Chapter 1

Sequences

There is often a finite way to represent an infinite sequence collectively. In particular, a

closed formula

is especially convenient. A

recursion rule

specifies later values in the sequence from earlier values.

1.1 SEQUENCES AS LISTS

In this section, we consider some common kinds of sequences and some of their attributes.

DEF: A ***sequence*** in a set S is a list of elements of that set S (called the ***range*** of the seq)

$$x_0 \quad x_1 \quad x_2 \quad \ldots$$

indexed by the non-negative integers, or sometimes by some other countable set.

NOTATION: Some of the most standard sets of numbers that serve as ranges for sequences are denoted here in *blackboard bold* typeface style:

$$\begin{aligned}
\mathbb{Z} &= \{\ldots, \quad -2, \quad -1, \quad 0, \quad 1, \quad \ldots\} \quad \text{integers} \\
\mathbb{Z}^+ &= \{1, \quad 2, \quad 3, \quad \ldots\} \quad \text{positive integers} \\
\mathbb{N} &= \{0, \quad 1, \quad 2, \quad \ldots\} \quad \text{natural numbers} \\
\mathbb{R} &= \text{real numbers} \\
\mathbb{Q} &= \text{rational numbers} \\
\mathbb{C} &= \text{complex numbers}
\end{aligned}$$

DEF: An algebraic expression in the argument n for the value of the general element x_n of a sequence $\langle x_n \rangle$ is called a ***closed formula*** for the (elements of the) sequence.

Example 1.1.1: The closed formula

$$x_n = n^3 - 5n$$

specifies the sequence

$$\langle x_n \rangle \ : \ 0 \quad -4 \quad -2 \quad 12 \quad 44 \quad 100 \quad 186 \quad \ldots$$

Example 1.1.2: The closed formula

$$y_n = 2^{n+2} - n^3$$

specifies the sequence

$$\langle y_n \rangle \ : \ 4 \quad 7 \quad 8 \quad 5 \quad 0 \quad 3 \quad 40 \quad \ldots$$

Fast-Growing Sequences

Rate of growth is understood in relation to the standard indexing sequence, i.e., the natural numbers.

Example 1.1.3: polynomial seq $\langle x_n = n^2 \rangle$

n	0	1	2	3	4	5	$\cdots$
n^2	0	1	4	9	16	25	$\cdots$

A polynomial sequence of degree greater than 1 grows more rapidly than the natural numbers.

Example 1.1.4: exponential seq $\langle x_n = 3^n \rangle$

n	0	1	2	3	4	5	$\cdots$
3^n	1	3	9	27	81	243	$\cdots$

Once a precise notion of comparative rate of growth is in hand in §1.4, it will be provable that any exponential sequence $\langle x_n = b^n \rangle$ with $b > 1$ "grows more rapidly" than any polynomial sequence. Of course, if $0 < b < 1$, then the sequence $\langle x_n = b^n \rangle$ decreases. For instance,

n	0	1	2	3	4	5	$\cdots$
$(1/3)^n$	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$	$\frac{1}{243}$	$\cdots$

Example 1.1.5: A seq that grows even more rapidly than an exponential sequence is the factorial sequence

n	0	1	2	3	4	5	$\cdots$
$n!$	1	1	2	6	24	120	$\cdots$

Slow-Growing Sequences

Various other increasing seqs grow slowly, relative to the integers. The first example here involves a fractional exponent. The second and third involve logarithms and *harmonic numbers.*

Example 1.1.6: A sequence $\langle x_n = n^r \rangle$ grows more slowly than $\mathbb{Z}^+$ if $0 < r < 1$. E.g.,

n	0	1	2	3	4	5	$\cdots$
$n^{1/2}$	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{5}$	$\cdots$

Example 1.1.7: The sequence

n	1	2	3	4	5	$\cdots$
$\lg n$	0	1	$\lg 3$	2	$\lg 5$	$\cdots$

grows even more slowly than the seq $\langle x_n = n^r \rangle$, for $r > 0$. (See Exercises.)

DEF: The ***harmonic number*** H_n is defined as the sum

$$\sum_{k=1}^{n} \frac{1}{k} \;=\; \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$$

with $H_0 = 0$ for the empty sum.

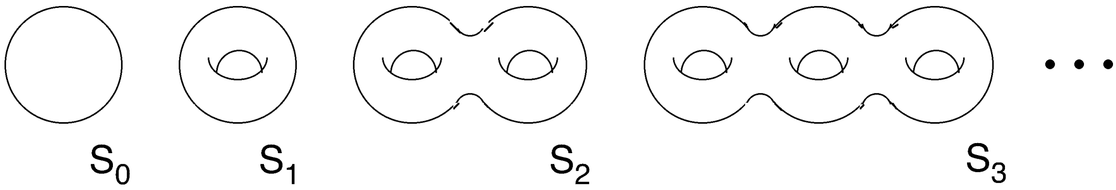
Example 1.1.8: The harmonic sequence

n	0	1	2	3	4	5	$\cdots$
H_n	0	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\cdots$

is closely related to the natural logarithm $\ln n$, as explained in §3.1.

Example 1.1.9: The values in a sequence need not be numbers.

PREVIEW OF §8.7: The surface S_g is the surface with g handles in the following sequence.



Bounded Sequences

DEF: A ***bounded sequence*** $\langle x_n \rangle$ is a sequence (typically of real numbers or integers) for which there is a number B (called a ***bound***), such that

$$|x_n| \leq B \qquad \text{for all } n$$

The sequence is bounded in *absolute* value.

Example 1.1.10: The real sequence

$$\left\langle x_n = 1 - \frac{1}{n+1} \right\rangle$$

is bounded. It is always non-negative, and its value never exceeds 1.

Periodic Sequences

DEF: A ***periodic sequence*** $\langle x_n \rangle$ is a sequence for which there is a positive integer P, such that

$$x_{j+P} = x_j \qquad \text{for all } j \in \mathbb{N}$$

The smallest such integer is called the ***period of the sequence***.

Example 1.1.11: An alternating sequence of 0's and 1's

$$0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad \cdots$$

is periodic with period 2.

DEF: The ***remainder function*** on a pair of integers $n \in \mathbb{N}$ and $d \in \mathbb{Z}^+$ is defined as

$$n \bmod d = n - d \left\lfloor \frac{n}{d} \right\rfloor$$

It is also called the ***mod function***. The arguments n and d are called the *dividend* and the *divisor*, respectively.

Example 1.1.12: The sequence $n \bmod 3$

n	0	1	2	3	4	5	$\cdots$
$n \bmod 3$	0	1	2	0	1	2	$\cdots$

is periodic with period 3. More generally, for any fixed divisor m, the sequence

$$\langle x_n = n \bmod m \rangle$$

is periodic with period m.

Generalizations

Example 1.1.13: The formula $x(n) = 3^n$ can also be regarded as a specification of the ***extended sequence*** $\langle 3^n \mid n \in \mathbb{Z} \rangle$:

n	$\cdots$	-2	-1	0	1	2	3	$\cdots$
3^n	$\cdots$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	$\cdots$

DEF: An ***array*** of dimension d in a set S is a function from the set of d-tuples of natural numbers to the set S.

NOTATION: Array elements are commonly written in the subscripted notation

$$\begin{matrix} x_{0,0} & x_{0,1} & x_{0,2} & \cdots \\ x_{1,0} & x_{1,1} & x_{1,2} & \cdots \\ x_{2,0} & x_{2,1} & x_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

DEF: The ***integer interval*** $[k : m]$ is the set

$$\{k,\ k+1,\ \ldots,\ m\}$$

The integer interval $[1 : n]$ is used as the standard set of cardinality n.

Eventual Behavior of Sequences

DEF: A sequence $\langle x_n \rangle$ ***eventually has property*** $\mathcal{P}$ if there is a number N such that the subsequence $\langle x_n \mid n > N \rangle$ has property $\mathcal{P}$.

Example 1.1.14: The sequence

$$\langle x_n \; = \; n^2 - 8n + 15 \rangle$$

is eventually increasing, as illustrated in Figure 1.1.1. Its shape is an upward parabola, with its minimum at $n = 4$, after which it is strictly increasing. Thus, it is eventually increasing.

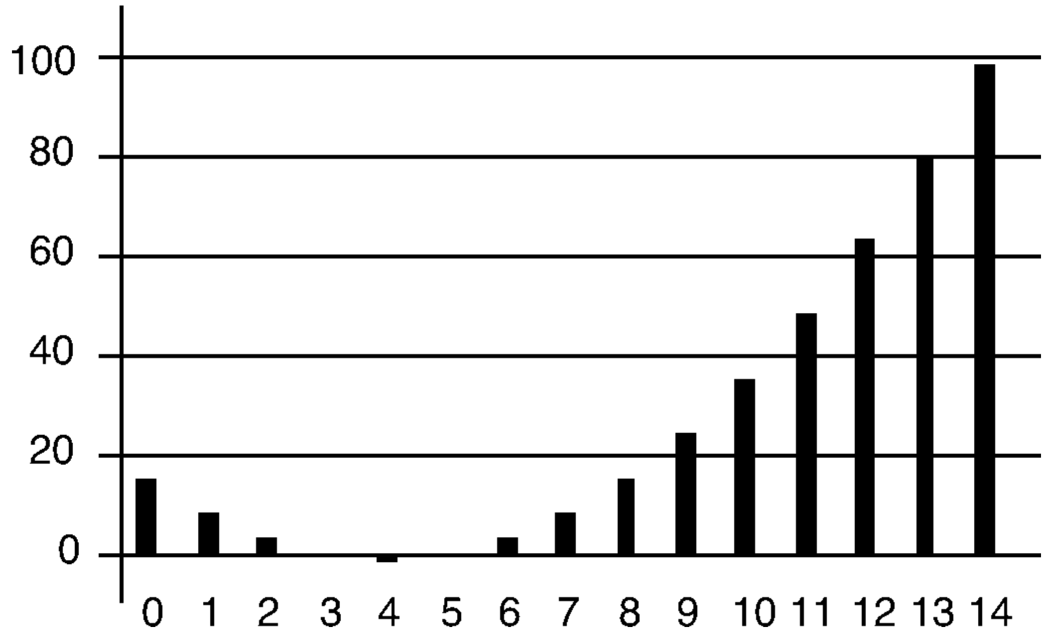


Fig 1.1.1 An eventually increasing sequence.

Example 1.1.15: The sequence $\langle x_n \; = \; 2n^3 - 2^n \rangle$ is eventually decreasing.

Remark: Every polynomial (except a constant) is eventually increasing or eventually decreasing, depending on the sign of its term of highest degree.

Example 1.1.16: The decimal digits of

$$\frac{4824}{8250} = 0.52412121212\ldots$$

are eventually periodic, as illustrated in Figure 1.1.2.

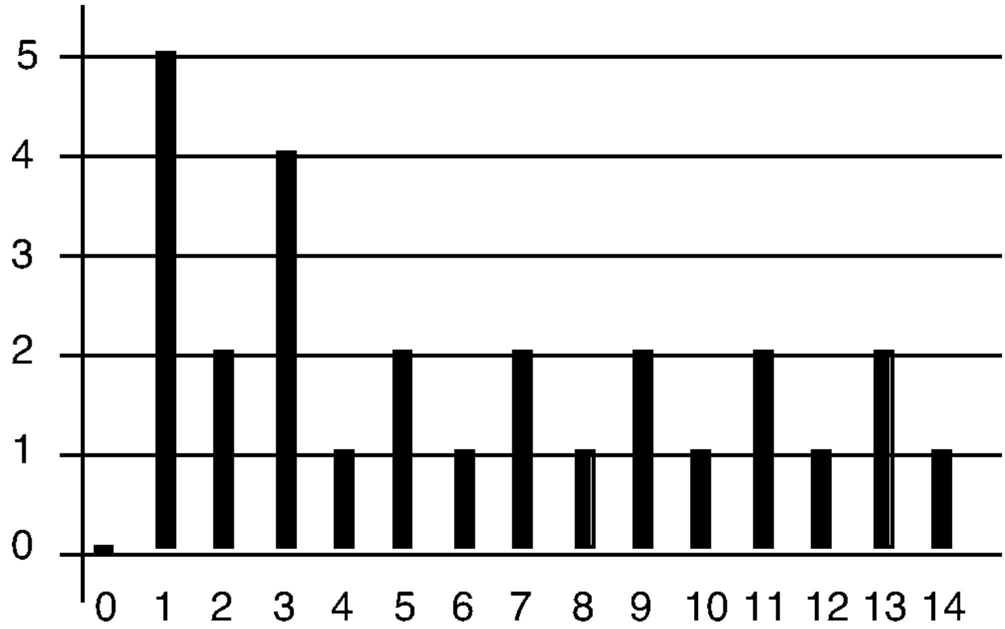


Fig 1.1.2 An eventually periodic sequence.

1.2 RECURRENCES

Most of the sequences considered in §1.1 were specified by a closed function $j \mapsto x_j$.

