Chapter 0

Introductory Survey

Most combinatorics problems have one of three fundamental objectives:

- counting or calculating a sum,
- constructing a configuration involving two or more discrete sets (usually two) — subject to a list of constraints,
- optimization, i.e., either finding the extreme values of a function or designing something with an optimal characteristic of some kind.

Chapter 0 begins with examples of problems of each type.

0.1 OBJECTIVES

We illustrate the three basic types of problems:

counting

constructing a configuration

optimization.

Combinatorial Enumeration

Prototype discrete measurement: sum the values of a function over a finite or countable set.

Analogy: calculate the area of a region in the plane between the x-axis and a curve.

Example 0.1.1: Evaluate this sum:

$$
\begin{aligned}
&1+\\
&2+1+\\
&3+2+1+\\
&\quad\cdots\\
&n+(n-1)+\ \cdots\ +1
\end{aligned}
$$

For $n = 12$, this sum corresponds to an English holiday song,* and the value of the sum is 364.

---

* *The Twelve Days of Christmas.*

Since the sum of the j^{th} row is

$$j + (j-1) + \cdots + 1 = \frac{(j+1)j}{2}$$

the value of the original sum equals

$$\sum_{j=1}^{n} \frac{(j+1)j}{2}$$

This latter sum rather neatly fits a standard form of what is called the *finite calculus* (see, especially, §3.4). It can be evaluated as follows:

$$\sum_{j=1}^{n} \frac{(j+1)j}{2} = \frac{(n+2)(n+1)n}{6}$$

For instance, for $n = 12$, the value is 364.

•

Generalization: sum the values of an arbitrary polynomial over a range of consecutive integers. (Stirling numbers are used.)

Such summation problems arise frequently in the analysis of algorithms, in which the time to execute the body of a loop might be roughly proportional to a polynomial-valued function.

Incidence Structures

An *incidence structure* is a combinatorial configuration that involves two or more discrete sets. Most commonly, there are exactly two sets:

a set P of *points* and a set L of *lines*

and an *incidence function* $\iota : P \times L \to \mathbb{Z}_2$.

Example 0.1.3: An abstract model for what is called a *simple graph* is an incidence structure in which every line has exactly two points and in which no two lines have the same two points.

$$
\begin{aligned}
P &= \{1, \quad 2, \quad 3, \quad 4, \quad 5\} \\
L &= \{12, \quad 14, \quad 15, \quad 23, \quad 25, \quad 45\}
\end{aligned}
$$

In a spatial model, the more intuitive model for a graph, each point of the graph is called a *vertex*, and each line of the graph is called an *edge*.

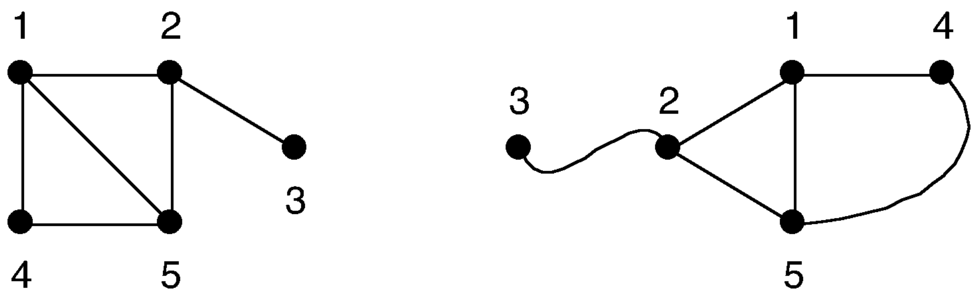

Fig 0.1.1 Two drawings of a simple graph.

Both for intrinsic interest and for their value in applications, graph theorists often solve problems of an enumerative character.

Example 0.1.4: Arthur Cayley encountered the problem of counting the number of different hydrocarbon isomers with the chemical formula

$$C_n H_{2n+2}$$

The two isomers for $n = 4$, called butane and isobutane, are illustrated in Figure 0.1.2. Graph enumeration is the principal concern of Chapter 9.

