad a_0b_2 + a_1b_1 + a_2b_0, \quad \ldots$$

- **Prop 1.7.3.** The product of the generating functions

$$A(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad B(z) = \sum_{n=0}^{\infty} b_n z^n$$

is the generating function

$$A(z)B(z) \;=\; \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) z^n$$

for the convolution of the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

The following example reviews how Prop 1.7.1 can be used to count unordered selections with ordinary generating functions.

Example 5.5.1: Let a_n and b_n be the numbers of ways to select n letters from the multi-sets represented by the strings

“ADD” and “SPICE”

respectively. Thus, the ordinary generating functions for the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ are

$$\begin{aligned} A(z) &= \sum_{n=0}^{\infty} a_n z^n = 1 + 2z + 2z^2 + z^3 \\ B(z) &= \sum_{n=0}^{\infty} b_n z^n = 1 + 5z + 10z^2 + 10z^3 + 5z^4 + z^5 \end{aligned}$$

The set of possibilities counted by the sequence $\langle a_i \rangle$ is completely disjoint from the set counted by the sequence $\langle b_n \rangle$, because the set of letters of "ADD" is disjoint from the set of letters of "SPICE". It follows that the number c_n of ways to choose n letters from the multi-set represented by the string

"ADDSPICE"

is the sum

$$a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

More generally, it follows that the sequence $\langle c_n \rangle$ is the convolution of the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$. Therefore, according to Proposition 1.7.3, the generating function for the sequence $\langle c_n \rangle$ is the product

$$\begin{aligned} A(z)B(z) &= 1 + 7z + 22z^2 + 41z^3 + 50z^4 + 41z^5 \\ &\quad + 22z^6 + 7z^7 + z^8 \end{aligned}$$

For instance, there are 21 ways to choose two letters from the seven different letters and 1 way to choose the same two letters, for a total of 22, the coefficient of z^2.

Counting Ordered Selections

To count ordered selections from a disjoint union of multisets, we use Proposition 1.7.2.

REVIEW FROM §1.7:

- **Proposition 1.7.2.** Let $\hat{G}(z)$ and $\hat{H}(z)$ be the exponential generating functions for counting ordered selections from two disjoint multisets S and T. Then $\hat{G}(z)\hat{H}(z)$ is the exponential generating function for counting ordered selections from the union $S \cup T$.

Example 5.5.2: Let r_n and s_n be the numbers of ways to select a sequence of n letters (without repetition) from the multi-sets represented by the strings

"ADD" and "SPICE"

respectively. Thus, the exponential generating functions for the sequences $\langle r_n \rangle$ and $\langle s_n \rangle$ are

$$\hat{R}(z) = \sum_{n=0}^{\infty} r_n \frac{z^n}{n!} = 1 + 2\frac{z}{1!} + 3\frac{z^2}{2!} + 3\frac{z^3}{3!}$$

$$\hat{S}(z) = \sum_{n=0}^{\infty} s_n \frac{z^n}{n!} = 1 + 5\frac{z}{1!} + 20\frac{z^2}{2!} + 60\frac{z^3}{3!} + 120\frac{z^4}{4!} + 120\frac{z^5}{5!}$$

The coefficient of z^2 in the product $\hat{R}(z)\hat{S}(z)$ is

$$\begin{aligned} & \frac{1\cdot 20}{0!\,2!} + \frac{2\cdot 5}{1!\,1!} + \frac{3\cdot 1}{2!\,0!} \\ =& \frac{1}{2!}\left[\binom{2}{0}1\cdot 20 + \binom{2}{1}2\cdot 5 + \binom{2}{2}3\cdot 1\right] \\ =& \frac{43}{2!} \end{aligned}$$

from which it follows that the coeff of $\frac{z^2}{2}$ in $\hat{R}(z)\hat{S}(z)$ is

$$43$$

This corresponds to $7^{\underline{2}} = 42$ possible ordered selections of two different letters from the seven in the string

ADDSPICE

plus 1 way to choose the same two letters, for a total of 43.

Giving a name to the construction appearing within Example 5.5.2 facilitates the use of a generalization of that method, via Proposition 5.5.1, which is analogous to Proposition 1.7.3.

DEF: The ***binomial convolution*** of two sequences $\langle r_n \rangle$ and $\langle s_n \rangle$ is the sequence $\langle t_n \rangle$ whose n^{th} entry is

$$t_n = \sum_{j=0}^{n} \binom{n}{j} r_j s_{n-j}$$

Prop 5.5.1. *The product of the EGFs for the sequences* $\langle r_n \rangle$ *and* $\langle s_n \rangle$ *is the EGF for their binomial convolution.*

Proof: The coefficient of z^n in the product of the exponential generating functions

$$\hat{R}(z) = \sum_{n=0}^{\infty} r_n \frac{z^n}{n!}$$

and

$$\hat{S}(z) = \sum_{n=0}^{\infty} s_n \frac{z^n}{n!}$$

is

$$\begin{aligned}
&\frac{r_0 s_n}{0!\,n!} + \frac{r_1 s_{n-1}}{1!\,(n-1)!} + \cdots + \frac{r_n s_0}{n!\,0!} \\
&= \frac{1}{n!}\left[\frac{n!}{0!\,n!} r_0 s_n + \frac{n!}{1!\,(n-1)!} r_1 s_{n-1} + \cdots + \frac{n!}{n!\,0!} r_n s_0\right] \\
&= \frac{1}{n!}\left[\binom{n}{0} r_0 s_n + \binom{n}{1} r_1 s_{n-1} + \cdots + \binom{n}{n} r_n s_0\right] \\
&= \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} r_j s_{n-j}
\end{aligned}$$

Thus, the coefficient of $\frac{z^n}{n!}$ in the product $\hat{R}(z)\hat{S}(z)$ is

$$\sum_{j=0}^{n} \binom{n}{j} r_j s_{n-j} \qquad \diamondsuit$$

We complete this section by considering several applications in which using EGF's is a highly convenient way to count.

Counting Certain Kinds of Strings

If a set of symbols has cardinality k, then, of course, there are k^n strings of length n. The examples in the sequence to follow impose various rules on the strings and count the strings that satisfy those rules. The first examples are easy enough, as an intended warmup, that solution without EGF's is well within grasp, and as the complications increase, the usefulness of EGF's becomes ever more clear.

Example 5.5.3: Let b_n be the number of binary strings of length n with at least one 1. Of the 2^n binary strings of length n, only one has no 1's. Thus,

$$b_n = 2^n - 1$$

Alternatively, we could observe that the EGF for the number of all-0 strings of length n is e^z. Accordingly, the EGF for the number of all-1 strings of length n with at least one 1 is $e^z - 1$. Thus, by Proposition 1.7.2, the EGF for b_n is

$$\hat{B}(z) = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!} = e^z \cdot (e^z - 1) = e^{2z} - e^z$$

The coefficient of z^n in $e^{2z} - e^z$ is

$$\frac{2^n}{n!} - \frac{1}{n!}$$

Thus, the coefficient of $\frac{z^n}{n!}$ is

$$b_n = 2^n - 1$$

Example 5.5.4: Let t_n be the number of ternary strings (i.e., base-3) of length n in which the digits 1 and 2 must each occur at least once. Of the 3^n ternary strings of length n, there are 2^n strings with no 1's and 2^n strings with no 2's and exactly 1 string with no 1's or 2's. Thus, by Inclusion-Exclusion,

$$t_n = 3^n - 2 \cdot 2^n + 1$$

Alternatively, we could write the EGF for t_n, which is

$$\hat{T}(z) = \sum_{n=0}^{\infty} t_n \frac{z^n}{n!} = e^z \cdot (e^z - 1)^2 = e^{3z} - 2e^{2z} + e^z$$

Thus, the coefficient of $\frac{z^n}{n!}$ is

$$t_n = 3^n - 2 \cdot 2^n + 1$$

For $n = 3$, for instance, the formula $t_n = 3^n - 2 \cdot 2^n + 1$ yields

$$\begin{aligned} t_3 &= 3^3 - 2 \cdot 2^3 + 1 \\ &= 27 - 16 + 1 \\ &= 12 \end{aligned}$$

This corresponds to $3! = 6$ arrangements of the digits within the string 012, plus 3 arrangements of the digits within the string 112, plus 3 arrangements of the digits within the string 122.