DEF: A ***standard recurrence*** for a sequence prescribes a set of ***initial values***

$$x_0 = b_0 \quad x_1 = b_1 \quad \ldots \quad x_k = b_k$$

and a ***recursion*** formula

$$x_n = \phi(x_{n-1}, x_{n-2}, \ldots, x_0) \quad \text{for } n > k$$

from which one may calculate the value of x_n, for any $n > k$, from the values of earlier entries.

Example 1.2.1: The recurrence

$$\begin{aligned} x_0 &= 0 && \text{initial value} \\ x_n &= x_{n-1} + 2n - 1 && \text{recursion} \end{aligned}$$

has as its first few values

$$x_0 = 0 \quad x_1 = 1 \quad x_2 = 4 \quad x_3 = 9 \quad x_4 = 16 \quad \ldots$$

We observe that the recursion formula here depends only on a fixed number of predecessors of x_n, specifically, only on x_{n-1}.

DEF: Inferring a closed formula from a recurrence is called ***solving the recurrence.***

Example 1.2.1, cont.: The first few values specified by the closed formula $x_n = n^2$ are

$$0^2 = 0 \quad 1^2 = 1 \quad 2^2 = 4 \quad 3^2 = 9 \quad 4^2 = 16 \quad \ldots$$

These coincide with those specified by the given recurrence

$$x_0 = 0 \quad x_1 = 1 \quad x_2 = 4 \quad x_3 = 9 \quad x_4 = 16 \quad \ldots$$

An induction can prove that $x_n = n^2$ solves the recurrence.

Method of Small Cases

Sometimes it is possible to guess the solution to a recurrence. More generally, the following approach goes a long way in mathematics.

1. Examine some small cases systematically.
2. Guess a pattern that covers all those cases.
3. Prove that the guess is correct.

We now describe recurrences and closed formulas for three well-known sequences: the *Tower of Hanoi sequence*, the *Fibonacci sequence*, and the *Catalan sequence*.

Tower of Hanoi

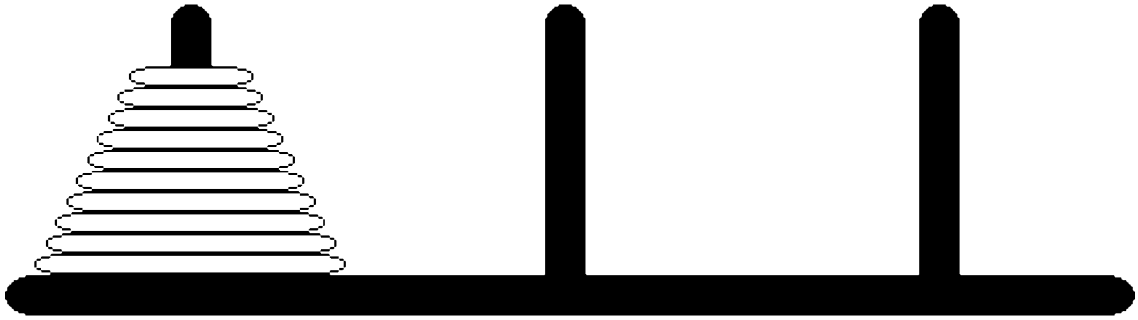


Fig 1.2.1 Tower of Hanoi puzzle.

(1) Only one disk may be transfered at a time.

(2) No disk may ever lie on top of a smaller disk.

The minimum number h_n of moves needed to transfer n disks satisfies the following recurrence:

RECURRENCE

$$\begin{aligned} h_0 &= 0 && \text{initial value} \\ h_n &= 2h_{n-1} + 1 && \text{recursion} \end{aligned}$$

We use the recursion to calculate the first few values of h_n, and then guess a closed formula.

SMALL CASES

$$\begin{aligned} h_0 &= 0 \\ h_1 &= 1 \\ h_2 &= 3 \\ h_3 &= 7 \\ h_4 &= 15 \end{aligned}$$

APPARENT PATTERN

$$h_n = 2^n - 1$$

Having guessed that $h_n = 2^n - 1$, we proceed to confirm the guess with a proof.

Thm 1.2.1. *The Tower of Hanoi recurrence*

$$h_0 = 0; \qquad h_n = 2h_{n-1} + 1 \;\; \textit{for } n \geq 1 \tag{1.2.1}$$

has the solution

$$h_n = 2^n - 1 \tag{1.2.2}$$

Proof: By induction.

BASIS: Applying the formula (1.2.2) yields the equation $h_0 = 2^0 - 1 = 1 - 1 = 0$, which agrees with the prescribed initial condition $h_0 = 0$.

IND HYP: Assume that $h_{n-1} = 2^{n-1} - 1$.

IND STEP: Starting with the recursion (1.2.1), we now complete the proof.

$$\begin{aligned} h_n &= 2h_{n-1} + 1 && \text{given recursion} \\ &= 2\left(2^{n-1} - 1\right) + 1 && \text{induction hypothesis} \\ &= 2^n - 2 + 1 \\ &= 2^n - 1 \end{aligned}$$

◊

Fibonacci Sequence

DEF: The ***Fibonacci sequence*** $\langle f_n \rangle$ is defined by the recurrence

$$\begin{array}{lll} f_0 = 0; \quad f_1 = 1 & \text{initial values} \\ f_n = f_{n-1} + f_{n-2} & \text{for } n \geq 2 \end{array} \tag{1.2.3}$$

Here are the first few entries:

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
f_n	0	1	1	2	3	5	8	13	21	34	$\cdots$

DEF: A ***Fibonacci number*** is any number that occurs in the Fibonacci sequence.

A closed formula for the Fibonacci recurrence is not easily guessed from the small cases. (But, once guessed, the solution is verifiable by a routine inductive proof.) See §2.5.

$$f_n = \frac{1}{\sqrt{5}}\left(\phi^n - \hat{\phi}^n\right) \tag{1.2.4}$$

$$\text{where } \phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

Example 1.2.2: For the time being, it is interesting to confirm an instance of the correctness of the formula (1.2.4) for the Fibo number f_n.

$$f_3 = \frac{1}{\sqrt{5}}\left(\frac{(1+\sqrt{5})^3}{8} - \frac{(1-\sqrt{5})^3}{8}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+3\sqrt{5}+15+5\sqrt{5}}{8}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-3\sqrt{5}+15-5\sqrt{5}}{8}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{6\sqrt{5}+10\sqrt{5}}{8}\right) = 2$$

Catalan Sequence

DEF: The ***Catalan sequence*** $\langle c_n \rangle$ is defined by the recurrence

$$\begin{aligned} c_0 &= 1; \\ c_n &= c_0 c_{n-1} + c_1 c_{n-2} + \cdots + c_{n-1} c_0 \end{aligned}$$

Here are the first few entries:

n	0	1	2	3	4	5	6	7	$\cdots$
c_n	1	1	2	5	14	42	132	429	$\cdots$

DEF: Any number that occurs in the Catalan sequence is called a ***Catalan number***.

Derivation of this closed formula appears in §4.4.

$$c_n = \frac{1}{n+1}\binom{2n}{n} \qquad (1.2.6)$$

Example 1.2.3: $c_3 = \frac{1}{4} \cdot \binom{6}{3} = \frac{20}{4} = 5.$

Proving Properties of Sequences

Proof that a sequence has some given property can be derived either from a closed formula or from a recursion, with the aid of mathematical induction. As an illustration, we consider the properties of *concavity* and *convexity.*

DEF: A sequence $\langle x_n \rangle$ is ***concave*** (on the integer interval $[a : b]$) if

$$x_n \geq \frac{x_{n-1} + x_{n+1}}{2} \qquad (\text{for } n = a+1, \ldots, b-1)$$

This means that the point (n, x_n) lies above the line segment joining the points $(n-1, x_{n-1})$ and $(n+1, x_{n+1})$ in the plane, as in Figure 1.2.2.

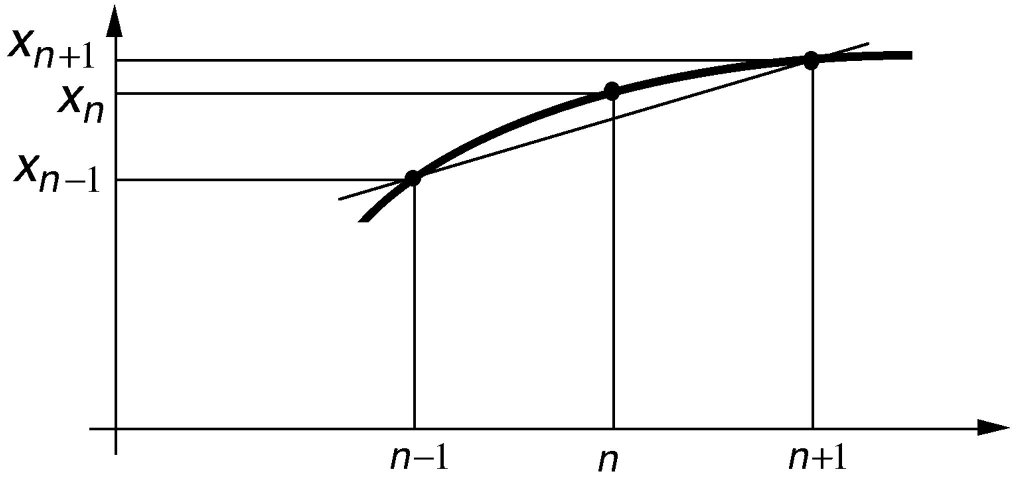


Fig 1.2.2 **Concavity in a sequence.**

Example 1.2.4: Concavity of the sequence

$$\langle x_n = 1 - \frac{1}{n} \rangle$$

follows from the observation that

$$\begin{aligned} 2x_n &= 2 - \frac{2}{n} = 2 - \frac{2n}{n^2} > 2 - \frac{2n}{n^2 - 1} \\ &= \left(1 - \frac{1}{n-1}\right) + \left(1 - \frac{1}{n+1}\right) = x_{n-1} + x_{n+1} \end{aligned}$$

Example 1.2.5: That the Fibonacci sequence $\langle f_n \rangle$ is eventually increasing, after $n = 2$, follows easily by mathematical induction. Moreover, it is a consequence for all $n > 3$ that $f_n < 2f_{n-1}$.

DEF: A sequence $\langle x_n \rangle$ is ***convex*** (on the integer interval $[a : b]$) if

$$x_n \leq \frac{x_{n-1} + x_{n+1}}{2} \qquad \text{for } n = a+1, \ldots, b-1$$

This means that the point (n, x_n) lies below the line segment joining the points $(n-1, x_{n-1})$ and $(n+1, x_{n+1})$ in the plane, as in Figure 1.2.3.

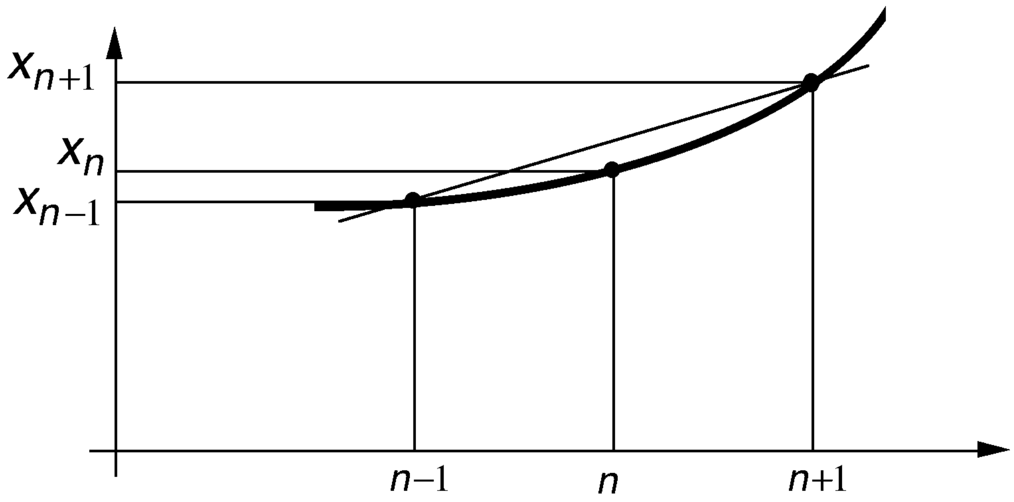


Fig 1.2.3 Convexity in a sequence.

Example 1.2.6: The Fibo seq is eventually convex, after $n = 2$. This is now confirmed:

$$\begin{aligned} f_{n+1} + f_{n-1} &= f_n + 2f_{n-1} && \text{(by the Fibo recursion)} \\ &\geq 2f_n && \text{(by Example 1.2.5)} \end{aligned}$$

which is equivalent to the defining condition for convexity.

1.3 PASCAL'S RECURRENCE

DEF: The ***combination coefficient*** $\binom{n}{k}$ is the number of ways (sometimes called *combinations*) to choose a subset of cardinality k from a set of n objects.