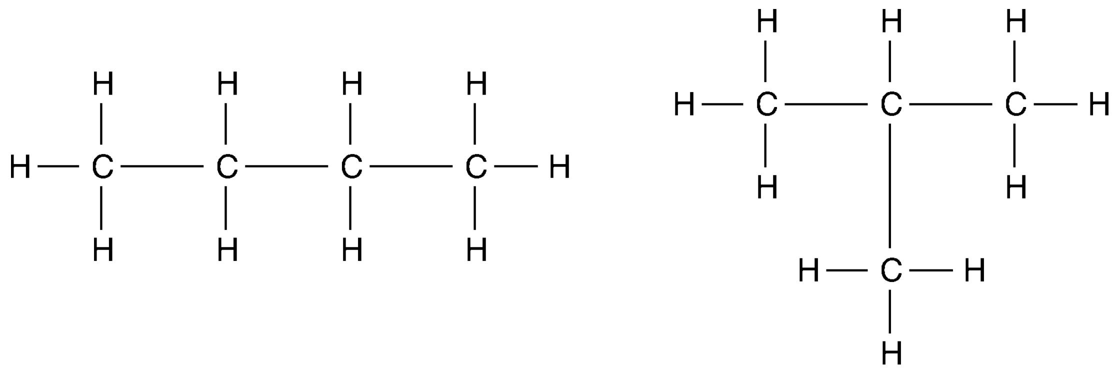

Fig 0.1.2 **Butane and isobutane.**

Optimization

DEF: *combinatorial optimization*:

> any discrete problem concerned with finding a maximum or a minimum.

Elsewhere, it may have a restricted meaning:

> finding the maximum value of a function on a region of a Euclidean space.

Example 0.1.5: For instance, if one is selecting subsets of size k from a set of size n, one may wish to know the value of k for which the number of different subsets is greatest.

In ***extremal graph theory***, one determines the max number of edges that a simple n-vertex graph G may have before some property necessarily holds.

Example 0.1.6: For instance, what is the maximum number of edges a simple n-vertex graph G may have before there must be a set of three mutually adjacent vertices? The following solution of this problem, due to Paul Turán, appears in §8.4.

$$|E_G| = \left\lfloor \frac{n^2}{4} \right\rfloor$$

0.2 ORDER AND SELECTION

Counting orderings and selections occurs throughout combinatorics.

DEF: An ***ordering of a set*** S of n objects is a bijection from the set

$$\{\ 1,\ 2,\ \ldots,\ n\ \}$$

to the set S. It serves as a formal model for an arrangement of the n objects into a row.

DEF: An ***(unordered) selection from a set*** S is a subset of S.

Example 0.2.1: In how many ways is it possible to arrange two of the letters

$$A \quad B \quad C \quad D \quad E$$

and two of the digits

$$0 \quad 1 \quad 2 \quad 3$$

into a row of four characters, such that no two digits are adjacent? For instance, the arrangement $C3A2$ meets that requirement.

SOL: There are 10 possible selections of two of the five letters and 6 possible selections of two of the four digits.

Thus, there are 60 possible selections of a combination of four symbols that meets the given requirement. An arrangements of four such symbols into a row meets the requirement if it has any of the three forms

$$LDLD \quad DLDL \quad \text{and} \quad DLLD$$

where D is a digit and L is a letter. Since there are four ways that two distinct letters and two distinct digits could be placed within one of the three forms, it follows that there are

$$12 \quad (= 4 \times 3)$$

ways that each of the 60 suitable selections of four symbols could be arranged so as to meet the requirement. Thus, the answer to the stated problem is 720 $(= 60 \times 12)$.

Sequences and Generating Functions

A generalization of Example 0.2.1 supposes that x_n is the number of ways to form an arrangement of four symbols when there are n letters, but still only four digits. We have just calculated that $x_5 = 720$. By similar analysis,

$$x_0 = 0,\ x_1 = 0,\ x_2 = 72,\ x_3 = 216,\ x_4 = 432,\ x_5 = 720,\ \ldots$$

We can encode a counting sequence

$$g_0 \quad g_1 \quad g_2 \quad \cdots$$

Multiply its entries by powers of z and sum:

$$g_0 + g_1 z + g_2 z^2 + \cdots$$

For this general version of Example 0.2.1, we would obtain

$$\begin{aligned} 0 + 0z + 72z^2 + 216z^3 + 432z^4 + \cdots \\ = 72z^2 + 216z^3 + 432z^4 + \cdots \end{aligned}$$

The resulting infinite polynomial may have a closed form, called a *generating function.*

Example 0.2.2: The closed form

$$\frac{1}{1 - 2z}$$

is equivalent to the infinite polynomial

$$1 + 2z + 4z^2 + 8z^3 + \cdots$$

In this context, the issue of convergence is rarely relevant.