Using EGFs on such simple problems seems not to expedite the calculation. However, for more complicated restrictions on the occurrences of some of the symbols in a string, EGF's are of considerable assistance.

Example 5.5.5: Let u_n be the number of ternary strings with at least one 1 and at least two 2's. Then the EGF for strings of 2's with at least two 2's is

$$e^z - z - 1$$

It follows that

$$\begin{aligned}\hat{U}(z) &= \sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = e^z(e^z - 1)(e^z - z - 1) \\ &= e^{3z} - 2e^{2z} - ze^{2z} + ze^z + e^z\end{aligned}$$

Therefore,

$$u_n = 3^n - 2^{n+1} - n2^{n-1} + n + 1$$

For instance,

$$\begin{aligned}u_3 &= 3^3 - 2^4 - 3 \cdot 2^2 + 3 + 1 \\ &= 27 - 16 - 12 + 3 + 1 \\ &= 3\end{aligned}$$

This corresponds to the three possible arrangements of the digits within the string 122.

Example 5.5.6: Let v_n be the number of ternary strings with evenly many 2's and at least one 1. Then the EGF for strings of evenly many 2's is

$$\begin{aligned} & 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots \\ = & \frac{e^z + e^{-z}}{2} \end{aligned}$$

Accordingly,

$$\begin{aligned} \hat{V}(z) \;=\; \sum_{n=0}^{\infty} v_n \frac{z^n}{n!} \;&=\; e^z(e^z - 1) \cdot \frac{e^z + e^{-z}}{2} \\ &=\; \frac{1}{2} \cdot \left(e^{3z} - e^{2z} + e^z - 1\right) \end{aligned}$$

Therefore,

$$v_n \;=\; \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{2}\left(3^n - 2^n + 1\right) & \text{if } n \geq 1 \end{cases}$$

For instance, this formula yields

$$v_3 \;=\; \frac{27 - 8 + 1}{2} \;=\; 10$$

which corresponds to the 7 binary strings (i.e., no 2's) with at least one 1, plus the 3 strings

$$122 \quad 212 \quad 221$$

An Application To Stirling Subset #s

Continuing as in the immediately previous examples, the EGF for the number of ternary strings with at least one 0, at least one 1, and at least one 2 is

$$\begin{aligned}(e^z-1)^3 &= e^{3z}-3e^{2z}+3e^z-1\\ &= \sum_{n=1}^{\infty}(3^n-3\cdot 2^n+3)\,\frac{z^n}{n!}\end{aligned}$$

If we identify the distinct positions $1,\ldots,n$ in the sequence with n distinct objects, then this is also the generating function for partitioning n distinct objects into three distinct boxes, with no box left empty. This is 3! times as many as if the boxes were indistinguishable, so that we were counting partitions into three subsets. Thus,

$$\frac{(e^z-1)^3}{3!} = \sum_{n=0}^{\infty}\left\{ {n \atop 3} \right\}\frac{z^n}{n!}$$

is an EGF for column 3 of Stirling's subset triangle. This calculation has an immediate generalization with a corollary that is equivalent to Theorem 3.6.4.

Prop 5.5.2. *Let n and k be non-negative integers. Then*

$$\frac{(e^z-1)^k}{k!} = \sum_{n=0}^{\infty}\left\{ {n \atop k} \right\}\frac{z^n}{n!} \qquad \diamondsuit$$

Corollary 5.5.3. *Let n and k be non-negative integers. Then*

$$\left\{ n \atop k \right\} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} j^n (-1)^{k-j}$$

Proof: The sides of the equation are the coefficients of

$$\frac{z^n}{n!}$$

in Proposition 5.5.2. ◇

Example 5.5.7: Applying the formula of Corollary 5.5.3 yields the evaluation

$$\begin{aligned}
\left\{ 4 \atop 2 \right\} &= \frac{1}{2!} \sum_{j=0}^{2} \binom{2}{j} j^4 (-1)^{2-j} \\
&= \frac{1}{2} \left[\binom{2}{0} 0^4 (-1)^{2-0} + \binom{2}{1} 1^4 (-1)^{2-1} \binom{2}{2} 2^4 (-1)^{2-2} \right] \\
&= \frac{1}{2} \left[1 \cdot 0 \cdot 1 + 2 \cdot 1 \cdot (-1) + 1 \cdot 16 \cdot 1 \right] \\
&= 7
\end{aligned}$$

which agrees with our previous calculations of

$$\left\{ 4 \atop 2 \right\}$$

An EGF for Derangement Numbers

We now show an example of how, sometimes, an EGF can be used in solving a linear recurrence with a variable coefficient. We then use this technique in finding a generating function for the derangement numbers.

Example 5.5.8: Consider the following recurrence of degree 2.

$$\begin{aligned} a_0 &= 0, \quad a_1 = 1; \\ a_n &= 3n\, a_{n-1} - 2n(n-1)\, a_{n-2} \quad \text{for } n \geq 2 \end{aligned}$$

Step 1. Multiplying both sides of the recursion by $\frac{z^n}{n!}$ and then summing from $n = 2$ to ∞ leads to the equation

$$\sum_{n=2}^{\infty} a_n \frac{z^n}{n!} = \sum_{n=2}^{\infty} 3n a_{n-1} \frac{z^n}{n!} - \sum_{n=2}^{\infty} 2n(n-1) a_{n-2} \frac{z^n}{n!}$$

which simplifies to the form

$$\sum_{n=2}^{\infty} (1)\, a_n \frac{z^n}{n!} = 3z \sum_{n=2}^{\infty} a_{n-1} \frac{z^{n-1}}{(n-1)!} - 2z^2 \sum_{n=2}^{\infty} a_{n-2} \frac{z^{n-2}}{(n-2)!}$$

Step 2. By substituting the EGF

$$\hat{A}(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$$

we obtain the equation

$$(2)\, \hat{A}(z) - a_1 z - a_0 = 3z\left(\hat{A}(z) - a_0\right) - 2z^2 \hat{A}(z)$$

Step 3. We then solve for $\hat{A}(z)$.

$$\hat{A}(z) - 1z - 0 \;=\; 3z\left(\hat{A}(z) - 0\right) \;-\; 2z^2\hat{A}(z)$$

$$(3)\ \hat{A}(z) \;=\; \frac{z}{1 - 3z + 2z^2}$$

Step 4. Use partial fractions to solve for a_n.

$$\hat{A}(z) \;=\; \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \;=\; \frac{1}{1-2z} \;-\; \frac{1}{1-z} \;=\; \sum_{n=0}^{\infty} (2^n - 1)\, z^n$$

$$(4)\ \Rightarrow\quad a_n \;=\; (2^n - 1)\, n!$$

Check the Answer: We now verify that the answer $a_n = (2^n - 1)n!$ satisfies the recurrence.

$$\begin{aligned}
a_0 &= (2^0 - 1)\, 0! \;=\; 0, && \text{(initial condition)}\\
a_1 &= (2^1 - 1)\, 1! \;=\; 1; && \text{(initial condition)}\\
a_n &= 3n\, a_{n-1} \;-\; 2n(n-1)\, a_{n-2} && \text{(recursion)}\\
&= 3n \cdot (2^{n-1} - 1)(n-1)! \;-\; 2n(n-1) \cdot (2^{n-2} - 1)(n-2)!\\
&= 3n! \cdot (2^{n-1} - 1) \;-\; 2n! \cdot (2^{n-2} - 1)\\
&= n! \cdot (3 \cdot 2^{n-1} - 3) \;-\; n! \cdot (2^{n-1} - 2)\\
&= n! \cdot (2 \cdot 2^{n-1}) \;-\; 3n! \;+\; 2n!)\\
&= n!\, (2^n - 1)
\end{aligned}$$

What enables the substitution of the EGF $\hat{A}(z)$ to lead to the successful conclusion of Example 5.5.8 is that in the recursion

$$a_n \;=\; 3na_{n-1} \;-\; 2n(n-1)a_{n-2} \quad \text{for } n \geq 2$$

the variable coefficients of a_{n-1} and a_{n-2} are the falling power monomials $3n^{\underline{1}}$ and $-2n^{\underline{2}}$, of degrees 1 and 2, respectively. Fortunately, the variable coefficient of the derangement recurrence has the same property. The nonhomogeneous part adds a small complication.