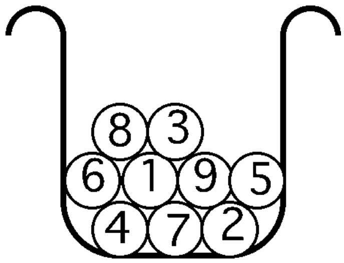


Fig 1.3.1 There are $\binom{9}{3}$ ways to choose 3 balls from the 9 in the urn.

Prop 1.3.1. *The combination coefficients* $\binom{n}{k}$ *satisfy* ***Pascal's recurrence***

$$(\mathrm{I}_1) \qquad \binom{n}{0} = 1 \qquad \textit{for all } n \geq 0$$

$$(\mathrm{I}_2) \qquad \binom{0}{k} = 0 \qquad \textit{for all } k \geq 1$$

$$(\mathrm{R}) \qquad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \text{for } n \geq 1$$

Proof: The initial values are easily confirmed. Our first proof for the recursion formula (R) is algebraic. The second is combinatorial.

Algebraic Proof: Algebraic proof of (R) starts with the right side of the equation and makes substitutions and arithmetic operations that result in the left side.

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-k-1)!} \\ &= \frac{k(n-1)!}{k!\,(n-k)!} + \frac{(n-k)(n-1)!}{k!\,(n-k)!} \\ &= \frac{[k+(n-k)](n-1)!}{k!\,(n-k)!} \\ &= \frac{n!}{k!\,(n-k)!} = \binom{n}{k} \qquad \diamondsuit\end{aligned}$$

Combinatorial Proof: The left side counts the number of ways to choose a subset of size k from the integer interval $[1:n]$. If such a subset includes object n, then it is counted by the summand $\binom{n-1}{k-1}$ on the right side. Alternatively, if such a subset excludes object n, then all k objects must be chosen from $[1:n-1]$, and it is counted by the summand $\binom{n-1}{k}$. This approach is called the ***Method of Distinguished Element***.

Binomial Coefficients

DEF: The coefficient $b_{n,k}$ of x^k in the expansion

$$(1+x)^n = \sum_{k=0}^{n} b_{n,k}\, x^k$$

is called a ***binomial coefficient***.

Example 1.3.1: Binomial coefficients can be calculated by iteratively multiplying by $1 + x$.

$$
\begin{aligned}
(1+x)^0 &= 1 \\
(1+x)^1 &= 1 + x \\
(1+x)^2 &= 1 + 2x + x^2 \\
(1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4
\end{aligned}
$$

Proposition 1.3.2. *The binomial coefficients $b_{n,k}$ satisfy Pascal's recurrence.*

Proof: The initial values of Pascal's recurrence are satisfied, since the values

$$
\begin{aligned}
b_{n,0} &= 1 \qquad \text{for all } n \geq 0 \\
b_{0,k} &= 0 \qquad \text{for all } k \geq 1
\end{aligned}
$$

can be verified by considering the direct expansions of $(1 + x)^0$ and $(1 + x)^n$, as in Example 1.3.1. To show that the recursion is satisfied, it is observed that

$$
\begin{aligned}
\sum_{k=0}^{n} b_{n,k}\, x^k &= (1+x) \sum_{k=0}^{n-1} b_{n-1,k}\, x^k \qquad (1.3.1) \\
&= \sum_{k=0}^{n-1} b_{n-1,k}\, x^k + x \sum_{k=0}^{n-1} b_{n-1,k}\, x^k \\
&= \sum_{k=0}^{n-1} b_{n-1,k}\, x^k + \sum_{k=0}^{n-1} b_{n-1,k}\, x^{k+1}
\end{aligned}
$$

$$= \sum_{k=0}^{n} b_{n-1,k}\, x^k \;+\; \sum_{k=1}^{n} b_{n-1,k-1}\, x^k$$

$$= \sum_{k=0}^{n} (b_{n-1,k} \;+\; b_{n-1,k-1})\, x^k \qquad (1.3.2)$$

Thus, the coefficient $b_{n,k}$ of x^k in the sum at the left of equation (1.3.1) must equal the coefficient of x^k in the sum at the right in equation (1.3.2), i.e., it must equal the sum

$$b_{n-1,k} \;+\; b_{n-1,k-1} \qquad \diamondsuit$$

Cor 1.3.3. *For all $n, k \geq 0$, the number $\binom{n}{k}$ of ways to choose k objects from a set of n distinct objects equals the binomial coefficient $b_{n,k}$.*

Proof: By Prop 1.3.2, the combination coefficients $\binom{n}{k}$ and the binomial coefficients $b_{n,k}$ satisfy the exact same recurrence system. An induction argument establishes that the values must be the same. $\diamondsuit$

TERMINOLOGY NOTE: The number $\binom{n}{k}$ is commonly called a ***binomial coefficient.*** From here on in this book, we shall refer to it as such.

DEF: If the zero values are left blank, then the array of binomial coefficients has a triangular shape and is called ***Pascal's triangle***.

Table 1.3.1 Pascal's △ for values of $\binom{n}{r}$.

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	Σ
0	1							1
1	1	1						2
2	1	2	1					4
3	1	3	3	1				8
4	1	4	6	4	1			16
5	1	5	10	10	5	1		32
6	1	6	15	20	15	6	1	64

In this form of Pascal's triangle, each number is the sum of the number directly above it and the number in the row above, one column to the left. Pascal's triangle also has a pyramid form:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

1.4 DIFFERENCES AND SUMS

DEF: Given a sequence $\langle a_n \rangle$, we define the ***difference sequence*** $\langle \triangle a_n \rangle$ by the rule

$$\triangle a_n \; = \; a_{n+1} - a_n$$

More generally, given a function $f : \mathbb{R} \to \mathbb{R}$, we define the ***difference function*** $\triangle f$ by the rule

$$\triangle f(x) = f(x+1) - f(x)$$

Example 1.4.1: If $a_n = n^2$, then

$$\triangle a_n \; = \; (n+1)^2 - n^2 \; = \; 2n+1$$

and

$$\triangle^{(2)} a_n \; = \; (2(n+1)+1) - (2n+1) \; = \; 2$$

These equations yield this ***difference table***.

$a_n = \mathbf{n^2}$	0	1	4	9	16	25	36	49	$\cdots$
$\triangle a_n$	1	3	5	7	9	11	13	$\cdots$	
$\triangle^{(2)} a_n$	2	2	2	2	2	2	$\cdots$		

Example 1.4.2: The sequence $\langle\, b_n \; = \; n^3 \,\rangle$ has the difference table, which was created by calculating its initial row and then iteratively taking differences.

$b_n = \mathbf{n^3}$	0	1	8	27	64	125	216	343	$\cdots$
$\triangle b_n$	1	7	19	37	61	91	127	$\cdots$	
$\triangle^{(2)} b_n$	6	12	18	24	30	36	$\cdots$		
$\triangle^{(3)} b_n$	6	6	6	6	6	$\cdots$			

Properties of the Difference Function

In Examples 1.4.1 and 1.4.2, we observe that the second and third rows of the difference tables for the sequences $\langle n^2 \rangle$ and $\langle n^3 \rangle$, respectively, have the constant values $2 = 2!$ and $6 = 3!$. An initial aspect of our exploration is to establish that this phenomenon holds generally.

Proposition 1.4.1. *The difference operator $\triangle$ is linear. That is,*

$$\triangle\,(f(n) + cg(n)) \;=\; (\triangle f)(n) + c(\triangle g)(n)$$

Proof: The details are straightforward.

$$\begin{aligned}\triangle\,(f(n) + cg(n)) &= (f(n+1) + cg(n+1)) \;-\; (f(n) + cg(n)) \\ &= (f(n+1) - f(n)) \;+\; c(g(n+1) - g(n)) \\ &= (\triangle f)(n) + c\,(\triangle g)(n) \qquad \diamondsuit\end{aligned}$$

Prop 1.4.2. *In the difference table for the seq*

$$\langle n^r \mid n \in \mathbb{N} \rangle$$

the r^{th} row has the constant value $r!$, and, accordingly, all subsequent rows are null.

Proof: By induction.

BASIS: The entries in the 0^{th} row of the sequence $\langle n^0 \rangle$ all have the value $1 = 0!$.

IND HYP: Assume that all the entries in the $(r-1)^{\text{st}}$ row of the table for n^{r-1} have the value $(r-1)!$ and that all higher order rows are null.

IND STEP: It follows from the expansion

$$\triangle(n^r) = (n+1)^r - n^r = rn^{r-1} + b_{r-2}n^{r-2} + \cdots + b_0$$

(for appropriate coefficients b_j) and from the linearity of $\triangle$ that

$$\begin{aligned}\triangle^{(r)}(n^r) &= \triangle^{(r-1)}(\triangle(n^r)) \\ &= \triangle^{(r-1)}(rn^{r-1} + b_{r-2}n^{r-2} + \cdots + b_0) \\ &= r\,\triangle^{(r-1)}\,(n^{r-1}) + \sum_{j=0}^{r-2} b_j\,\triangle^{(r-1)}\,(n^j)\end{aligned}$$

By the induction hypothesis, $\triangle^{(r-1)}(n^j) = 0$, for $j \leq r-2$, from which it follows that every term in the sum on the right has value 0. Thus,

$$\triangle^{(r)}(n^r) = r\,\triangle^{(r-1)}\,(n^{r-1})$$

It follows that the r^{th} row of the difference table for $\langle n^r \rangle$ equals r times the $(r-1)^{\text{st}}$ row of the table for n^{r-1}, in which every entry has the value $(r-1)!$, by the induction hypothesis. ♢

Summation Operator

DEF: Let $\langle x_n \rangle$ be a sequence with values in an algebraic structure with an addition. Then the expression

$$\sum_{j=0}^{n} x_j$$

is called the n^{th} ***partial sum.***

DEF: The ***summation operator*** maps a sequence $\langle\, x_n \mid n \in \mathbb{N}\,\rangle$ to the sequence of partial sums

$$\left\langle \sum_{j=0}^{n} x_j \;\middle|\; n \in \mathbb{N} \right\rangle$$

Example 1.4.3: Under the summation operator, the integer sequence

$$\langle x_n \rangle \;=\; 1 \quad 3 \quad 5 \quad 7 \quad \ldots$$

is mapped to the integer sequence of its n^{th} partial sums

$$\left\langle u_n \;=\; \sum_{j=0}^{n} (2j+1) \right\rangle$$

which begins with the values

$$1 \quad 4 \quad 9 \quad 16 \quad \ldots$$

It may be guessed that $u_n = (n+1)^2$, which is readily proved by induction. If one now defines

$$a_n = \sum_{j=0}^{n-1} (2j+1)$$

then the sequence $\langle a_n = n^2 \rangle$ has the values

$$0 \quad 1 \quad 4 \quad 9 \quad 16 \quad \ldots$$

which inverts Example 1.4.1. We recall from §0.3 that the empty sum is defined to be zero. This accounts for the value

$$a_0 = \sum_{j=0}^{n-1} (2j+1) = \sum_{j=0}^{-1} (2j+1) = 0$$

The next theorem establishes that the inversion is not at all a coincidence.

NOTATION: From time to time, it is convenient to use the notation x_j' as an alternative to $\triangle x_j$.

Thm 1.4.3(a). *Let* $\langle x_n \mid n \in \mathbb{N} \rangle$ *be a seq. Then*

$$\sum_{j=0}^{n-1} x_j' = x_n - x_0 \tag{1.4.1}$$

Proof: Another straightforward calculation.

$$\begin{aligned}\sum_{j=0}^{n-1} x_j' &= \sum_{j=0}^{n-1} (x_{j+1} - x_j) \\ &= (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \\ &= (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - x_0) \\ &= x_n - x_0 \qquad \diamondsuit\end{aligned}$$

The upper limit of the sum in equation (1.4.1) must be $n-1$, rather than n, to get the correct result. Figure 1.4.1 illustrates the proof of Theorem 1.4.3(a). The sum

$$x_0 + \sum_{j=0}^{3} x_j'$$

of the lengths along the y-axis clearly equals the height x_4 of the rightmost rectangle. Thus,

$$x_4 - x_0 = \sum_{j=0}^{3} x_j'$$

which is the total vertical distance from the top of the leftmost rectangle to the top of the rightmost rectangle.

$$(x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + (x_4 - x_3) \;=\; x_4 - x_0$$

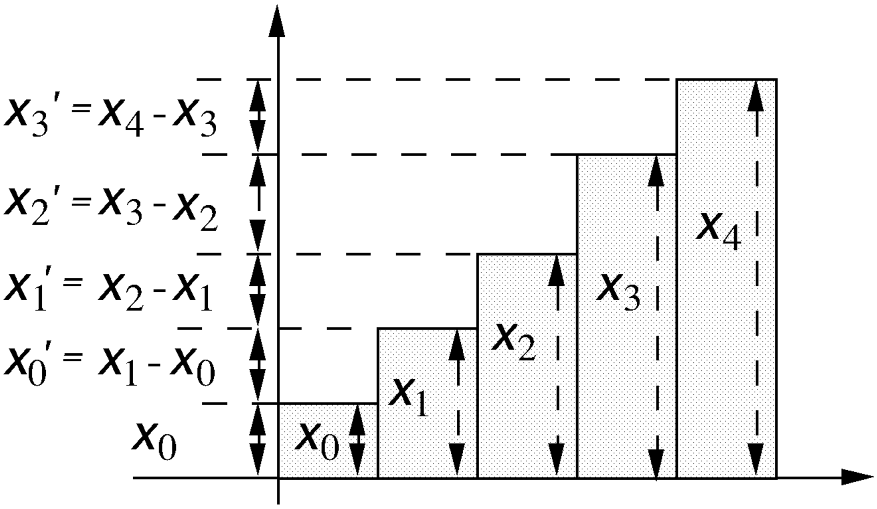


Fig 1.4.1 Accumulating consecutive differences, as in Theorem 1.4.3(a).