Generating functions are the main topic of §1.7. It is described there how they are used to solve various kinds of counting problems.

Recurrences

A sequence can be specified by giving some initial values and a *recurrence* that says how each later entry can be calculated from earlier entries.

Example 0.2.3: Famously, the recurrence

$$\begin{array}{lll} f_0 = 0; \quad f_1 = 1 & & \text{initial values} \\ f_n = f_{n-1} + f_{n-2} & & \text{for } n \geq 2 \end{array}$$

gives the *Fibonacci sequence*:

n	0	1	2	3	4	5	6	7	8	9 $\cdots$
f_n	0	1	1	2	3	5	8	13	21	34 $\cdots$

Generating functions are used in Chapter 2 to derive the formula

$$f_n = \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

A ***closed formula*** for that sequence for a recurrence is called a *solution* to the recurrence, in the same sense that a differentiable function might be a solution to a differential equation.

Combination Coefficients

The number of possible subsets of size k within a set of size n ("n-choose-k") is denoted

$$\binom{n}{k}$$

and called a *combination coefficient.* Its value

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \tag{0.2.1}$$

is derived in §0.4. Its alternative name of *binomial coefficient* is justified in Chapter 1.

Example 0.2.4: The sequence of combination coefficients

$$\binom{0}{2} \quad \binom{1}{2} \quad \binom{2}{2} \quad \binom{3}{2} \quad \cdots$$

has the generating function

$$\frac{x^2}{(1-x)^3}$$

To verify this observation, one may expand the denominator and divide it into the numerator, using long division of polynomials, which is described in more detail in §1.7.

0.3 RULES FOR COUNTING

Having meaningful names for concepts, even for very simple concepts, makes it possible to say what method is being used.

NOTATION: The cardinality of a set U is denoted $|U|$. The most common binary operations on two sets U and V are denoted

$$\begin{array}{ll} U \cup V & \text{for union} \\ U \cap V & \text{for intersection} \\ U - V & \text{for difference, and} \\ U \times V & \text{for cartesian product} \end{array}$$

Rules of Sum and Product

C. L. (Dave) Liu [Liu1968] gave popularity to now-standard names of two principles that relate elementary arithmetic operations to the counting of set unions and set products.

DEF: ***Rule of Sum***: Let U and V be disjoint sets. Then

$$|U \cup V| \;=\; |U| + |V|$$

DEF: ***Rule of Product***: Let U and V be sets. Then

$$|U \times V| \;=\; |U| \cdot |V|$$

Example 0.3.2: Three dice colored blue, red, and yellow are rolled. In how many ways can the outcome be different numbers on the three dice?

SOL: There are 6 possibilities for the blue die, leaving 5 for the red die, and then 4 for the yellow die. Thus, the total number of possibilities is

$$6 \cdot 5 \cdot 4$$

Rule of Quotient

Another simple rule sometimes applies to counting the cells in a set *partition.*

DEF: A ***partition*** of a set U is a collection of mutually exclusive subsets

$$U_1, \ldots, U_p$$

called ***cells of the partition***, whose union is U.

DEF: ***Rule of Quotient***: Let $\mathcal{P}$ be a partition of a set U into cells, each of the same cardinality k. Then the number of cells equals the quotient

$$\frac{|U|}{k}$$

Example 0.3.3: Fig 0.3.1 shows 20 objects partitioned into cells of four each. By Rule of Quotient, # cells is

$$\frac{20}{4} = 5$$

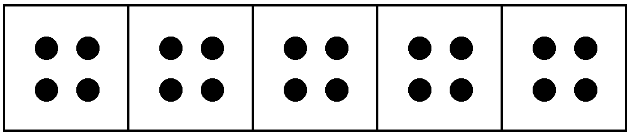

Fig 0.3.1 Partitioning a set of 20 objects into cells of size 4.