Thm 5.5.4. *Let $\hat{D}(z)$ be the EGF for the derangement numbers D_n. Then*

$$\hat{D}(z) \;=\; \frac{e^{-z}}{1-z}$$

Proof: This pf follows the paradigm of Example 5.5.8.

$$\begin{aligned}
D_n &= nD_{n-1} + (-1)^n \qquad \text{(Prop 5.4.1)} \\
\Rightarrow \sum_{n=1}^{\infty} D_n \frac{z^n}{n!} &= \sum_{n=1}^{\infty} nD_{n-1}\frac{z^n}{n!} + \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n!} \\
\Rightarrow \hat{D}(z) - D_0\frac{z^0}{0!} &= z\sum_{n=1}^{\infty} D_{n-1}\frac{z^{n-1}}{(n-1)!} + e^{-z} - 1 \\
\Rightarrow \hat{D}(z) - 1 &= z\hat{D}(z) + e^{-z} - 1 \\
\Rightarrow \hat{D}(z) &= \frac{e^{-z}}{1-z}
\end{aligned}$$

◇

Corollary 5.5.5. *Let $\langle D_n \rangle$ be the derangement sequence. Then*

$$D_n \;=\; n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right]$$

Proof: To rederive Theorem 5.4.2, this time as a corollary to Theorem 5.5.5, we proceed as follows:

$$\hat{D}(z) = \sum_{n=0}^{\infty} D_n \frac{z^n}{n!} = \frac{e^{-z}}{1-z} = \frac{1}{1-z}\left[\frac{1}{0!} - \frac{z}{1!} + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots\right]$$

We recognize $(1-z)^{-1}$ as a summing operator.

$$\Rightarrow \frac{D_n}{n!} = \left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

$$\Rightarrow D_n = n!\left[\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right] \diamondsuit$$

5.6 POSETS AND LATTICES

A *lattice* is a highly structured kind of poset.

FROM APPENDIX A3:

- A ***partial ordering*** on a set P is a binary relation $\preceq$ with the following properties, for all $x, y, z \in P$:
  - i. $x \preceq x$ (*reflexive*)
  - ii. if $x \preceq y$ and $y \preceq x$ then $x = y$ (*antisymmetric*)
  - iii. if $x \preceq y$ and $y \preceq z$ then $x \preceq z$ (*transitive*)
- Two elements x, y in a poset P such that either $x \preceq y$ or $y \preceq x$ are said to be ***comparable***.
- If $x \preceq y$, we may say that y ***dominates*** x.
- We write $x \prec y$ if $x \preceq y$ and $x \neq y$.
- The structure $\mathcal{P} = \langle P, \preceq \rangle$ is called a ***partially ordered set*** or a ***poset***. The set P is the ***domain***.
- Writing or saying "the poset P" (giving the *domain* of the poset, rather than the complete structure) is commonplace and convenient.
- The ***order of a poset*** $\mathcal{P} = \langle P, \preceq \rangle$ is the cardinality of its domain P. Informally, the word *size* is also used.
- A ***subposet*** of a poset $\langle P, \preceq \rangle$ is a subset $S \subseteq P$, in which $x \preceq_S y$ if and only if $x \preceq_P y$.

Products of Sets

One way a poset arises is when subjects are scaled in more than one attribute, e.g., College Board scores.

Example 5.6.1: The cartesian product $[m : n] \times [r : s]$ of two integer intervals is partially ordered under the rule

$$(a, b) \leq (c, d) \text{ if } a \leq c \text{ and } b \leq d$$

This construction can also be generalized to an iterated product over arbitrarily many integer intervals or, indeed, over arbitrarily many posets.

Cover Digraph

Several digraphs and graphs are associated with a poset. The most useful is the *cover digraph*.

DEF: If $x \prec t \prec y$ in a poset $\langle P, \preceq \rangle$, then t is called an ***intermediate element*** between x and y.

DEF: If $x \prec y$ and if there is no intermediate element t, then y ***covers*** x.

DEF: The ***cover digraph*** of a poset $\langle P, \preceq \rangle$ has the elements of the set P as its vertices. There is an arc from x to y if and only if x is covered by y. The ***cover graph*** is its underlying graph. A ***Hasse diagram*** for the poset is a drawing of the cover graph in which the dominant of any two comparable elements must appear above the other.

Example 5.6.1, cont.: Figure 5.6.1 illustrates the cover digraph and Hasse diagram of the poset $[0:1] \times [0:2]$.

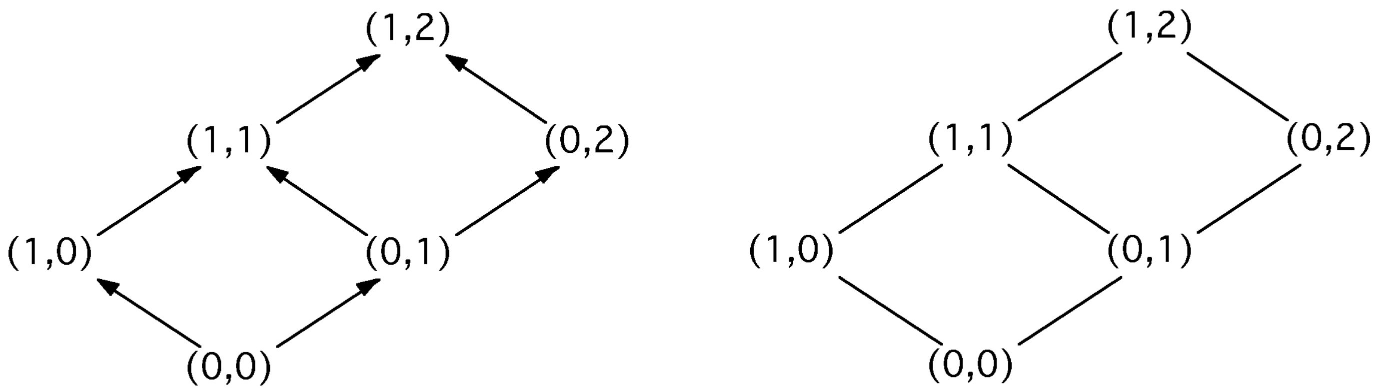

Fig 5.6.1 Cover digraph and Hasse diagram of $[0:1] \times [0:2]$**.**

For any poset $\mathcal{P} = \langle P, \preceq \rangle$, we may observe that $x \preceq y$ if and only if there is a *directed path* from x to y in the cover digraph. The digraph corresponding directly to the partial ordering itself is called the *comparability digraph*, and its underlying graph is called the *comparability graph*.

The Boolean Poset

The *boolean poset* is among the most familiar partially ordered structures.

DEF: The ***boolean poset***

$$\mathcal{B}_n = \langle 2^{[1:n]}, \subseteq \rangle$$

has as its domain the set of subsets of $[1:n]$. They are partially ordered by set-theoretic inclusion.

Example 5.6.2: Figure 5.6.2 shows a cover digraph for the boolean poset $\mathcal{B}_4$.