Theorem 1.4.3(b). *Let* $\langle x_n \mid n \in \mathbb{N}\rangle$ *be a sequence. Then*

$$\left(\sum_{j=0}^{k-1} x_j\right)'_n \;=\; x_n$$

Proof: By the def of the difference operator,

$$\left(\sum_{j=0}^{k-1} x_j\right)'_n \;=\; \sum_{j=0}^{(n+1)-1} x_j \;-\; \sum_{j=0}^{n-1} x_j$$

$$=\; x_n \qquad \diamondsuit$$

Fig 1.4.2 illustrates the proof of Thm 1.4.3(b).

$$(x_0 + x_1 + x_2 + x_3 + x_4) - (x_0 + x_1 + x_2 + x_3) \quad = \quad x_4$$

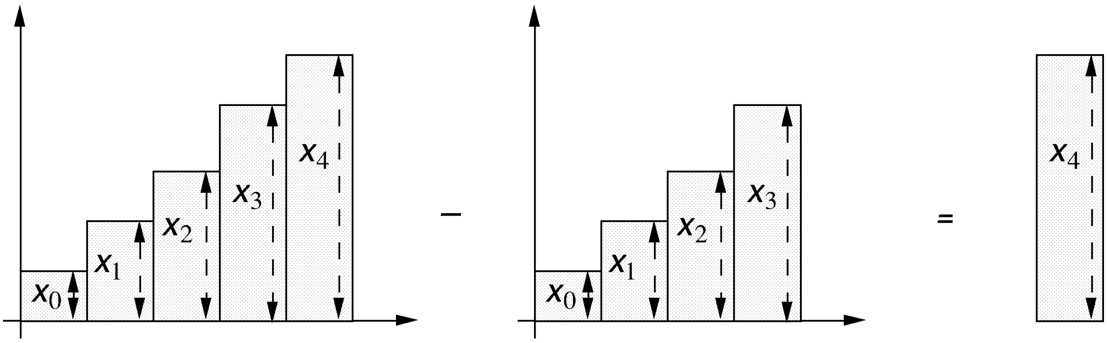


Fig 1.4.2 Subtracting consecutive sums, as in Theorem 1.4.3(b).

The difference of the sum $x_0 + \cdots + x_4$ of the areas of the consecutive rectangle including x_4 and the sum $x_0 + \cdots + x_3$ of the areas excluding x_4 clearly equals the area x_4 of the rightmost rectangle.

TERMINOLOGY: Thm 1.4.3 is a form of what is commonly called the ***Fundamental Theorem of Finite Calculus***. One sees a direct analogy to the Fundamental Theorem of Infinitessimal Calculus:

$$(a) \quad \int_0^x \tfrac{d}{dt} f(t)\, dt = f(x) - f(0);$$

$$(b) \quad \tfrac{d}{dx} \int_0^x f(t)\, dt = f(x)$$

Growth Rate of Sequences

The most common criterion for comparing the long term behavior of two sequences is called *asymptotic dominance.* However, by way of analogy to differential calculus, a possible measure of the growth rate of a seq is its difference seq.

Example 1.4.4: To establish, in the sense of finite differences, that the sequence $\langle n^3 \rangle$ grows faster than the sequence $\langle cn^2 \rangle$, for any constant value of c, we make the following calculations.

$$\begin{aligned} \triangle n^3 &= (n+1)^3 - n^3 = 3n^2 + 3n + 1 \\ \triangle cn^2 &= c(n+1)^2 - cn^2 = 2cn + c \end{aligned}$$

For $n > c$, we have

$$3n^2 + 3n > 3cn + 3c > 2cn + c$$

Thus, $\triangle n^3$ eventually dominates $\triangle cn^2$.

Another possible measure of the growth rate of a sequence of positive values is the sequence of ratios

$$\left\langle \frac{x_{n+1}}{x_n} \,\middle|\, n \in \mathbb{Z}^+ \right\rangle$$

of consecutive terms.

Example 1.4.4, cont.: The successive ratios of n^3 are

$$\frac{(n+1)^3}{n^3} = \frac{n^3 + 3n^2 + 3n + 1}{n^3} = 1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}$$

and the successive ratios of cn^2 are

$$\frac{c(n+1)^2}{cn^2} = \frac{cn^2 + 2cn + c}{cn^2} = 1 + \frac{2}{n} + \frac{1}{n^2}$$

which are clearly smaller.

1.5 FALLING POWERS

REVIEW FROM §0.2: The n^{th} ***falling power*** of a real number x is the product

$$x^{\underline{n}} = \overbrace{x\,(x-1)\;\cdots\;(x-n+1)}^{n \text{ factors}} \qquad \text{for } n \in \mathbb{N}$$

We recall that the differential calculus has nice formulas:

$$\frac{d}{dx}x^2 = 2x \qquad \frac{d}{dx}x^3 = 3x^2 \quad \text{etc.}$$

So does the calculus of finite differences, but these are *not* examples of them:

$$\begin{aligned} \triangle(x^2) &= 2x + 1 \\ \triangle(x^3) &= 3x^2 + 3x + 1 \end{aligned}$$

In the calculus of finite differences, the falling monomial $x^{\underline{n}}$ lends itself quite naturally to nice formulas that are analogous to those of the ordinary monomial x^n.

Example 1.5.1: A "nice formula".

$$\begin{aligned} \triangle(x^{\underline{3}}) &= (x+1)^{\underline{3}} - x^{\underline{3}} \\ &= (x+1)\,x\,(x-1) - x\,(x-1)(x-2) \\ &= [(x+1) - (x-2)]\,x\,(x-1) \\ &= 3x(x-1) = 3x^{\underline{2}} \end{aligned}$$

Now generalize Example 1.5.1.

Theorem 1.5.1. $\triangle(x^{\underline{r}}) = rx^{\underline{r-1}}$.

Proof: A straightforward approach.

$$\begin{aligned}\Delta(x^{\underline{r}}) &= (x+1)^{\underline{r}} - x^{\underline{r}} \\ &= (x+1)\,x^{\underline{r-1}} - x^{\underline{r-1}}(x-r+1) \\ &= [(x+1) - (x-r+1)]\,x^{\underline{r-1}} = rx^{\underline{r-1}}\end{aligned}$$

◊

Corollary 1.5.2. *For every non-negative integer r and every positive integer n,*

$$\sum_{j=0}^{n-1} j^{\underline{r}} = \frac{n^{\underline{r+1}}}{r+1}$$

Proof: We have $j^{\underline{r}} = \triangle\left(\frac{j^{\underline{r+1}}}{r+1}\right)$ by Theorem 1.5.1. Thus, by the Fundamental Theorem of Finite Calculus, it follows that

$$\sum_{j=0}^{n-1} j^{\underline{r}} = \left.\frac{j^{\underline{r+1}}}{r+1}\right|_{j=0}^{n} = \frac{n^{\underline{r+1}}}{r+1}$$

◊

Example 1.5.2: Direct addition and the formula of Cor 1.5.2 give the same result when summing $k^{\underline{2}}$.

$$\sum_{k=0}^{4} k^{\underline{2}} = 0\cdot(-1) + 1\cdot 0 + 2\cdot 1 + 3\cdot 2 + 4\cdot 3 = 20$$

$$\frac{5^{\underline{3}}}{3} = \frac{5\cdot 4\cdot 3}{3} = 20$$

Unimodal Sequences

DEF: A sequence $\langle x_n \rangle$ is ***unimodal*** if there is an index M such that

$$x_0 \leq x_1 \leq \cdots \leq x_{M-1} \leq x_M$$

and that $\langle x_n \rangle$ is non-increasing after index M. The value x_M is called the ***mode*** and M the ***mode index***. (A tie is permitted at the mode value.)

Example 1.5.3: Most of the unimodal seqs of interest in the present context are eventually 0. Figure 1.5.1 illustrates that the sequence $\binom{8}{r}$ is unimodal.

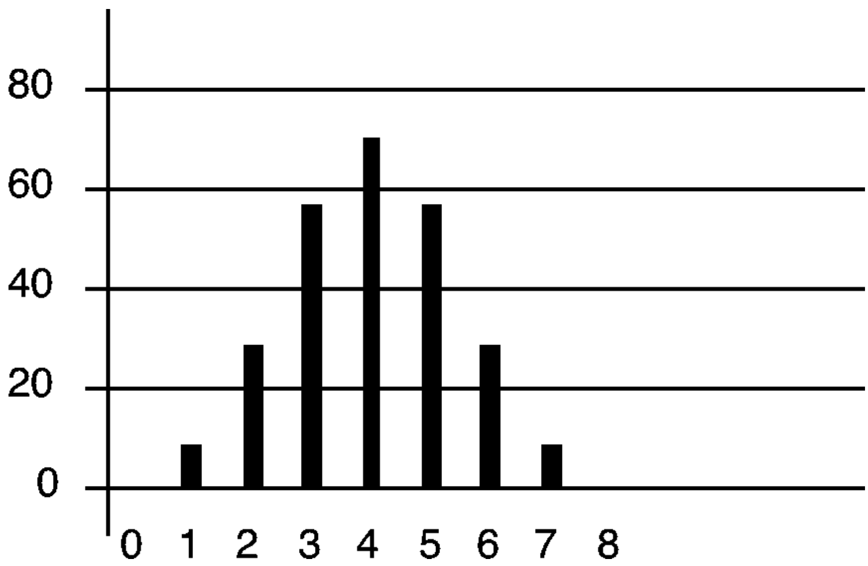


Fig 1.5.1 The unimodal sequence $\binom{8}{r}$.

Thm 1.5.3. *For any fixed non-negative integer n, the binomial sequence*

$$\left\langle \binom{n}{r} \;\middle|\; r = 0, 1, \ldots \right\rangle$$

is unimodal with mode index $\lfloor n/2 \rfloor$ and is eventually 0.

Proof: We observe that for $r \le \left\lfloor \frac{n}{2} \right\rfloor$, we have $\frac{n}{r} \ge 2$, and, hence, $\frac{n+1}{r} > 2$.
Thus, $\frac{n-r+1}{r} > 1$. Accordingly,

$$\begin{aligned} \binom{n}{r-1} &= \frac{n^{\underline{r-1}}}{(r-1)!} \\ &< \frac{n^{\underline{r-1}}}{(r-1)!} \cdot \frac{n-r+1}{r} \\ &= \frac{n^{\underline{r}}}{r!} = \binom{n}{r} \end{aligned}$$

Moreover, for $r \ge \left\lfloor \frac{n}{2} \right\rfloor$, we have $n \le 2r+1$, and it follows that $n - r \le r + 1$. Thus, $\frac{n-r}{r+1} \le 1$. It follows that

$$\binom{n}{r+1} = \frac{n^{\underline{r+1}}}{(r+1)!} = \frac{n^{\underline{r}}}{r!} \cdot \frac{n-r}{r+1} \le \frac{n^{\underline{r}}}{r!} = \binom{n}{r}$$

Of course, the sequence is zero for $r > n$. ♢

Remark: Unimodality of a sequence may make it possible to find the maximum by hill-climbing, for which there exist highly efficient computational strategies.