N.B. The Rule of Quotient does not apply when the cells of the partition are of different sizes.

Example 0.3.4: Fill each square of a 3×3 tic-tac-toe board with an "X" or an "O", or a blank. The total number of possible configurations is 3^9. It is natural to regard two such configurations as equivalent if one could be obtained from the other by a rotation or a reflection. The equivalence classes are not all of the same size. For instance, Figure 0.3.2 illustrates an equivalence class of size four.

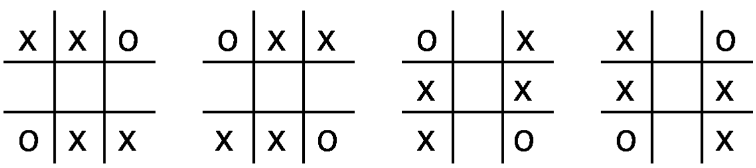

Fig 0.3.2 Four equivalent tic-tac-toe configurations.

On the other hand, the configurations that are all blank, all "X", or all "O" are in equivalence classes of size one. There are also some equivalence classes of sizes two and eight. Thus, the Rule of Quotient cannot be applied.

Counting equivalence classes that are defined by symmetries is frequently with the aid of Burnside-Pólya counting. This method of counting is developed in Chapter 9.

Reals to Integers

Three standard functions for converting a real number into a nearby integer are convenient when one wants to apply integer methods.

DEF: The ***floor of a real number*** x is the largest integer that is not larger than x. It is denoted $\lfloor x \rfloor$.

DEF: The ***ceiling of a real number*** x is the smallest integer that is not smaller than x. It is denoted $\lceil x \rceil$.

DEF: The ***nearest integer*** to a real number x is $round\,(x)$

$$= \begin{cases} \lfloor x \rfloor & \text{if } x - \lfloor x \rfloor \ < \ \frac{1}{2}; \\ \lfloor x \rfloor & \text{if } x - \lfloor x \rfloor \ = \ \frac{1}{2} \text{ and } \lfloor x \rfloor \text{ is even}; \\ \lceil x \rceil & \text{if } x - \lfloor x \rfloor \ > \ \frac{1}{2}; \\ \lceil x \rceil & \text{if } x - \lfloor x \rfloor \ = \ \frac{1}{2} \text{ and } \lceil x \rceil \text{ is even} \end{cases}$$

Pigeonhole Principle

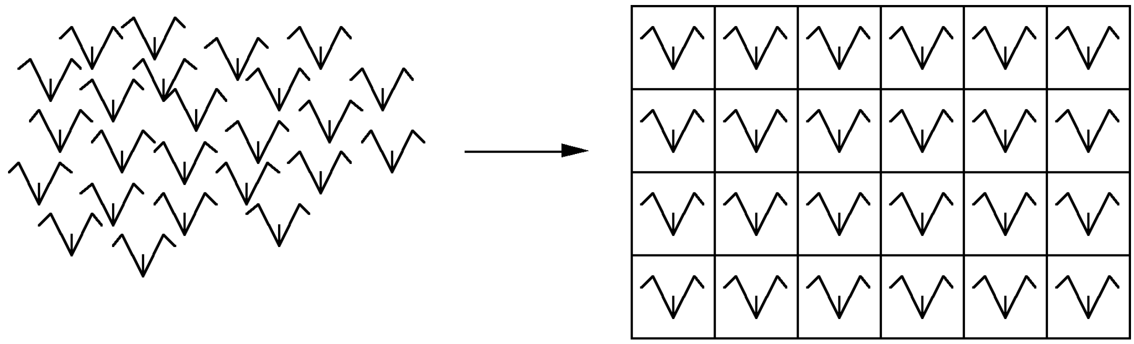

Fig 0.3.3 Flock of pigeons neatly fills the pigeonholes.

DEF: ***Pigeonhole Principle***: Let $f : U \to V$ be a function with finite domain and finite codomain. Let any two of the following three conditions hold:

1. f is one-to-one.
2. f is onto.
3. $|U| = |V|$.

Then the third condition also holds.

Example 0.3.7: In any collection of 13 people, there must be two of them who were born in the same month. In this elementary example, the people are the pigeons, and the months are the pigeonholes.