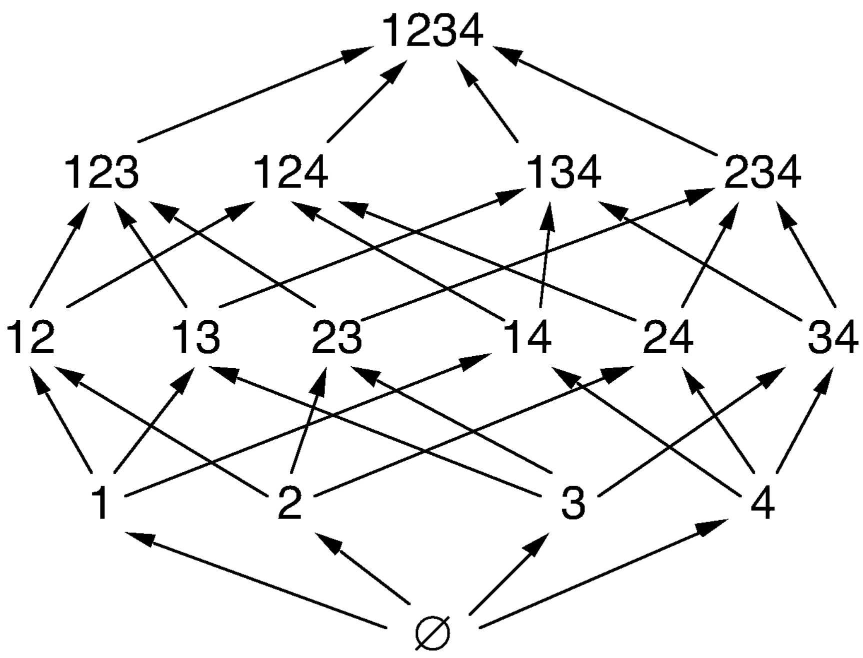

Fig 5.6.2 The boolean poset $\mathcal{B}_4$.

Various properties of the boolean poset $\mathcal{B}_n$ can be observed in Figure 5.6.2. For instance, at level k, the number of subsets is the binomial coefficient $\binom{n}{k}$. Also, the subset Y covers the subset X if $X \subseteq Y$ and if $Y - X$ is a single element of $[1 : n]$.

The Divisibility Poset

REVIEW FROM §3.1:

- The notation $k \setminus n$ means that the integer k divides the integer n.

DEF: In the ***divisibility poset*** $\mathcal{D}_n = \langle D_n, \setminus \rangle$, the domain is the set

$$D_n = \{k \in [1:n] \mid k \setminus n\}$$

and the relation is *divisor of.* The ***infinite divisibility poset*** $\mathcal{D} = \langle \mathbb{Z}^+, \setminus \rangle$ has as its domain the set of all positive integers.

Under the divisibility relation, y covers x if the quotient $\frac{y}{x}$ is prime.

Example 5.6.3: Figure 5.6.3 illustrates the divisibility poset $\mathcal{D}_{72}$.

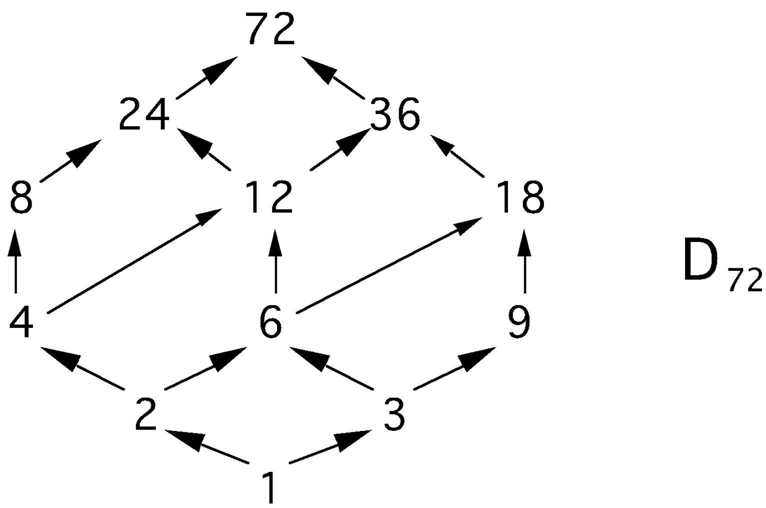

Fig 5.6.3 Cover diagram of the divisibility poset $\mathcal{D}_{72}$.

The Partition Poset

DEF: A partition V of a set S is a ***refinement of the partition*** U if every cell of V is a subset of some cell of U. This relation is denoted $U \sqsupseteq V$.

DEF: In the ***partition poset*** $\mathcal{P}_n = \langle P_n, \sqsupseteq \rangle$, the subsets of the integer interval $[1 : n]$ are partially ordered by the refinement relation.

Example 5.6.4: We now consider an *ad hoc* calculation of a Stirling subset number. The integer interval $[1 : 4]$ can be partitioned into 3 cells in 6 ways:

12–3–4	13–2–4	14–2–3
23–1–4	24–1–3	34–1–2

$$\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} = 6$$

Example 5.6.5: A partition V covers a partition U if it splits a single cell of U into two non-empty subcells. Figure 5.6.4 illustrates a cover diagram for the partition lattice $\mathcal{P}_4$. Hyphens are used to delimit the cells.

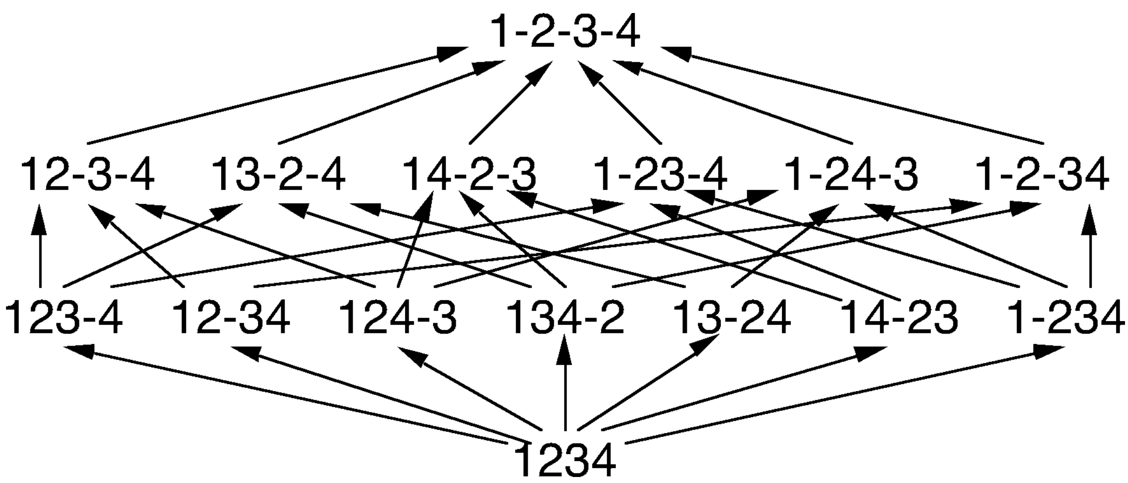

Fig 5.6.4 **The partition poset $\mathcal{P}_4$.**

Inversion-Dominance Ordering on Perms

NOTATION: The set of all permutations of the integer interval $[1 : n]$ is denoted Σ_n. (Under the composition of permutations, it is a group, in the sense of Appendix A2, called the *symmetric group*.)

DEF: The ***inversion-dominance relation***

$$\pi \preceq \tau$$

on Σ_n means that every inversion of π is also an inversion of τ.

Example 5.6.6: The permutation $\pi = 1342$ has two inversions, namely

$$\pi(4) < \pi(2) \quad \text{and} \quad \pi(4) < \pi(3)$$

In addition to those inversions, the permutation $\tau = 3142$ has both those inversions and the inversion

$$\tau(2) < \tau(1)$$

as well. Thus, $1342 \preceq 3142$.

DEF: The ***inversion poset*** $\mathcal{I}_n = \langle \Sigma_n, \preceq \rangle$ is the partially ordered set whose domain is the set of partitions on $[1 : n]$, with the inversion-dominance relation $\pi \preceq \tau$ as its partial ordering.