Log-Concavity and Log-Convexity

In trying to establish unimodality, curiously enough, it is often easier to prove the stronger property called *log-concavity.*

DEF: A sequence $\langle x_n \rangle$ of positive real numbers is ***log-concave*** (on the integer interval $[a:b]$) if, for $n = a+1, \ldots, b-1$,

$$\log x_n \;\geq\; \frac{\log x_{n-1} + \log x_{n+1}}{2} \qquad (1.5.1)$$

and is ***log-convex*** if

$$\log x_n \;\leq\; \frac{\log x_{n-1} + \log x_{n+1}}{2} \qquad (1.5.2)$$

Prop 1.5.4. *A sequence $\langle x_n \rangle$ of positive real numbers is* log-concave *(on the integer interval $[a:b]$) if and only if, for $n = a+1, \ldots, b$,*

$$x_n^2 \;\geq\; x_{n-1}x_{n+1} \qquad (1.5.3)$$

It is log-convex if and only if

$$x_n^2 \;\leq\; x_{n-1}x_{n+1} \qquad (1.5.4)$$

Proof: The defining condition (1.5.1) for log-concavity

$$\log x_n \;\geq\; \frac{\log x_{n-1} + \log x_{n+1}}{2}$$

is equivalent to the inequality

$$2\log x_n \;\geq\; \log x_{n-1} + \log x_{n+1} \tag{1.5.5}$$

Exponentiating both sides of inequality (1.5.5) leads to inequality (1.5.3), i.e.,

$$x_n^2 \;\geq\; x_{n-1}x_{n+1}$$

A similar argument establishes the equivalence of inequalities (1.5.2) and (1.5.4). ♢

Thm 1.5.5. *Let $\langle x_n \rangle$ be a log-concave sequence (over the integer interval $[a:b]$). Then it is unimodal (over that integer interval).*

Proof: It follows from Proposition 1.5.4 that the sequence of ratios

$$\frac{x_1}{x_0} \quad \frac{x_2}{x_1} \quad \frac{x_3}{x_2} \quad \cdots$$

(wherever defined) is non-increasing. That is,

$$x_n^2 \;\geq\; x_{n-1}x_{n+1} \quad \Rightarrow \quad \frac{x_n}{x_{n-1}} \;\geq\; \frac{x_{n+1}}{x_n}$$

Let M be the largest number k in the integer interval $[a:b]$ such that

$$\frac{x_k}{x_{k-1}} \;>\; 1$$

or $M = a$ if no such number k exists. Then the initial subsequence

$$x_a \quad x_{a+1} \quad \ldots \quad x_M$$

is increasing and the terminal subsequence

$$x_M \quad x_{M+1} \quad \ldots \quad x_b$$

is non-increasing, precisely the conditions for unimodality with mode index M. ♢

Theorem 1.5.6. *The binomial sequence*

$$\left\langle \binom{n}{r} \,\middle|\, r = 0, 1, \ldots \right\rangle$$

is log-concave on the integer interval $[0 : n]$.

Proof: The falling-power formula for binomial coeffs is

$$\binom{n}{r}^2 = \left(\frac{n^{\underline{r}}}{r!}\right)^2$$

Since $\dfrac{r}{r+1} < 1$ and $\dfrac{n-r}{n-r+1} < 1$, it follows that

$$\left(\frac{n^{\underline{r}}}{r!}\right)^2 > \frac{n^{\underline{r}}}{r!} \cdot \frac{n^{\underline{r}}}{r!} \cdot \frac{r}{r+1} \cdot \frac{n-r}{n-r+1}$$

and, in turn, that

$$\begin{aligned} \binom{n}{r}^2 &> \frac{n^{\underline{r}}}{r!} \cdot \frac{n^{\underline{r}}}{r!} \cdot \frac{r}{r+1} \cdot \frac{n-r}{n-r+1} \\ &= \frac{n^{\underline{r-1}}}{(r-1)!} \cdot \frac{n^{\underline{r+1}}}{(r+1)!} = \binom{n}{r-1} \cdot \binom{n}{r+1} \end{aligned}$$

Accordingly, by Proposition 1.5.4, the binomial sequence is log-concave. ♢

Remark: Theorems 1.5.6 and 1.5.5 can be used together to reconfirm Theorem 1.5.3, that the sequence of binomial coefficients $\binom{n}{k}$, for $k = 0, \ldots, n$, is unimodal.

1.6 STIRLING NUMBERS: PREVIEW

Stirling numbers are highly useful in counting partitions and permutations.

Falling into Ordinary Powers

The following theorem provides a recursive method for converting a falling power into ordinary powers.

Theorem 1.6.1. *Any falling power $x^{\underline{n}}$ can be expressed as a linear combination of ordinary powers, i.e., in the form*

$$x^{\underline{n}} = \sum_{k=0}^{n} s_{n,k}\, x^k \text{ with } s_{n,n} = 1 \text{ and } s_{n,0} = 0 \text{ for } n \geq 1$$

Proof: By induction on the exponent n.

BASIS: For $n = 0$ and $n = 1$, we have

$$\begin{aligned} x^{\underline{0}} &= 1x^0 \\ x^{\underline{1}} &= 1x^1 + 0x^0 \end{aligned}$$

Thus, we take $s_{0,0} = 1$, $s_{1,1} = 1$, and $s_{1,0} = 0$.

IND HYP: Suppose for some $n > 1$ that there exist integer coefficients

$$s_{n-1,0} \quad s_{n-1,1} \quad \cdots \quad s_{n-1,n-1}$$

for which

$$x^{\underline{n-1}} = \sum_{k=0}^{n-1} s_{n-1,k}\, x^k \text{ with } s_{n-1,n-1} = 1 \text{ and } s_{n-1,0} = 0$$

IND STEP: It follows that

$$\begin{aligned}
x^{\underline{n}} &= x^{\underline{n-1}} \cdot (x - n + 1) && \text{(def of falling power } x^{\underline{n}}\text{)} \\
&= (x - n + 1) \sum_{r=0}^{n-1} s_{n-1,k}\, x^k && \text{(ind hyp)} \\
&= x \sum_{k=0}^{n-1} s_{n-1,k}\, x^k \;-\; (n-1) \sum_{k=0}^{n-1} s_{n-1,k}\, x^k \\
&= -(n-1) s_{n-1,0}\, x^0 \;+\; \sum_{k=1}^{n-1} \left(s_{n-1,k-1} - (n-1) s_{n-1,k}\right) x^k \\
&\qquad + \; s_{n-1,n-1} x^n \\
&= 0\, x^0 \;+\; \sum_{k=1}^{n-1} \left(s_{n-1,k-1} - (n-1) s_{n-1,k}\right) x^k \;+\; 1\, x^n
\end{aligned}$$

Thus, we may take $s_{n,0} = 0$, $s_{n,n} = 1$, and $s_{n,k} = s_{n-1,k-1} - (n-1)s_{n-1,k}$, for $0 < k < n$. ♢

DEF: The coefficients $s_{n,k}$ in the summation

$$x^{\underline{n}} = \sum_{k=0}^{n} s_{n,k}\, x^k$$

are called ***Stirling numbers of the first kind***. For $k > n$ or $k < 0$, the Stirling number $s_{n,k}$ is taken to be 0, corresponding to letting the upper and lower limits of the sum go to ∞.

The Stirling numbers $s_{n,k}$ can be calculated by multiplying the factors in the expansion

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$$

Example 1.6.1:

$$\begin{aligned} x^{\underline{2}} &= x^2 - x^1 \\ x^{\underline{3}} &= x^3 - 3x^2 + 2x^1 \\ x^{\underline{4}} &= x^4 - 6x^3 + 11x^2 - 6x^1 \\ x^{\underline{5}} &= x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1 \end{aligned}$$

Thus, $s_{5,2} = -50$ and $s_{3,1} = 2$. We observe the alternating signs in each equation.

Ordinary into Falling Powers

Expressing an ordinary power as a sum of falling powers is an analogous task.

Thm 1.6.2. *Any ordinary power x^n can be expressed as a linear combination of falling powers, i.e., in the form*

$$x^n \;=\; \sum_{k=0}^{n} S_{n,k}\, x^{\underline{k}} \quad \textit{with} \quad S_{n,n} = 1 \textit{ and } S_{n,0} = 0 \textit{ for } n \geq 1$$

Proof: Once again, we use induction on the exponent n.

BASIS: For $n = 0$ and $n = 1$, we have

$$\begin{aligned} x^0 &= 1x^{\underline{0}} \\ x^1 &= 1x^{\underline{1}} + 0x^{\underline{0}} \end{aligned}$$

We take $S_{0,0} = 1$, $S_{1,1} = 1$, and $S_{1,0} = 0$.

IND HYP: Suppose that for some $n > 1$, the monomial x^{n-1} can be expressed as a linear combination

$$x^{n-1} \;=\; \sum_{k=0}^{j} S_{j,k}\, x^{\underline{k}}$$

of falling-power monomials $S_{j,k}\, x^{\underline{k}}$, each of degree less than or equal to j.

IND STEP: Then

$$
\begin{aligned}
x^n &= x \cdot x^{n-1} \\
&= x \cdot \sum_{k=0}^{n-1} S_{n-1,k}\, x^{\underline{k}} \qquad \text{(inductive hypothesis)} \\
&= \sum_{k=0}^{n-1} S_{n-1,k}\, x \cdot x^{\underline{k}} \\
&= \sum_{k=0}^{n-1} S_{n-1,k}\, (x-k) \cdot x^{\underline{k}} \; + \; \sum_{k=0}^{n-1} S_{n-1,k}\, k \cdot x^{\underline{k}} \\
&= \sum_{k=0}^{n-1} S_{n-1,k}\, x^{\underline{k+1}} \; + \; \sum_{k=0}^{n-1} k\, S_{n-1,k}\, x^{\underline{k}} \\
&= \sum_{k=1}^{n} S_{n-1,k-1}\, x^{\underline{k}} \; + \; \sum_{k=0}^{n-1} k\, S_{n-1,k}\, x^{\underline{k}} \\
&= S_{n-1,n-1}\, x^{\underline{n}} \; + \; \sum_{k=1}^{n-1} (S_{n-1,k-1} + k\, S_{n-1,k})\, x^{\underline{k}} \; + \; 0\, S_{n-1,0}\, x^{\underline{0}}
\end{aligned}
$$

Thus, we may take

$$S_{n,0} = 0, \;\; S_{n,n} = 1, \;\; \text{and} \;\; S_{n,k} = s_{n-1,k-1} + kS_{n-1,k}$$

for $0 < k < n$. ♢

DEF: The coefficients $S_{n,k}$ in the sum

$$x^n = \sum_{k=0}^{n} S_{n,k}\, x^{\underline{k}}$$

are called ***Stirling numbers of the second kind***. For $k > n$ or $k < 0$, the Stirling number $S_{n,k}$ is 0, which corresponds to letting the upper and lower limits of the sum go to ∞.

Example 1.6.2:

$$\begin{aligned} x^2 &= x^{\underline{2}} + x^{\underline{1}} \\ x^3 &= x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}} \\ x^4 &= x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} \\ x^5 &= x^{\underline{5}} + 10x^{\underline{4}} + 25x^{\underline{3}} + 15x^{\underline{2}} + x^{\underline{1}} \end{aligned}$$

Thus, $S_{5,3} = 25$ and $S_{4,2} = 7$.

Corollary 1.5.2 provides a simple formula for the sum of the values of any falling power $n^{\underline{r}}$, over an interval of integer values of the base n. Accordingly, due to the linearity of the difference operator (Proposition 1.4.1), we could calculate the sum of the values of any ordinary power n^r, over a range of values of n, if we first express n^r as a linear combination of falling powers.

Example 1.6.3: Notice, in particular, in Example 1.6.2, that $n^2 = n^{\underline{2}} + n^{\underline{1}}$. It follows from Theorem 1.6.2 that

$$\begin{aligned}\sum_{j=0}^{n} j^2 &= \sum_{j=0}^{n} j^{\underline{2}} + \sum_{j=0}^{n} j^{\underline{1}} \\ &= \frac{(n+1)^{\underline{3}}}{3} + \frac{(n+1)^{\underline{2}}}{2}\end{aligned}$$

E.g., $0+1+4+9+16+25+36 = \frac{7^{\underline{3}}}{3} + \frac{7^{\underline{2}}}{2} = 70+21 = 91.$

In turn, this enables us to calculate the sum of the sequential values of a polynomial, since a polynomial is a linear combination of ordinary powers. This method of summing the values of polynomials will be further explored in §3.4.

Partitions

DEF: A ***partition*** of a set S is a family $\mathcal{F} = \{S_1, \ldots, S_n\}$ of mutually disjoint subsets of S, called the ***cells of the partition*** $\mathcal{F}$, whose union is S.

NOTATION: Cells of partitions of a set may be indicated by the use of hyphens. If the set is small enough, then its elements can be represented by single characters, thereby avoiding potential ambiguities latent in juxtapositions of the characters.

Example 1.6.4: The partition $\{\{1,3\}, \{2,5\}, \{4\}\}$ of the integer interval $[1:5]$ may be denoted

$$13 - 25 - 4$$

or also, for instance, by $4-52-13$, since the cells of a partition and the order within cells are taken to be unordered.

Stirling Subset Numbers

DEF: The ***Stirling subset number***

$$\left\{ {n \atop k} \right\}$$

is the number of ways to partition the integer interval $[1:n]$ into k non-empty non-distinct cells.*

In §5.1, we establish that the Stirling number $S_{n,k}$ of the second kind equals the Stirling subset number $\left\{ {n \atop k} \right\}$.