Evaluating Sums

This formula of infinitessimal calculus

$$\int_{x=a}^{b} x^r\,dx \;=\; \left.\frac{x^{r+1}}{r+1}\right|_{x=a}^{b}$$

is used for integrating a monomial (i.e., a polynomial with only one term). The next definition facilitates an analogous formula for finite sums.

DEF: The r^{th} ***falling power*** of a real number x is the product

$$x^{\underline{r}} \;=\; \overbrace{x\,(x-1)\;\cdots\;(x-r+1)}^{r\text{ factors}} \qquad \text{for } r \in \mathbb{N}$$

Remark: For nonnegative integers n and $r \le n$,

$$n^{\underline{r}} \;=\; \frac{n!}{(n-r)!} \tag{0.3.1}$$

Example 0.3.11: Here are some falling power evaluations.

$$\begin{aligned} x^{\underline{3}} &= x\,(x-1)(x-2) \;=\; x^3 - 3x^2 + 2x \\ 6^{\underline{3}} &= 6 \cdot 5 \cdot 4 \;=\; 120 \\ \left(\frac{2}{5}\right)^{\underline{3}} &= \frac{2}{5} \cdot \frac{-3}{5} \cdot \frac{-8}{5} \;=\; \frac{48}{125} \end{aligned}$$

The formula for summing a falling-power monomial is

$$\sum_{k=a}^{b} k^{\underline{r}} \;=\; \left.\frac{k^{\underline{r+1}}}{r+1}\right|_{k=a}^{b+1} \tag{0.3.2}$$

Example 0.3.12: For exponent $r = 2$ and limits of summation $a = 3$ and $b = 5$, Formula (0.3.2) yields

$$\begin{aligned}
\sum_{k=3}^{5} k^{\underline{2}} \;&=\; 3^{\underline{2}} + 4^{\underline{2}} + 5^{\underline{2}} \\
&=\; 6 + 12 + 20 \;=\; 38 \\
\left.\frac{k^{\underline{3}}}{3}\right|_{k=3}^{6} \;&=\; \frac{6^{\underline{3}}}{3} - \frac{3^{\underline{3}}}{3} \\
&=\; 40 - 2 \;=\; 38
\end{aligned}$$

Summations of ordinary powers can be achieved via a preliminary conversion to falling powers. For instance,

$$x^2 \;=\; x^{\underline{2}} + x^{\underline{1}} \quad \text{and} \quad x^3 \;=\; x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}}$$

The coefficients used in the conversion, which are called *Stirling numbers.* See §1.6 and Chapter 5.

Empty Sums and Empty Products

In manipulating expressions with iterated sums and products, such as

$$\sum_{x_j \in S} x_j \quad \text{or} \quad \prod_{x_j \in S} x_j$$

we sometimes encounter a sum or product over the *empty set* $\emptyset$.

DEF: A sum over an empty set of numbers is called an ***empty sum***. Its value is taken to be 0, the additive identity of the number system.

DEF: A product over an empty set of numbers is called an ***empty product***. Its value is taken to be 1, the multiplicative identity of the number system.

Multisets

One of the many applications of the Rule of Quotient is to counting arrangements of *multisets*. Informally, a multiset is often described as a "set in which the same element may occur more than once".

DEF: A ***multiset*** is a pair (S, ι) in which S is a set and $\iota : S \to \mathbb{Z}^+$ is a function that assigns to each element $s \in S$ a number $\iota(s)$ called its ***multiplicity***. (The Greek letter iota is a mnemonic for *instances*.)

Example 0.3.13: The letters of SYZYGY form a multiset in which Y occurs three times. Each of the other three letters occurs once. If the six letters were all different, then the number of ways of arranging them into a row of six would be $6! = 720$.

We may model this by artificially attaching distinct subscripts to each of the copies of the letter Y, so that they become Y_1, Y_2, and Y_3. We regard two arrangements of the six elements of the resulting set as equivalent if the positions of the letters G, S, and Z are the same in both arrangements.