Example 5.6.7: A digraph representing the cover digraph of $\mathcal{I}_4$ is drawn in Figure 5.6.5 so as to embody the shape of the truncated octahedron, whose 1-skeleton is the underlying graph. Observe that the direction of the arcs is away from 1234, the least inverted permutation, and toward 4321, the most inverted.

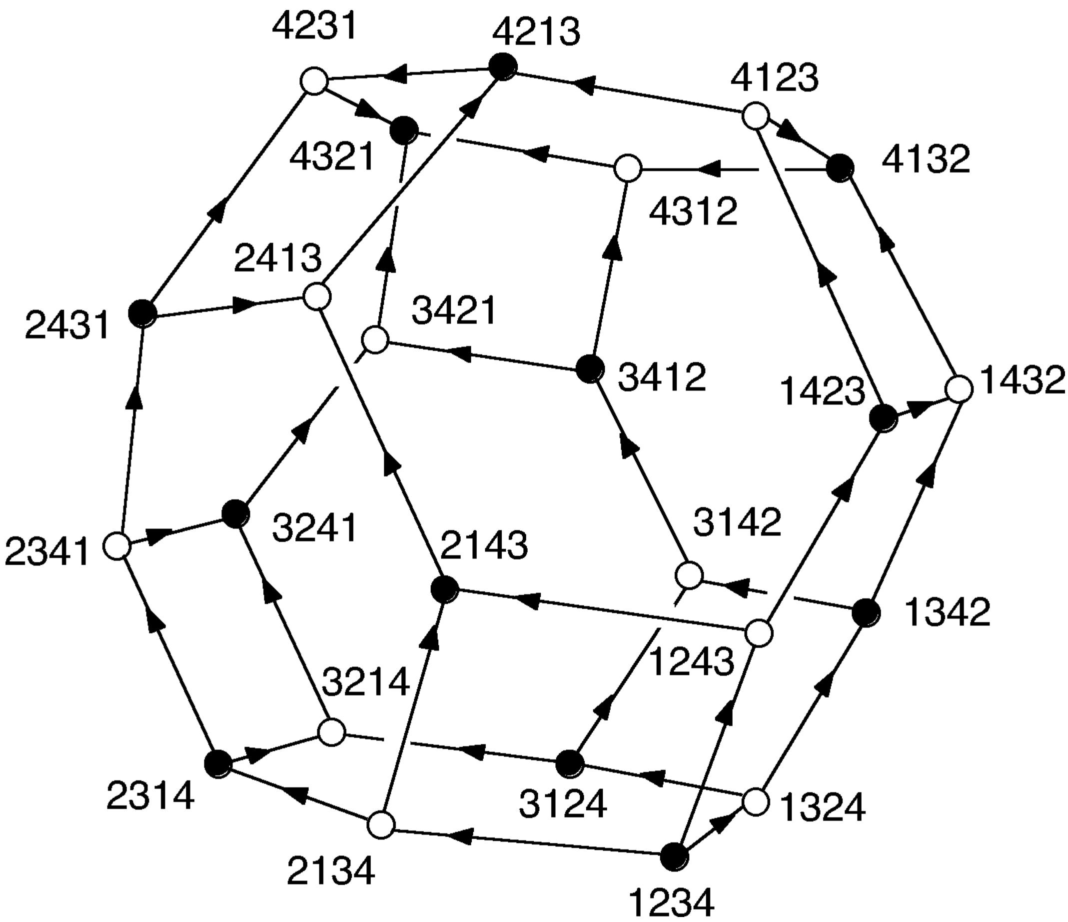

Fig 5.6.5 Cover digraph of the inversion poset $\mathcal{I}_4$.

The underlying graph is obtained by drawing an edge between two permutations whose one-line representations differ only by a single transposition of adjacent integers. The direction reflects increasing the number of inversions.

Minimal and Maximal Elements

DEF: A ***minimal element in a poset*** P is an element x such that there is no element w with $w \prec x$. If $x \preceq y$ for every $y \in P$, then x is the ***minimum element***.

DEF: A ***maximal element in a poset*** P is an element y such that there is no element w with $y \prec w$. If $x \preceq y$ for every $x \in P$, then y is the ***maximum element***.

Example 5.6.8: In Figure 5.6.6, there is no minimum or maximum element. However, the elements a and b are maximal, and the elements d, j, and k are minimal.

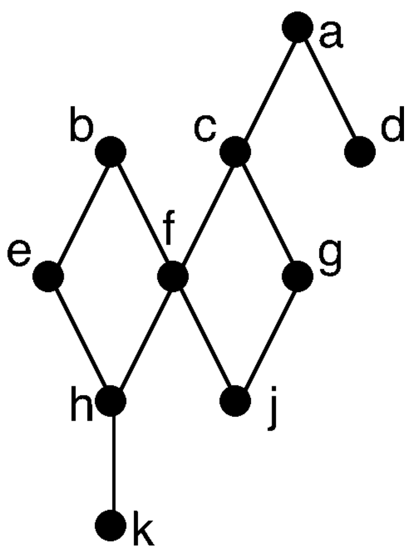

Fig 5.6.6 A poset with two maximal elements and three minimal elements.

Example 5.6.2, cont.: The minimum element of the boolean poset $\mathcal{B}_n$ is the empty set $\emptyset$, and the maximum element is the entire set $\{1, 2, \ldots, n\}$.

Example 5.6.3, cont.: The min of the divisibility poset $\mathcal{D}_n$ is the number 1, and the max is the number n. The infinite divisibility poset $\mathcal{D}$ has no maximum.

Example 5.6.5, cont.: The min of partition poset $\mathcal{P}_n$ is the unpartitioned set $[1:n]$. The max is the partition

$$1 - 2 - \cdots - n$$

into singletons.

Example 5.6.7, cont.: The minimum element of the inversion poset $\mathcal{I}_n$ is the permutation $12 \ldots n$, and the maximum element is the permutation $n(n-1) \ldots 1$.

Lattice Property

A *lattice* is a poset with upper and lower bounds for pairs of elements, as per this list of definitions.

DEFINITIONS:

- An ***upper bound*** for a subset S of a poset P is an element u such that $s \preceq u$ for all $s \in S$.
- A ***lower bound*** for a subset S of a poset P is an element w such that $w \preceq s$ for all $s \in S$.
- A ***least upper bound*** for a subset S of a poset P is an upper bound u such that if z is any other upper bound for S, then $u \preceq z$. We commonly write $lub(x, y)$ for the least upper bound of a subset of two elements, which, if it exists, must be unique, by the antisymmetry property.

- A ***greatest lower bound*** for a subset S of a poset P is a lower bound w for S such that if z is any other lower bound for S, then $z \preceq w$. We commonly write $glb(x, y)$ for the greatest lower bound of a subset of two elements, which, if it exists, must be unique, by the antisymmetry property.
- A ***lattice*** is a poset such that every pair of elements has a *lub* and a *glb*.

Example 5.6.2, cont.: The boolean poset $\mathcal{B}_n$ is a lattice, in which the least upper bound of two subsets is their union and the greatest lower bound is their intersection.

Example 5.6.3, cont.: The divisibility lattices $\mathcal{D}$ and $\mathcal{D}_n$ are lattices, in which the lub of two numbers is their LCM and the glb is their GCD.

Proving that the partition poset is a lattice involves a few details regarding the least upper and greatest lower bounds.

Example 5.6.5, cont.: The partition poset $\mathcal{P}_n$ is a lattice. The constructions of the least upper bound and the greater lower bound are now given.

NOTATION: In the partition lattice $\mathcal{P}_n$, let $U \vee V$ denote the set of non-empty intersections of a cell of a partition U with a cell of another partition V.

Example 5.6.9: Let U be the partition $123 - 45 - 678$ and let V be the partition $14 - 235 - 67 - 8$. Then $U \vee V = 1 - 23 - 4 - 5 - 67 - 8$.