Example 1.6.2, cont.: The value $S_{4,2} = 7$ is consistent with the following list of 7 partitions of $[1:4]$ into 2 cells, as an *ad hoc* calculation of $\left\{ {4 \atop 2} \right\}$.

$$1 - 234, \quad 2 - 134, \quad 3 - 124, \quad 4 - 123$$
$$12 - 34, \quad 13 - 24, \quad 14 - 23$$

---

* Wikipedia acknowledges D. E. Knuth for promoting usage of the user-friendly notations, $\left\{ {n \atop k} \right\}$ and $\left[{n \atop k} \right]$, of the Serbian mathematician J. Karamata (1902-1967) for Stirling numbers.

Stirling Cycle Number

DEF: The ***Stirling cycle number*** $\left[{n \atop k}\right]$ is the number of ways to partition the integer interval $[1 : n]$ into k non-empty non-distinct cycles.

In §5.2, we establish that the Stirling number $s_{n,k}$ of the first kind equals the absolute value of the Stirling cycle number $\left[{n \atop k}\right]$.

Example 1.6.1, cont.: The value $s_{4,2} = 11$ of the Stirling number of the first kind is consistent with the following list of 11 partitions of the integer interval $[1 : 4]$ into 2 cycles, as an *ad hoc* calculation of the Stirling cycle number $\left[{4 \atop 2}\right]$.

$$
\begin{array}{ccc}
(1)(2\ 3\ 4), & (2)(1\ 3\ 4), & (3)(1\ 2\ 4) \\
(1)(2\ 4\ 3), & (2)(1\ 4\ 3), & (3)(1\ 4\ 2) \\
\multicolumn{3}{c}{(4)(1\ 2\ 3), \quad (4)(1\ 3\ 2)} \\
(1\ 2)(3\ 4), & (1\ 3)(2\ 4), & (1\ 4)(2\ 3)
\end{array}
$$

Remark: Since the Stirling cycle numbers

$$\begin{bmatrix} n \\ 1 \end{bmatrix} \quad \begin{bmatrix} n \\ 2 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} n \\ n \end{bmatrix}$$

correspond to an inventory of all permutations of the integer interval $[1 : n]$, according to the number of cycles in their disjoint cycle representation, it follows that

$$\sum_{j=1}^{n} \begin{bmatrix} n \\ j \end{bmatrix} = n!$$

1.7 ORDINARY GENERATING FNS

A sequence $\langle g_n \rangle$ can be represented by the polynomial

$$\sum_{n=0}^{\infty} g_n z^n \;=\; g_0 + g_1 z + g_2 z^2 + \cdots$$

DEF: An ***(ordinary) generating function*** (abbr. ***OGF***) for the sequence $\langle g_n \rangle$ is any closed form $G(z)$ such that

$$G(z) \;=\; \sum_{n=0}^{\infty} g_n z^n$$

or, sometimes, it means the polynomial itself.

Exponential Generating Functions

Another kind of generating function, called an *exponential generating function*, is also used directly for counting and in solving recurrences. More extensive development of exponential generating functions appears in §5.5.

DEF: An ***exponential generating function*** (abbr. ***EGF***) for a sequence $\langle g_n \rangle$ is any closed form $\hat{G}(z)$ corresponding to the infinite polynomial

$$\sum_{n=0}^{\infty} g_n \frac{z^n}{n!}$$

or, sometimes, the polynomial itself.

Direct Counting with OGFs

Ordinary generating functions are readily applicable to counting unordered selections. We return to a counting problem first raised in Example 0.3.13.

Example 1.7.1: A combination of letters from the word SYZYGY may contain at most one S. Thus, an ordinary generating function for the number of possible combinations containing no letters that are not S's is

$$1 + s$$

Similarly, ordinary generating functions for combinations containing no letters except Z's and no letters except G's are, respectively

$$1 + z \quad \text{and} \quad 1 + g$$

Since the word SYZYGY contains three Y's, the OGF for counting combinations containing no letters except Y's is

$$1 + y + y^2 + y^3$$

which signifies that there is one choice with no Y's, one choice with one Y, one with two Y's, and one with three Y's. In the product

$$(1 + s)(1 + z)(1 + g)(1 + y + y^2 + y^3)$$

of these four generating functions, the terms of degree d provide an itemization of the ways to select d letters from SYZYGY. For instance, the seven terms of degree 2 are

$$sz \quad sg \quad sy \quad zg \quad zy \quad gy \quad y^2$$

It follows that if each of the indeterminates s, z, g, and y is replaced by a single indeterminate, say x,

$$(1+x)^3(1+x+x^2+x^3)$$

then the coefficient of x^d in the expansion

$$1 + 4x + 7x^2 + 8x^3 + 7x^4 + 4x^5 + x^6$$

is the number of ways to select d letters from SYZYGY. The general principle is articulated by the following proposition.

Prop 1.7.1. *Let $G(z)$ and $H(z)$ be the OGFs for counting unordered selections from two disjoint multisets S and T. Then $G(z)H(z)$ is the OGF for counting unordered selections from the union $S \cup T$.*

Proof: This is a direct application of the Rule of Sum and Rule of Product. ♢

Direct Counting with EGFs

Exponential generating functions are readily applicable to counting ordered selections. We continue the analysis of Example 1.7.1.

Example 1.7.1, cont.: An ordered selection of letters from SYZYGY may contain at most one S. Thus, an exponential generating function for the number of possible combinations containing no letters that are not S's is

$$1+s$$

Similarly, exponential generating functions for ordered selections containing no letters except Z's and no letters except G's are, respectively

$$1 + z \quad \text{and} \quad 1 + g$$

Since the word SYZYGY contains three Y's, the exponential generating function for counting ordered selections containing no letters except Y's is

$$1 + y + \frac{y^2}{2!} + \frac{y^3}{3!}$$

which signifies that there is one way with no Y's, one way with one Y, one with two Y's, and one with three Y's. In the product

$$(1 + s)(1 + z)(1 + g)\left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!}\right)$$

of these four generating functions, the terms of degree d provide an itemization of the ways to select d letters from SYZYGY. Suppose that the multivariate indeterminate monomial of a term of degree d is given the denominator of $d!$. For instance, this would give the transformation

$$\frac{zgy^2}{2!} \quad \longrightarrow \quad \frac{4!}{2!\,1!\,1!} \cdot \frac{zgy^2}{4!} = \binom{4}{2\;1\;1} \cdot \frac{zgy^2}{4!}$$

in which the multinomial coefficient $\binom{4}{2\;1\;1}$ is the number of ways to order the selection ZGYY represented by the monomial zgy^2. It follows that if each of the indeterminates s, z, g, and y is replaced by a single indeterminate, say x,

$$(1+x)^3\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)$$

then the coefficient of x^d in the expansion

$$1 + 4\frac{x}{1!} + 13\frac{x^2}{2!} + 34\frac{x^3}{3!} + 72\frac{x^4}{4!} + 120\frac{x^5}{5!} + 120\frac{x^6}{6!}$$

is the # of ordered selections of d letters from SYZYGY. The general principle is as follows.

Prop 1.7.2. *Let $\hat{G}(z)$ and $\hat{H}(z)$ be the EGFs for counting ordered selections from two disjoint multisets S and T. Then $\hat{G}(z)\hat{H}(z)$ is the EGF for counting ordered selections from the union $S \cup T$.*

Proof: This is a another direct application of the Rule of Sum and Rule of Product. ♢

Analyzing a Generating Function

To use generating functions effectively, either for direct counting or for solving recurrences, one needs to be able to analyze generating functions so as to recover a closed-form function for the list of entries. We now indicate briefly how this might be done, deferring most of the details to Chapter 2.

Example 1.7.2: When a closed-form generating function is a quotient of polynomials, such as

$$\frac{z}{1-3z+2z^2} \tag{1.7.1}$$

one way to extract the entries of the sequence is by *long division of polynomials.*

$$\begin{array}{r|l} & z + 3z^2 + 7z^3 + 15z^4 + \cdots \\ \hline 1-3z+2z^2 & z \\ & \underline{z - 3z^2 + 2z^3} \\ & \quad 3z^2 - 2z^3 \\ & \quad \underline{3z^2 - 9z^3 + 6z^4} \\ & \qquad\quad 7z^3 - 6z^4 \end{array}$$

This yields the coefficients of smaller powers z^n, but not a closed form. Factoring the denominator of the expression (1.7.1) and splitting the fraction into two parts, like this

$$\begin{aligned} \frac{z}{(1-z)(1-2z)} &= \frac{1}{1-2z} - \frac{1}{1-z} \\ &= (1 + 2z + 2^2z^2 + 2^3z^3 + \cdots) \\ &\quad - (1 + z + z^2 + z^3 + \cdots) \\ &= \sum_{n=0}^{\infty} (2^n - 1)z^n \end{aligned}$$

is the standard way to recover a closed-form. See §2.3.

Remark: Example 1.7.2 uses the familiar algebraic identity

$$\frac{1}{1-ay} = 1 + ay + a^2y^2 + \cdots$$

which can be justified either by long division or by multiplying $1 - ay$ and $1 + ay + a^2y^2 + \cdots$.

Rational Functions

DEF: A quotient of two polynomials in z (each with finitely many terms) is called a ***rational function*** in z. If the degree of the numerator is less than the degree of the denominator, then it is called a ***proper rational function***.

Long division of the denominator into the numerator transforms a generating function $G(z)$ represented as a rational function

$$G(z) = \frac{b_0 + b_1 z + \cdots + b_s z^s}{c_0 + c_1 z + \cdots + c_t z^t}$$

into its power series

$$G(z) = g_0 + g_1 z + g_2 z^2 + \cdots$$

as in Example 1.7.2. Moreover, it will be shown in Chapter 2 how to use factoring of the denominator, as in Example 1.7.2, to represent the values of the sequence by a closed function. For the time being, we consider another case of this phenomenon.

Example 1.7.3: Here is an additional illustration of the effect of factoring the denominator and splitting the fraction into a sum of fractions with linear polynomials as denominators

$$\begin{aligned} G(z) &= \frac{z-1}{1-5z+6z^2} = \frac{-2}{1-3z} + \frac{1}{1-2z} \\ &= -2\left(1+3z+3^2z^2+\cdots\right) + \left(1+2z+2^2z^2+\cdots\right) \\ &= \sum_{n=0}^{\infty}(2^n - 2\cdot 3^n)z^n \end{aligned}$$

Taylor Series

The fact that a rational function can be reconverted into a power series motivates the use of the terminology *generating function*, because a rational function may be regarded as *generating* its coefficients by the process of long division. Another sense in which a function $G(z)$ can generate the coefficients of a power series is by application of a ***Taylor series*** expansion at $z = 0$.

$$G(z) = G(0) + G'(0)\frac{z}{1!} + G''(0)\frac{z^2}{2!} + G'''(0)\frac{z^3}{3!} + \cdots$$

that assigns to the infinitely differentiable function $G(z)$ the power series

$$G(z) = g_0 + g_1 z + g_2 z^2 + \cdots$$

where

$$g_n \;=\; \frac{G^{(n)}(0)}{n!}$$

Using Taylor series permits an interpretation of a wide range of infinitely differentiable functions as generating functions.

Example 1.7.4: For the function $G(z) = -ln(1-z)$, the value of the n^{th} derivative at $z = 0$ is

$$G^{(n)}(0) \;=\; \left.\frac{(n-1)!}{(1-z)^n}\right|_{z=0} \;=\; (n-1)! \qquad \text{for } n \geq 1$$

and, thus,

$$G(z) \;=\; 0 + 0!\,\frac{z}{1!} + 1!\,\frac{z^2}{2!} + 2!\,\frac{z^3}{3!} + \cdots \;=\; \sum_{n=1}^{\infty} \frac{1}{n}\, z^n$$

That is, the function $-ln(1-z)$ is the OGF for the sequence $\langle x_n = \frac{1}{n}\rangle$.

Addition and Scalar Multiplication

There is a correspondence between various operations on sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ and some operations on their associated generating functions

$$A(z) \;=\; \sum_{j=0}^{\infty} a_j z^j \quad \text{and} \quad B(z) \;=\; \sum_{j=0}^{\infty} b_j z^j$$

DEF: The ***sum of two sequences*** $\langle a_n \rangle$ and $\langle b_n \rangle$ is the sequence

$$a_0 + b_0, \quad a_1 + b_1, \quad a_2 + b_2, \quad \ldots$$

This corresponds to the sum of their generating functions, i.e., to the generating function

$$(A+B)(z) \;=\; \sum_{j=0}^{\infty} (a_j + b_j) z^j$$

DEF: ***Multiplying the sequence*** $\{a_n\}$ ***by the scalar*** c yields the sequence

$$ca_0, \quad ca_1, \quad ca_2, \quad \ldots$$

This corresponds to the generating function

$$cA(z) \;=\; \sum_{j=0}^{\infty} ca_j z^j$$

that results from multiplying the generating function $A(z)$ by that scalar.