There are then $6 = 3!$ equivalent arrangements in each equivalence class. By the Rule of Quotient, the number of equivalence classes is

$$\frac{6!}{3!} = 120$$

More generally, the Rule of Quotient implies that the number of ways to arrange the elements of a finite multiset (S, ι) is

$$\frac{\left(\sum_{s\in S} \iota(s)\right)!}{\prod_{s\in S}(\iota(s)!)}$$

DEF: The ***cardinality of a multiset*** (S, ι) is taken to be the sum

$$\sum_{s\in S} \iota(s)$$

of the multiplicities of its elements. It is denoted $|(S, \iota)|$.

0.4 COUNTING SELECTIONS

An *ordered selection* assigns an order to the elements of the selected subset.

Ordered Selections

DEF: An ***ordered selection*** of k objects from a set of n objects is a function from the set

$$\{\ 1,\ 2,\ \ldots,\ k\ \}$$

to the set S.

Prop 0.4.1. *Let $P(n,k)$ be the number of possible ordered selections of k objects from a set S of n objects. Then*

$$P(n,k) \;=\; n^{\underline{k}} \tag{0.4.1}$$

Proof: By induction on k.

BASIS: For $k = 0$, the only possible ordered selection is the empty list. Thus,

$$P(n,0) \;=\; 1 \;=\; n^{\underline{0}}$$

IND HYP: Assume that $P(n,k) = n^{\underline{k}}$, for some $k \geq 0$.

IND STEP: After the first k objects have already been selected from S, the number of remaining objects from which to choose the $k+1^{\text{st}}$ object is $n-k$. Thus,

$$\begin{aligned} P(n, k+1) &= P(n,k) \cdot (n-k) & \text{(Rule of Product)} \\ &= n^{\underline{k}} \cdot (n-k) & \text{(ind hyp)} \\ &= n(n-1)\cdots(n-k+1)\cdot(n-k) & \\ &= n^{\underline{k+1}} & \diamondsuit \end{aligned}$$

Unordered Selections

To evaluate $\binom{n}{k}$, which counts unordered selections, we regard the unordered selections as equivalence classes of ordered selections, in which two ordered selections of k objects are considered to be equivalent if they contain the exact same k objects.

Prop 0.4.2. *The # of unordered selections of k objects from a set of n objects is given by*

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!\,(n-k)!} \qquad (0.4.2)$$

Proof: By Prop 0.4.1, the number of ordered selections of k objects from S is $n^{\underline{k}}$. Since the number of orderings of k objects is $k!$, there are $k!$ ordered selections corresponding to each unordered selection. The conclusion follows from the Rule of Quotient. $\diamondsuit$

Selections with Repetitions

Couting *ordered selections* from a set S *with unlimited repetition allowed* is easy.

DEF: An ***ordered selection with unlimited repetition*** of k objects from a set S of size n is a finite sequence

$$x_1, \quad x_2, \quad \ldots, \quad x_k$$

of k objects, each of which is an element of S.

Prop 0.4.3. *The number of ordered selections of k objects from a set S of n objects is n^k.*

Proof: This is easily proved by an induction argument, involving the Rule of Product. $\diamondsuit$

Counting *unordered selection with unlimited repetitions allowed* seems difficult, if approached directly.

DEF: An ***unordered selection with unlimited rep*** of k objects from a set S of size n is a multiset (S, ι) of size k, with domain S.

Example 0.4.1: Consider unordered selections, with unlimited rep allowed, of four objects from the set $\{1, 2, 3, 4\}$. There are these four selections containing only one distinct digit

$$1111 \quad 2222 \quad 3333 \quad 4444$$

these 18 with two different digits

1112	1113	1114	2221	2223	2224
3331	3332	3334	4441	4442	4443
1122	1133	1144	2233	2244	3344

these 12 with three different digits

1123	1124	1134	2213	2214	2234
3312	3314	3324	4412	4413	4423

and only one with four different digits

1234

for a total of 35 possibilities.

To simplify the task of counting unordered selections with unlimited repetitions, we represent multisets as binary strings.

DEF: The ***bitcode for a multiset*** (S, ι) of cardinality k, with domain $\{1, 2, \ldots, n\}$, is defined recursively:

- If $n = 1$, then the bitcode is a string of k 0-bits.
- For $n > 1$, the bitcode for (S, ι) is the bitcode for the submultiset $(S - \{n\}, \iota)$, followed by a 1-bit, followed by a suffix of $\iota(n)$ 0-bits.