Prop 5.6.1. *In the partition poset $\mathcal{P}_n$, the partition $U \vee V$ is the least upper bound of partitions U and V.*

Proof: See Exercises. ♢

NOTATION: Let U and V be two partitions of the integer interval $[1 : n]$. Then

- Let $K_{U,V}$ denote the bipartite graph whose partite sets are the cells of U and the cells of V, respectively, and where a cell of U is adjacent to a cell of V if they have a vertex in common.
- Let $U \wedge V$ denote the partition of $[1 : n]$, each of whose cells is the union of the vertices in a component of $K_{U,V}$.

Example 5.6.9, cont.: Let U be the partition $123 - 45 - 678$ and V the partition $14 - 235 - 67 - 8$. The graph $K_{U,V}$ is shown in Figure 5.6.7.

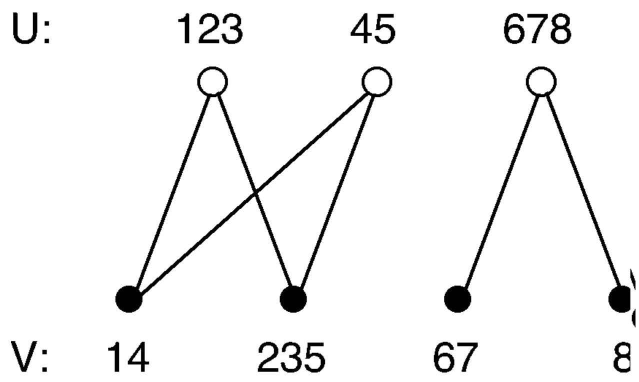

Fig 5.6.7 The bipartite graph $K_{U,V}$ for two partitions.

Then $U \wedge V = 12345 - 678$.

Prop 5.6.2. *In the partition poset $\mathcal{P}_n$, the partition $U \wedge V$ is the greatest lower bound of partitions U and V.*

Proof: See Exercises. ◇

Example 5.6.10: The poset whose cover diagram appears in Figure 5.6.8 is not a lattice, because although d and e are both common lower bounds for b and c, neither is a lower bound for the other.

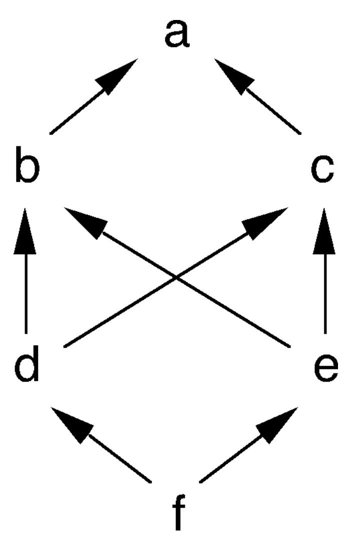

Fig 5.6.8 A poset that is not a lattice.

Poset Isomorphism

DEF: An ***isomorphism of posets*** $\langle P, \preceq_P\rangle$ and $\langle Q, \preceq_Q\rangle$ is a bijection

$$f : P \to Q$$

such that $x \preceq_P y$ in P if and only if $f(x) \preceq_Q f(y)$ in Q.

Example 5.6.11: The divisibility poset $\mathcal{D}_{12}$ is isomorphic to the poset of integer pairs $[0:1] \times [0:2]$, under the bijection

$$\begin{array}{lll} 1 \to (0,0) & 2 \to (0,1) & 3 \to (1,0) \\ 4 \to (0,2) & 6 \to (1,1) & 12 \to (1,2) \end{array}$$

Figure 5.6.9 shows the Hasse diagram for the poset $\mathcal{D}_{12}$.

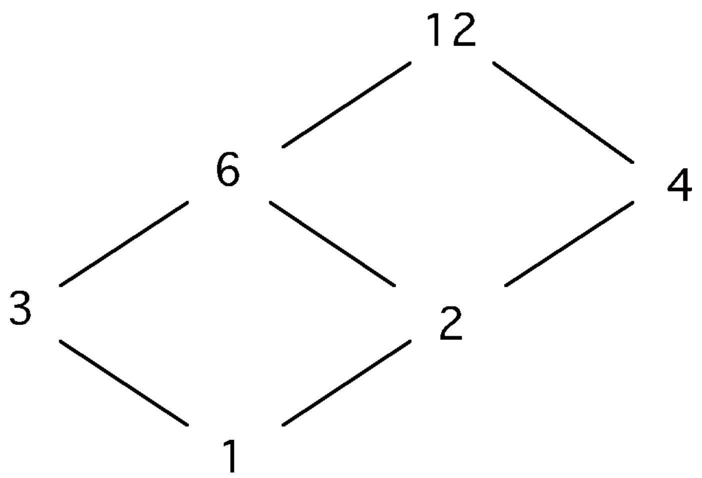

Fig 5.6.9 Hasse diagram of the divisibility poset $\mathcal{D}_{12}$.

Example 5.6.12: The divisibility poset $\mathcal{D}_{30}$ is isomorphic to the boolean poset $\mathcal{B}_3$ under the bijection

$$\begin{array}{llll} 1 \to \emptyset & 2 \to \{1\} & 3 \to \{2\} & 5 \to \{3\} \\ 6 \to \{1,2\} & 10 \to \{1,3\} & 15 \to \{2,3\} & 30 \to \{1,2,3\} \end{array}$$

Example 5.6.13: Figure 5.6.10 shows Hasse diagrams for the five isomorphism types of posets of size 3. The only one of them that is a lattice is at the far right.

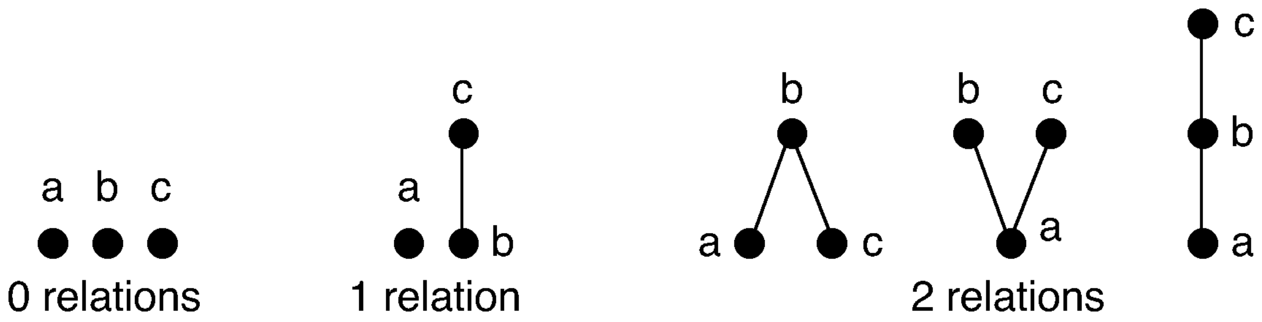

Fig 5.6.10 Hasse diagrams of the posets of size 3.

Observe that two of the posets of size 3 with 2 relations have isomorphic cover graphs (see §7.4). This complicates classifying the isomorphism types of posets of a given size. Also observe that not all simple graphs can occur as cover graphs, as indicated by Proposition 5.6.3.

Proposition 5.6.3. *The cover graph of a poset* $\langle P, \preceq \rangle$ *cannot contain a 3-cycle.*

Proof: Suppose that elements $u, v, w \in P$ form a 3-cycle in the cover graph. Then in each pair, one element must cover the other. By transitivity, there cannot be a cycle in the cover digraph, so one of them, say u, must cover neither of the others, and another, say w, must cover both the others. But then $u \prec v \prec w$, which implies that w does not cover u. ♢

Chains and Antichains

There are two extreme forms of posets. At one extreme, in a *chain*, every pair of elements is comparable.

At the other, in an *antichain*, no two elements are comparable.

DEFINITIONS: Here are a few related definitions:

- If every two elements of a poset $\langle P, \preceq \rangle$ are comparable, then $\langle P, \preceq \rangle$ is said to be ***totally ordered***, ***linearly ordered***, or a ***chain***.
- A poset in which all elements are incomparable is called a ***clutter*** or an ***antichain***.
- The ***height of a poset*** is the cardinality of a longest chain.
- The ***width of a poset*** is the cardinality of a max-size antichain.