Example 1.7.5: Since the ordinary generating functions

$$A(z) = \frac{1}{1-5z} \quad \text{and} \quad B(z) = \frac{1}{1-7z}$$

generate the sequences $\langle a_n = 5^n \rangle$ and $\langle b_n = 7^n \rangle$, respectively, it follows that the ordinary generating function

$$2A(z) + 3B(z) \;=\; \frac{2}{1-5z} + \frac{3}{1-7z} \;=\; \frac{5-29z}{(1-5z)(1-7z)}$$

generates the sequence $\langle 2 \cdot 5^n + 3 \cdot 7^n \rangle$.

Products and Convolutions

The following two examples illustrate how one might use products of OGFs in direct computations.

Example 1.7.6: Consider counting the number p_n of ways to make $n¢$ postage from $3¢$ and $5¢$ stamps. If one had nothing but $3¢$ stamps, the OGF would be

$$\sum_{n=0}^{\infty} a_n x^n = 1 + x^3 + x^6 + x^9 + \cdots = \frac{1}{1-x^3}$$

since there is exactly one way from $3¢$ stamps alone to make each multiple of 3, and no way to make any other postage. Similarly, if one had nothing but $5¢$ stamps, the OGF would be

$$\sum_{n=0}^{\infty} b_n y^n = 1 + y^5 + y^{10} + y^{15} + \cdots = \frac{1}{1-y^5}$$

In the product of these two OGFs, the number of terms of degree n would be the number of ways of making $n¢$ postage. For instance, the terms of degree 23 (i.e., the terms whose exponents have 23 as their sum) are

$$x^{18}y^5 \quad \text{and} \quad x^3y^{20}$$

It follows that if z is substituted for x and y, then the coefficient of z^{23} is the number of ways. Thus, the OGF is

$$\frac{1}{1-z^3} \cdot \frac{1}{1-z^5} = \sum_{n=0}^{\infty} p_n z^n = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} a_j b_{n-j}$$

That is, the only way to get $n¢$ postage is to find an $a_j = 1$ and a $b_{n-j} = 1$. The sequence $\langle p_n \rangle$ is not monotonic. For instance,

$$p_{14} = 1 \quad p_{15} = 2 \quad p_{16} = 1$$

COMPUTATIONAL NOTE: In trying to obtain actual values for such a sequence, it is useful to have the aid of a computational engine such as *Mathematica.*

DEF: The ***convolution of the sequences*** $\langle u_n \rangle$ and $\langle v_n \rangle$ is the sequence

$$u_0v_0, \quad u_0v_1 + u_1v_0, \quad u_0v_2 + u_1v_1 + u_2v_0, \quad \ldots$$

Example 1.7.6, cont.: Thus, the sequence $\langle p_n \rangle$ is the convolution of the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

Example 1.7.7: Four distinguishable six-sided dice are rolled, each marked with the numbers 1, 2, 3, 4, 5, 6. Then the generating function for the number of ways that sum of the outcomes could be n is the coefficient of z^n in the expansion of

$$(z + z^2 + z^3 + z^4 + z^5 + z^6)^4$$

Proposition 1.7.3. *The product of the generating functions*

$$U(z) = \sum_{n=0}^{\infty} u_n z^n \quad \text{and} \quad V(z) = \sum_{n=0}^{\infty} v_n z^n$$

is the generating function

$$U(z)V(z) = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} u_j v_{n-j}$$

for the convolution of the sequences $\langle u_n \rangle$ *and* $\langle v_n \rangle$. ♢

Example 1.7.8: The rational functions

$$\frac{1}{1-2z} \quad \text{and} \quad \frac{1}{1-3z}$$

generate the sequences $\langle u_n = 2^n \rangle$ and $\langle v_n = 3^n \rangle$, respectively. Their product is the generating function

$$\begin{aligned} \frac{1}{(1-2z)(1-3z)} &= \frac{-2}{1-2z} + \frac{3}{1-3z} \\ &= \sum_{n=0}^{\infty} z^n (3^{n+1} - 2^{n+1}) \\ &= 1 + 5z + 19z^2 + 69z^3 + \cdots \end{aligned}$$

The convolution of the sequences $\langle u_n = 2^n \rangle$ and $\langle v_n = 3^n \rangle$ is the sequence whose n^{th} element (counting from the 0^{th} element) is

$$2^0 \cdot 3^n + 2^1 \cdot 3^{n-1} + \cdots + 2^n \cdot 3^0$$

Thus, the convolution sequence begins

$$1, \quad 5, \quad 19, \quad 69, \quad \ldots$$

in affirmation of Proposition 1.7.3.

Sums and Generating Functions

Prop 1.7.3 has a slue of useful consequences. An immediate consequence is that it provides a method for going from a counting sequence to its sequence of partial sums.

Theorem 1.7.4. *Let $B(z)$ be the OGF for a sequence $\langle b_n \rangle$. Then the OGF for the sequence*

$$\left\langle \sum_{j=0}^{n} b_j \mid n = 0, 1, \ldots \right\rangle$$

of partial sums is

$$\frac{B(z)}{1-z}$$

Proof: We observe that the total coefficient of z^n in

$$\frac{B(z)}{1-z} = \left(b_0 + b_1 z + b_2 z^2 + \cdots\right)\left(1 + z + z^2 + \cdots\right)$$

equals the sum $\sum_{j=0}^{n} b_j$, as per the following calculation:

$$\begin{array}{r} b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \\ \times \qquad 1 + z + z^2 + z^3 + \cdots \\ \hline \end{array}$$

$$\begin{array}{l} b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \\ \qquad b_0 z + b_1 z^2 + b_2 z^3 + b_3 z^4 + \cdots \\ \qquad\qquad b_0 z^2 + b_1 z^3 + b_2 z^4 + b_3 z^5 + \cdots \\ \qquad\qquad\qquad \cdots \\ \hline b_0 + (b_1 + b_0) z + (b_2 + b_1 + b_0) z^2 + \cdots \end{array}$$

This is just a special case of Proposition 1.7.3. ♢

Corollary 1.7.5. $$\frac{1}{(1-z)^r} = \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} z^n$$

Proof: By induction on r.

BASIS: For $r = 1$, we have

$$\frac{1}{(1-z)^1} = \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} \binom{n+1-1}{1-1} z^n$$

since the value of each of the coefficients $\binom{n}{0}$ is 1.

IND HYP: Next, suppose for some $r \geq 1$ that

$$\frac{1}{(1-z)^{r-1}} = \sum_{n=0}^{\infty} \binom{n+r-2}{r-2} z^n$$

IND STEP: Then

$$\begin{aligned}
\frac{1}{(1-z)^r} &= \frac{1}{1-z} \cdot \frac{1}{(1-z)^{r-1}} \\
&= \frac{1}{1-z} \sum_{n=0}^{\infty} \binom{n+r-2}{r-2} z^n && \text{(ind hyp)} \\
&= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} \binom{j+r-2}{r-2} && \text{(Thm 1.7.4)} \\
&= \sum_{n=0}^{\infty} \frac{z^n}{(r-2)!} \sum_{j=0}^{n} (j+r-2)^{\underline{r-2}} && \text{(factor } j\text{-sum)} \\
&= \sum_{n=0}^{\infty} \frac{z^n}{(r-2)!} \frac{(n+r-1)^{\underline{r-1}} - (r-2)^{\underline{r-1}}}{r-1} && \text{(Cor 1.5.2)}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{(r-2)!} \frac{(n+r-1)^{\underline{r-1}}}{(r-1)}$$

$$= \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} z^n \qquad \diamondsuit$$

Table 1.7.1 **OGFs for some sequences.**

sequence	closed form
$1,\quad 1,\quad 1,\quad 1,\quad \ldots$	$\dfrac{1}{1-z}$
$1,\quad -1,\quad 1,\quad -1,\quad \ldots$	$\dfrac{1}{1+z}$
$1,\quad 0,\quad 1,\quad 0,\quad \ldots$	$\dfrac{1}{(1-z^2)}$
$1,\quad 0,\quad 0,\quad 1,\quad 0,\quad 0,\quad \ldots$	$\dfrac{1}{(1-z^3)}$
$1,\quad a,\quad a^2,\quad a^3,\quad \ldots$	$\dfrac{1}{1-az}$
$0,\quad a,\quad 2a^2,\quad 3a^3,\quad \ldots$	$\dfrac{z}{1-az}$
$1,\quad 2,\quad 3,\quad 4,\quad \ldots$	$\dfrac{1}{(1-z)^2}$
$1,\quad \binom{m+1}{1},\quad \binom{m+2}{2},\quad \binom{m+3}{3},\quad \ldots$	$\dfrac{1}{(1-z)^{m+1}}$
$\frac{1}{0!},\quad \frac{1}{1!},\quad \frac{1}{2!},\quad \frac{1}{3!}\quad \ldots$	e^z
$0,\quad 1,\quad \frac{1}{2},\quad \frac{1}{3}\quad \ldots$	$\ln(1-z)$

Example 1.7.9: The rational function $\frac{1}{(1-z)^2}$ generates the sequence

$$\binom{n+1}{1}: \quad 1, \quad 2, \quad 3, \quad 4, \quad \cdots$$

Example 1.7.10: The rational function $\frac{1}{(1-z)^3}$ generates the sequence

$$\binom{n+2}{2}: \quad 1, \quad 3, \quad 6, \quad 10, \quad \cdots$$

Corollary 1.7.6. $$\frac{1}{(1-az)^r} = \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} a^n z^n$$

Proof: Substitute az for z in Corollary 1.7.5. ♢

Example 1.7.11: The rational function $\frac{1}{(1-2z)^2}$ generates the sequence

$$\binom{n+1}{1} 2^n: \quad 1, \quad 4, \quad 12, \quad 32, \quad \cdots$$

Example 1.7.12: The rational function $\frac{1}{(1-2z)^3}$ generates the sequence

$$\binom{n+2}{2} 2^n: \quad 1, \quad 6, \quad 24, \quad 80, \quad \cdots$$

1.8 SYNTHESIZING OGF'S

Synthesizing an OGF for a given sequence is a skill, like analyzing them, that is fundamental to solving counting problems with them. The approach is to recognize fundamental patterns in the sequence and to perceive how these patterns were combined.

Example 1.8.1: In the sequence

$$-4, \quad 2, \quad 5, \quad 2, \quad -6, \quad 2, \quad 7, \quad 2 \cdots \tag{1.8.1}$$

the two fundamental patterns are

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6 \quad \cdots \tag{1.8.2}$$

and

$$2, \quad 2, \quad 2, \quad 2, \quad \cdots \tag{1.8.3}$$

It seems that sequence (1.8.2) acquired negative signs on its even elements, that the entries preceding the entry 4 were truncated, and that it was then interwoven with sequence (1.8.3) by strict alternation.

Example 1.8.1 serves as a running example for this section. Our objective is to construct its generating function.

Substitution

Prop 1.8.1 *Substitution Rule.* *If $G(z)$ is an OGF for the sequence $\langle g_n \rangle$, then $G(bz)$ is an OGF for the sequence $\langle b^n g_n \rangle$.*

Proof: $$\sum_{n=0}^{\infty} g_n (bz)^n = \sum_{n=0}^{\infty} b^n g_n z^n. \qquad \diamondsuit$$

Example 1.8.1, cont.: By Example 1.7.9, the OGF for the sequence (1.8.2): 1, 2, 3, 4, ... is

$$\frac{1}{(1-z)^2}$$

Substitute $(-1)z$ for z, according to Proposition 1.8.1, to obtain the OGF

$$\frac{1}{(1+z)^2}$$

for the sequence

$$1, \quad -2, \quad 3, \quad -4, \quad 5, \quad -6 \quad \cdots \tag{1.8.4}$$

Shifting Right and Left

DEF: ***Shifting the sequence*** $\langle a_n \rangle$ ***to the right*** by k places yields the sequence

$$\overbrace{0,\ 0,\ \ldots,\ 0}^{k \text{ zeroes}},\ a_0,\ a_1,\ a_2,\ \ldots$$

The corresponding generating function is

$$z^k A(z) \;=\; \sum_{j=0}^{\infty} a_j z^{j+k}$$

DEF: ***Nullifying*** the j^{th} element of the sequence $\langle a_n \rangle$ means replacing a_j by 0. The corresponding generating function is

$$A(z) \,-\, a^j z^j$$

DEF: ***Shifting the sequence*** $\langle a_n \rangle$ ***to the left*** by k places yields the sequence

$$a_k,\; a_{k+1},\; a_{k+2},\; \ldots$$

The corresponding generating function is

$$z^{-k}\left[A(z) - \sum\nolimits_{j=0}^{k-1} a_j z^j\right] \;=\; \sum\nolimits_{j=k}^{\infty} a_j z^{j-k}$$

The terms $a_0, a_1, \ldots, a_{k-1}$ are nullified, so that they do not end up as non-zero coefficients of negative powers of z.