A collection of elements of a poset forms a chain if and only if there is a directed path in the cover digraph from the vertex corresponding to one of them to the vertex corresponding to another of them, with the vertices corresponding to all the others as interior vertices along the way. A collection of elements of a poset forms an antichain if the corresponding vertices are mutually unreachable in the cover digraph.

NOTATION: It is common practice to refer to a poset, at times, by its domain, that is, writing simply P for $\langle P, \preceq \rangle$.

Posets have some general structural properties. The following two are among the most easily proved.

Proposition 5.6.4. *Let $\langle P, \preceq \rangle$ be a poset, let C be a chain in P, and let A be an antichain. Then the intersection $A \cap C$ contains at most one element.*

Proof: Let x and y be any elements of the poset $\langle P, \preceq \rangle$. If $x, y \in C$, then they are comparable. If $x, y \in A$, then they are incomparable. ♢

Theorem 5.6.5. *Let $\langle P, \preceq \rangle$ be a finite poset of height h. Then P can be partitioned into h antichains, and into no fewer than h antichains.*

Proof: By Proposition 5.6.4, it follows that an antichain contains at most one element of a longest chain C. Thus, the number of antichains whose union contains C is at least h, the number of elements in chain C.

Proof that the poset $\langle P, \preceq \rangle$ can be partitioned into h antichains is by induction on the height h.

BASIS: If $h = 1$, then the poset $\langle P, \preceq \rangle$ itself is an antichain.

IND HYP: Assume that such a partition exists for $h = n-1$.

IND STEP: Suppose that height $h = n$. Let A_1 be the antichain containing all minimal elements of the poset P. Then the longest chain in the subposet $P - A_1$ is of length $n - 1$. By the induction hypothesis, it follows that the subposet $P - A_1$ can be partitioned into $n - 1$ antichains.
♢

Example 5.6.2, cont.: A chain in the boolean poset $\mathcal{B}_n$ is a sequence of sets, each nested in its successor. Thus the height of the poset $\mathcal{B}_n$ is $n+1$, corresponding to starting with the empty set and including one additional element at a time. An antichain is a collection of subsets, no two of which are nested. The collection U_k of subsets of size k is an antichain. Clearly, the boolean poset $\mathcal{B}_n$ can be partitioned into these $n+1$ collections U_k, for $k = 0, \ldots, n$.

Example 5.6.3, continued: A chain in the divisibility poset $\mathcal{D}_n$ is a sequence of numbers, each of which is a multiple of its predecessor. It follows that the height of the divisibility poset $\mathcal{D}_n$ is 1 plus the sum of the exponents in the prime factorization of n. The subset E_k of numbers whose exponent sum is k is an antichain. Clearly, the divisibility poset $\mathcal{D}_n$ can be partitioned into these collections E_k, as illustrated in Figure 5.6.11. For instance, $12 = 2^2 3^1$, so the exponent sum is $3 = 2+1$, which implies that four antichains are necessary and sufficient.

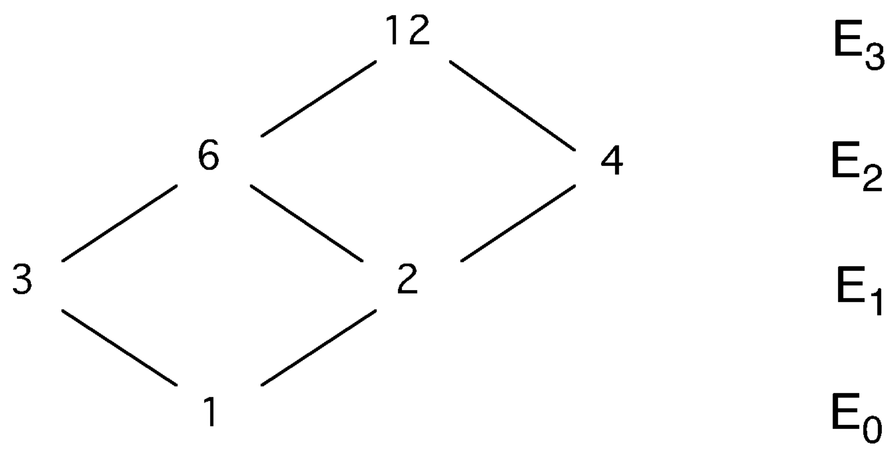

Fig 5.6.11 Partitioning poset $\mathcal{D}_{12}$ into four antichains.

Ranked Posets

Boolean posets, divisibility poset, partition posets, and inversion posets all appear to be layered, so that any traversal of the cover graph is between adjacent ranks. The formal name used for these layers is *ranks*.

DEFINITIONS: Here is another list of related definitions:

- A ***rank function*** on a poset $\langle P, \preceq \rangle$ is a function $\rho : P \to \mathbb{N}$ such that if the element y covers the element x then $\rho(y) = \rho(x) + 1$.
- A ***ranked poset*** is a poset with a rank function.
- The ***k*^th *rank*** of a ranked poset P is the antichain P_k of elements of rank k.
- The ***k*^th *Whitney number*** $N_k(P)$ of a ranked poset $\langle P, \preceq \rangle$ is the cardinality of the k^{th} rank of P.

Example 5.6.2, cont.: The rank function of the boolean poset $\mathcal{B}_n$ assigns to every subset of $[1 : n]$ its number of elements. Thus, the Whitney number $N_k(B_n)$ is $\binom{n}{k}$.

Example 5.6.3, cont.: The rank function of the divisibility poset $\mathcal{D}_n$ assigns to every divisor of n the sum of the exponents in its prime power factorization.

Example 5.6.3, cont.: The rank function of the permutation poset $\mathcal{P}_n$ is the number of cells in the partition. The Whitney number $N_r(P_n)$ is the *Stirling subset number* $\left\{ {n \atop r} \right\}$. For instance, $\mathcal{P}_4$ has $\left\{ {4 \atop 2} \right\} = 7$ elements of rank 2 at the middle level of the cover diagram.

DEF: A poset is ***graded*** if all maximal chains have the same length.

Prop 5.6.6. *A graded poset can be ranked.*

Proof: Assign rank $\rho(x) = 0$ to every minimal element x. Then, proceeding recursively, assign rank $\rho(x) + 1$ to an element that covers x. ♢

Prop 5.6.7. *The inversion poset $\mathcal{I}_n$ is a graded poset.*

Proof: All the maximal chains extend from $12 \cdots n$ to $n(n-1) \cdots 1$ and are of length n. The rank of each permutation is the number of inversions. ♢

Some posets cannot be ranked.

Example 5.6.14: The poset of Figure 5.6.12 is unrankable. Indeed, any poset with an odd cycle in its cover graph is unrankable.

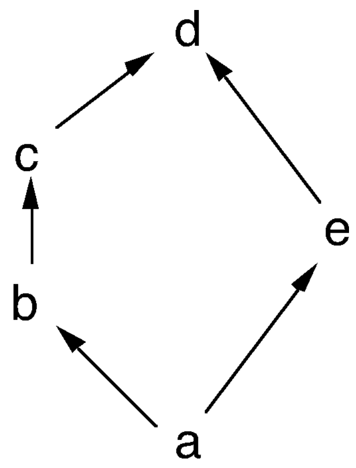

Fig 5.6.12 An unrankable poset.

Linear Extensions

Linear extension of a poset is a way to make a partially ordered set into a totally ordered set.

DEF: An ***extension of a poset*** $\langle P, \preceq \rangle$ is a poset $\langle P, \preceq^* \rangle$ with the same domain, such that $x \preceq^* y$ whenever $x \preceq y$. Thus, an extension adds one or more relations.

DEF: A ***linear extension of a poset*** $\langle P, \preceq \rangle$ is an extension that is totally ordered.

Example 5.6.15: The partial orderings on a set P are partially ordered by extension. The linear extensions are the maximal orderings. The clutter is the minimum ordering.

Proposition 5.6.8. *Every finite poset $\langle P, \preceq \rangle$ has a linear extension.*

Proof: Suppose that $|P| = n$.

BASIS: If $n = 1$, then $\langle P, \preceq \rangle$ is linearly ordered.

IND HYP: Assume that any poset of size $n - 1$ has a linear extension.

IND STEP: Let x be a minimal element of $\langle P, \preceq \rangle$. By the induction hypothesis, there is a linear extension of the poset $P - \{x\}$. Complete the linear extension of P by making x precede every element in the linear extension of the poset $P - \{x\}$. ♢

Example 5.6.16: Here are three linear extensions of the boolean poset $\mathcal{B}_3$.

$$\emptyset \leq 1 \leq 2 \leq 3 \leq 12 \leq 13 \leq 23 \leq 123$$
$$\emptyset \leq 3 \leq 1 \leq 2 \leq 23 \leq 13 \leq 12 \leq 123$$
$$\emptyset \leq 1 \leq 2 \leq 12 \leq 3 \leq 23 \leq 13 \leq 123$$

DEF: A ***topological sort*** is an algorithm whose input is a poset $\langle P, \preceq \rangle$, and whose output is a list $\langle x_j \rangle$ of the elements of the domain P of that poset that is consistent with a linear extension of the poset.

In the following algorithm for a topological sort, we take $Min(P)$ to be a function on a non-empty poset that returns a minimal element of the poset.

Algorithm 5.6.1: Topological Sort

Input: a finite poset $\langle P, \preceq \rangle$ of size n
Output: a roster $\langle x_j \rangle$ of P such that $x_i \preceq x_j$ for $0 \leq i < j < n$

Initialize $j = 0$
while $P \neq \emptyset$
  $x_j := Min(P)$ {returns a minimal element of P}
  $P := P - x_j$
  $j := j + 1$
  continue

Dilworth's Theorem

Whereas Thm 5.6.5 concerns the decomposition of a poset into antichains, there is a complementary theorem of Robert P. Dilworth (1914-1993) that concerns a decomposition into chains. There are two preliminary lemmas.

Lemma 5.6.9. *Let $\langle P, \preceq \rangle$ be a poset, and let L be the set containing all the minimal elements of P. Then L is a maximal antichain.*

Proof: Every element of L is a min element in P, so no two are comparable. Thus, L is an antichain. If $y \notin L$, then since y is not a min element, there is an element $x \in L$ such that $x \prec y$. It follows that $L \cup \{y\}$ is not an antichain. $\diamondsuit$

Lemma 5.6.10. *Let $\langle P, \preceq \rangle$ be a poset, and let U be the set that contains all the maximal elements of P. Then U is a maximal antichain.*

Proof: The proof parallels the proof of Lemma 5.6.9. $\diamondsuit$

Theorem 5.6.11 [Dilw1950]. *Let $\langle P, \preceq \rangle$ be a finite poset of width w. Then P can be partitioned into w chains, and into no fewer than w chains.*

Proof: By Prop 5.6.4, each chain contains at most one element of any antichain, in particular, of a largest antichain. It follows that the width w is a lower bound on the total number of chains in a partition of P into chains.

Proof that a partition into w chains exists is by induction on the width w, with a secondary induction on the size of the poset P.

BASIS: If $w = 1$, then P itself is a chain.

IND HYP: Assume that such a partition exists, for $w = n - 1$.

IND STEP: Suppose that width $w = n$. If $|P| = n$, then each of the n elements of P serves as a chain. Assume that this is also true for all posets of width n whose size is less than the size of P.

Now let A be a maximum antichain, that is, an antichain of size n.

Case 1. Suppose the following two conditions hold:

(1.1) The antichain A is not the set of all maximal elements.

(1.2) The antichain A is not the set of all minimal elements.

We define the subposets

$$\begin{aligned} {}^{\geq}A &= \{x \in P \mid (\exists a \in A)\,[x \geq a]\} \\ {}^{\leq}A &= \{x \in P \mid (\exists a \in A)\,[x \leq a]\} \end{aligned}$$

Observe that the following two properties hold.

(i) $|{}^{\geq}A| < |P|$.
Proof of (i). If every min element of P were in A,

then the subset of A containing only those min elements of P would, by Lemma 5.6.9, already in itself be a max antichain. This would imply that that subset is the antichain A, which violates condition (1.2). Thus, some min element of P cannot be in A. Since it is a min element, it cannot dominate any element of A. Hence, that min element also cannot be in ${}^{\geq}A$.

(ii) $|{}^{\leq}A| < |P|$.
Proof of (ii). If every maximal element of P were in A, then the subset of A containing only those maximal elements of P would, by Lemma 5.6.10, already in itself be a maximal antichain. As before, this would imply that that subset is the antichain A, in violation of condition (1.1). Thus, some maximal element of P cannot be in A. Since it is a maximal element, it cannot be dominated by any element of A. Thus, that maximal element cannot be in ${}^{\leq}A$.

Any antichain in the subposet ${}^{\geq}A$ is also an antichain in the poset P. By construction, $A \subseteq {}^{\geq}A$. Thus, the antichain A is a maximum antichain in ${}^{\geq}A$. By the induction hypothesis, it follows from (i) that the subposet ${}^{\geq}A$ can be partitioned into n chains, $B_1, \ldots, B_n$. Since every element of ${}^{\geq}A$ dominates some element of A, and since $A \subseteq {}^{\geq}A$, it follows that the minimal element of each of these chains B_j is some element $b_j \in A$. Since $\{B_1, \ldots, B_n\}$ is a partition, the elements $b_1, \ldots, b_n$ are distinct.

Similarly, it follows from (ii) that the subposet $^{\leq}A$ can be partitioned into n chains, $C_1, \ldots, C_n$, that the maximal element of each of these chains C_j is some element $c_j \in A$, and that the elements $c_1, \ldots, c_n$ are distinct.

Since $|A| = n$, it follows that $A = \{b_1, \ldots, b_n\}$ and that $A = \{c_1, \ldots, c_n\}$. Hence, the minimal element b_j of each chain B_j is the maximal element $c_{\pi(j)}$ of some chain $C_{\pi(j)}$, and the union of the two chains is a chain $B_j \cup C_{\pi(j)}$ in poset P, as illustrated in Figure 5.6.13. The chains

$$B_1 \cup C_{\pi(1)}, \ \ldots, \ B_n \cup C_{\pi(n)}$$

are a partition of P.

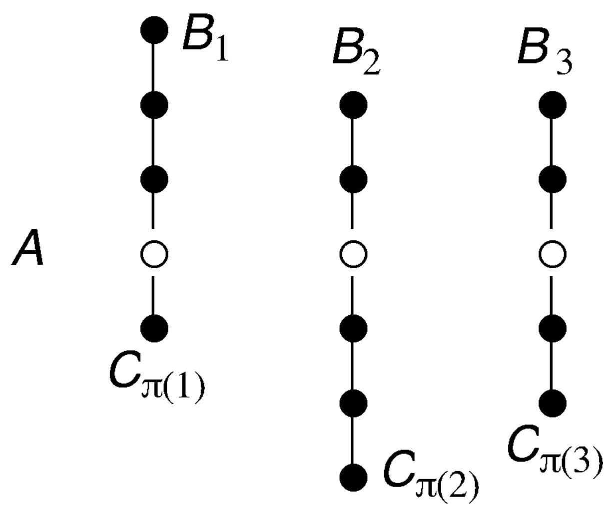

Fig 5.6.13 Partitioning poset P into 3 chains.

Case 2. Suppose, alternatively, that there are no antichains of maximum size n, except for either the set of all maximal elements of P or the set of all minimal elements of P (or both). In this case, let u be a minimal element and v a maximal element. Then the size of the largest antichain in the poset $P - \{u, v\}$ is $n-1$. By the ind hyp, the subposet $P - \{u, v\}$ can be partitioned into $n-1$ chains. These $n-1$ chains, along with the chain $\{u, v\}$ give a partition of poset P into n chains. $\diamondsuit$