Example 1.8.1, cont.: Shifting sequence (1.8.4) to the left by three places yields the sequence

$$-4, \quad 5, \quad -6 \quad 7, \quad -8, \quad 9, \cdots \tag{1.8.5}$$

which corresponds to the OGF

$$z^{-3}\left(\frac{1}{(1+z)^2} - 1 + 2z - 3z^2\right) \;=\; \frac{-4-3z}{(1+z)^2}$$

Spacing Out

DEF: ***Spacing a sequence*** $\langle a_n \rangle$ by k units yields the sequence

$$a_0, \overbrace{0, \ldots, 0}^{k \ \ 0's}, a_1, \overbrace{0, \ldots, 0}^{k \ \ 0's}, a_2, \overbrace{0, \ldots, 0}^{k \ \ 0's}, \ldots$$

The corresponding generating function is

$$A(z^{k+1})$$

Example 1.8.1, cont.: Spacing sequence (1.8.5) by 1 place yields the sequence

$$-4, \quad 0, \quad 5, \quad 0, \quad -6, \quad 0, \quad 7, \quad 0, \quad -8, \quad 0, \quad 9, \cdots \tag{1.8.6}$$

which corresponds to the OGF

$$\left.\frac{-4-3z}{(1+z)^2}\right|_{z \to z^2} = \frac{-4-3z^2}{(1+z^2)^2}$$

Isolating a Subsequence

DEF: ***Isolating the subsequence*** $n \equiv k \bmod m$ of the sequence $\langle a_n \rangle$ yields the sequence in which all terms are nullified, except those whose index is congruent to $k \bmod m$.

For modulus $m = 2$, the corresponding generating function is

$$\begin{cases} \dfrac{A(z) + A(-z)}{2} & \text{if } k = 0 \\ \dfrac{A(z) - A(-z)}{2} & \text{if } k = 1 \end{cases}$$

Example 1.8.1, cont.: Since the rational function $\frac{1}{1-z}$ generates a sequence of 1's, the generating function for the sequence (1.8.3) is

$$\frac{2}{1-z}$$

Isolating the 1 mod 2 subsequence from sequence (1.8.3) yields the sequence

$$0, \quad 2, \quad 0, \quad 2, \quad \cdots \qquad (1.8.7)$$

which corresponds to the OGF

$$\frac{1}{2}\left(\frac{2}{1-z} - \frac{2}{1+z}\right) \;=\; \frac{2z}{1-z^2}$$

which might also have been obtained by spacing sequence (1.8.3) out by 1 unit and shifting right 1 place. Sequence (1.8.1) is the sum of sequences (1.8.6) and (1.8.7). Thus, its OGF is the sum of their OGF's, i.e.,

$$\frac{-4-3z^2}{(1+z^2)^2} + \frac{2z}{1-z^2} \;=\; \frac{2z^5 + 3z^4 + 4z^3 + z^2 + 2z - 4}{(1-z^4)(1+z^2)}$$

Differentiation

DEF: The ***derivative of the generating function***

$$G(x) \;=\; \sum_{n=0}^{\infty} g_n z^n$$

is the generating function

$$G'(x) \;=\; \sum_{n=1}^{\infty} n\, g_n z^{n-1} \;=\; \sum_{n=0}^{\infty} (n+1)\, g_{n+1}\, z^n$$

Example 1.8.2: Consider the generating function

$$G(z) \;=\; \frac{1}{1-2z} \;=\; \sum_{n=0}^{\infty} 2^n z^n$$

Then taking its derivative yields the equation

$$G'(z) \;=\; \frac{2}{(1-2z)^2} \;=\; \sum_{n=0}^{\infty} (n+1)\, 2^{n+1} z^n$$

which is consistent with Corollary 1.7.6.

1.9 ASYMPTOTIC ESTIMATES

The growth rate of a function is customarily reckoned via comparison to benchmarks. The focus is on the long term.

DEF: Let $f(n)$ be a function such that $f(n) \neq 0$ for sufficiently large n. The sequence x_n is ***asymptotic*** to $f(n)$ if

$$\lim_{n\to\infty} \frac{x_n}{f(n)} = 1$$

It is often reasonably straightforward to guess or to find a well-understood function $f(n)$ such that the ratio

$$\frac{x_n}{f(n)}$$

converges.

Example 1.9.1: How large is the Catalan number c_n? From the expansion

$$\begin{aligned} c_n &= \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1} \cdot \frac{(2n)^{\underline{n}}}{n!} \\ &= \frac{1}{n+1} \cdot \frac{2n}{n} \cdot \frac{2n-1}{n-1} \cdot \cdots \cdot \frac{(n+1)}{1} \end{aligned}$$

one sees that the Catalan number c_n is a product of the value of $\frac{1}{n+1}$ and the values of n other factors, whose values form an increasing sequence from 2 to $n+1$. One surmises that

$$\frac{2^n}{n+1} < c_n < \frac{(n+1)^n}{n+1} = (n+1)^{n-1}$$

which is a very wide range of possibilities, since the lower and upper bounds are far apart. Narrowing that gap is a primary need toward improved understanding of the behavior of the Catalan sequence.

Ratio Method

Ratio Method: Some information about the asymptotic behavior of a sequence x_n lies in the ratio

$$\frac{x_n}{x_{n-1}}$$

of successive terms. We calculate the limit of that ratio.

Example 1.9.1, cont.: The ratio of successive entries of the Catalan sequence is

$$\begin{aligned}\frac{c_n}{c_{n-1}} &= \frac{1}{n+1}\binom{2n}{n} \Big/ \frac{1}{n}\binom{2n-2}{n-1} \\ &= \frac{1}{n+1} \cdot \frac{(2n)^{\underline{n}}}{n!} \Big/ \frac{1}{n} \cdot \frac{(2n-2)^{\underline{n-1}}}{(n-1)!} \\ &= \frac{1}{n+1} \cdot \frac{(2n)^{\underline{n}}}{(2n-2)^{\underline{n-1}}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n+1} \cdot \frac{(2n)(2n-1)}{n} \\
\Rightarrow \quad \frac{c_n}{c_{n-1}} &= \frac{4n-2}{n+1} \qquad (1.9.1) \\
\Rightarrow \quad \lim_{n\to\infty} \frac{c_n}{c_{n-1}} &= \lim_{n\to\infty} \frac{4n-2}{n+1} \\
&= \lim_{n\to\infty} \frac{4n+4}{n+1} - \lim_{n\to\infty} \frac{6}{n+1} = 4-0 \\
\Rightarrow \quad \lim_{n\to\infty} \frac{c_n}{c_{n-1}} &= 4 \qquad (1.9.2)
\end{aligned}
$$

Since the ratio $\frac{c_n}{c_{n-1}}$ is everywhere less than its asymptotic upper limit of 4, and since $c_1 = 1 < 4$, it is possible to narrow the estimating range of c_n to

$$\frac{2^n}{n+1} < c_n = \frac{1}{n+1}\binom{2n}{n} < 4^n \qquad (1.9.3)$$

Tightening Bounds on Estimates

Concrete Substitution: Concrete early values of a sequence can often be used to improve asymptotic upper and lower bounds.

Example 1.9.1, cont.: Sharpening the lower bound of (1.9.3) for the Catalan number c_n, including eliminating the denominator of $n+1$, can begin with an observation regarding the ratio $\frac{c_n}{c_{n-1}}$ after $n = 5$.

$$\begin{aligned} 4n-2 &\geq 3n+3 \qquad \text{for } n \geq 5 \\ \Rightarrow \quad \frac{4n-2}{n+1} &\geq \frac{3n+3}{n+1} = 3 \end{aligned}$$

Recalling (1.9.1), we have

$$\frac{c_n}{c_{n-1}} \geq 3 \tag{1.9.4}$$

Using (1.9.4) and the fact that

$$c_n = c_4 \cdot \frac{c_5}{c_4} \cdot \frac{c_6}{c_5} \cdot \cdots \cdot \frac{c_n}{c_n - 1}$$

we infer that

$$\begin{aligned} c_n &\geq c_4 \cdot 3^{n-4} = 14 \cdot 3^{n-4} \\ \Rightarrow \quad c_n &= \frac{14}{81} \cdot 3^n > \frac{1}{6} \cdot 3^n \quad \text{for } n \geq 4 \end{aligned} \tag{1.9.5}$$

The inequality (1.9.5) also holds for c_0, c_1, c_2, and c_3.

Recalling the inequality (1.9.3), it follows that

$$\frac{1}{6} \cdot 3^n \;<\; c_n \;<\; 4^n \qquad \text{for } n \geq 0 \tag{1.9.6}$$

We shall now show that the coefficient of $\frac{1}{6}$ can be removed from the lower bound of (1.9.6) for sufficiently large values of n. Since the ratio

$$\frac{c_n}{c_{n-1}}$$

is increasing monotonically to 4, it eventually exceeds $\frac{7}{2}$, say, for all $n > P$. Since $\frac{7}{6} > 1$, there is a number Q such that

$$\left(\frac{7}{2}\right)^q \;>\; \frac{3^P}{c_P} \qquad \text{for all } q \geq Q - P$$

which implies that

$$\begin{aligned} c_n \;&=\; c_P \cdot \left(\frac{c_{P+1}}{c_P} \cdot \frac{c_{P+2}}{c_{P+1}} \cdot \;\cdots\; \cdot \frac{c_Q}{c_{Q-1}}\right) \cdot \left(\frac{c_{Q+1}}{c_Q} \cdot \frac{c_{Q+2}}{c_{Q+1}} \cdot \;\cdots\; \cdot \frac{c_n}{c_{n-1}}\right) \\ &>\; c_P \cdot \left(\frac{7}{2}\right)^{Q-P} \cdot \left(\frac{7}{2}\right)^{n-Q} \\ &>\; 3^P \cdot 3^{Q-P} \cdot 3^{n-Q} \end{aligned}$$

$$\Rightarrow \;\; c_n \;>\; 3^n \tag{1.9.7}$$

Combining (1.9.6) and (1.9.7) yields the desired result

$$3^n \;<\; c_n \;<\; 4^n \qquad \text{for } n \geq Q \tag{1.9.8}$$

Remark: In fact, this lower bound is further improvable. Since the ratio

$$\frac{c_n}{c_{n-1}}$$

is increasing monotonically to 4, it eventually surpasses $4 - \epsilon$ for any $\epsilon > 0$, say, for all $n > N(\epsilon)$. It follows, by an argument similar to that used in the derivation of (1.9.6), that

$$\frac{c_{N(\epsilon)}}{4^{N(\epsilon)}} \cdot (4-\epsilon)^n \; < c_n \; < \; 4^n \qquad \text{for } n \geq N(\epsilon)$$

The coefficient could be removed, once again, as in the derivation of (1.9.8), to yield the asymptotic estimate

$$(4-\epsilon)^n \; < \; c_n \; < \; 4^n$$

which is adequate for present purposes.

The following proposition formulates the method used in Example 1.9.1 as a general principle.

Proposition 1.9.1. *Let x_n be a sequence such that*

$$lim_{n\to\infty} \frac{x_n}{x_{n-1}} \; = \; K > 0$$

Then, for $\epsilon > 0$ and sufficiently large values of n,

$$(K-\epsilon)^n \; < c_n \; < \; (K+\epsilon)^n \qquad (1.9.9)$$

If the ratio $\frac{x_n}{x_{n-1}}$ is bounded above by K, then (1.9.9) can be sharpened to

$$(K-\epsilon)^n \; < c_n \; < \; x_0 K^n \qquad (1.9.10)$$

If bounded below by K, then (1.9.9) can be sharpened to

$$x_0 K^n < c_n < (K+\epsilon)^n \tag{1.9.11}$$

Proof: Details from Example 1.9.1 are readily transformed into a proof. This is left to the Exercises. ♢

Asymptotic Dominance

DEF: If there is a positive number c such that

$$f(n) \leq cg(n) \quad \text{for all } n \geq N$$

then we may write

$$f(n) \in \mathcal{O}(g(n))$$

and say "$f(n)$ is in ***big-oh*** of $g(n)$". The numbers c and N are called ***witnesses*** to the relationship.

TERMINOLOGY NOTE: Although $\mathcal{O}(g(n))$ is defined here as the class of functions that are eventually dominated by a multiple of $g(n)$, the usage "f *is* big-oh of g" (omitting the preposition "in") is quite common. The rationale is that membership in the class may be regarded as an adjectival property.

Example 1.9.2: One way to prove that $7n^2 \in \mathcal{O}(n^3)$ is to choose the witnesses $N = 7$ and $c = 1$. Then

$$7n^2 \;\leq\; 1 \cdot n^3 \;\text{ for } n \geq 7$$

Another proof uses the witnesses $N = 1$ and $c = 7$. Then

$$7n^2 \;\leq\; 7 \cdot n^3 \;\text{ for } n \geq 1$$

In general, there tends to be a tradeoff in the size of the witnesses N and c. Choosing a larger value of witness c may enable one to choose a smaller value of witness n.

Example 1.9.3: To prove that $n^3 \notin \mathcal{O}(7n^2)$, we observe that for any witness c, and for any number $n > 8c$,

$$n^3 \;>\; (8c)n^2 \;>\; 7c \cdot n^2$$