Example 0.4.1, cont.: For the domain $\{1, 2, 3, 4\}$, the bitcode for the multiset

$$\{1, 1, 3, 4\}$$

is 0011010.

Remark: In reconstructing a multiset from its bitcode, we may regard the 1-bits as separating the bitstring into n substrings of 0-bits, some of which may be nullstrings. The lengths of the n consecutive substrings of 0-bits are the multiplicities on the corresponding integers in the domain. This may be depicted as in Figure 0.4.1.

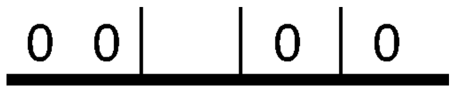

Fig 0.4.1 A representation of the bitstring 0011010.

Prop 0.4.4. *The correspondence between the set of multisets of cardinality k with domain $\{1, 2, \ldots, n\}$ and the set of bitstrings of length $n + k - 1$ with exactly $n - 1$ 1-bits is a bijection.*

Proof: One possible proof of this proposition is that the encoding of multisets as bitcodes is clearly invertible, which could be established by generalizing the inversion in Example 0.4.1. Another alternative is by induction. $\diamondsuit$

Cor 0.4.5. *The number of different multisets of cardinality k with domain $\{1, 2, \ldots, n\}$ is*

$$\binom{n+k-1}{n-1}$$

Proof: By Prop 0.4.2, the number of bitstrings of length $n+k-1$ with exactly $n-1$ 1-bits is

$$\binom{n+k-1}{n-1}$$

It follows from the Pigeonhole Principle, in view of Prop 0.4.4, that the number of different multisets of cardinality k with domain $\{1, 2, \ldots, n\}$ is the same as the number of bitstrings of length $n+k-1$ with exactly $n-1$ 1-bits. $\diamondsuit$

Example 0.4.1, cont.: By Cor 0.4.5, the number of multisets of cardinality four with domain $\{1, 2, 3, 4\}$ is

$$\binom{4+4-1}{3} = \binom{7}{3} = 35$$

Thus, Corollary 0.4.5 can greatly reduce the effort needed to count multisets with repetitions.

Example 0.4.2: Consider counting the number of possible outcomes of rolling three cubic dice, with the six sides of each die marked with 1 to 6 spots. Any two outcomes with the exact same number of instances of each of the six numbers of spots are regarded as equivalent. How many different possible outcomes are there? According to Corollary 0.4.5, the answer is

$$\binom{6+3-1}{5} = \binom{8}{5} = 56$$

Distributions into Labeled Cells

DEF: A *multicombination* from a set S of n objects is a distribution of the elements of S into k labeled cells

$$B_1 \quad B_2 \quad \ldots \quad B_k$$

(sometimes) called *boxes*. Although this does not distinguish the order of the objects with a cell, the cells are distinct.

DEF: The ***multicombination coefficient***

$$\binom{n}{r_1 \quad r_2 \quad \cdots \quad r_k}$$

is the number of ways to distribute a set of n objects into k labeled cells

$$B_1 \quad B_2 \quad \ldots \quad B_k$$

of respective sizes $r_1, r_2, \ldots, r_k$.

Proposition 0.4.6. *The values of the multicombination coefficients are given by the rule*

$$\binom{n}{r_1 \quad r_2 \quad \cdots \quad r_k} = \frac{n!}{r_1!\, r_2! \,\cdots\, r_k!} \tag{0.4.3}$$

Proof: The number of ways to select r_1 for box B_1 is

$$\binom{n}{r_1}$$

The number of ways to subsequently select r_2 for box B_2 from the remaining $n - r_1$ objects is

$$\binom{n - r_1}{r_2}$$

And so on. By the Rule of Product, it follows that the number of ways to complete the distribution is

$$\begin{aligned}
&\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\cdots-r_{k-1}}{r_k} \\
&= \frac{n!}{r_1!\,(n-r_1)!} \cdot \frac{(n-r_1)!}{r_2!\,(n-r_1-r_2)!} \cdot \cdots \\
&\qquad \cdots \quad \cdot \frac{(n-r_1-r_2-\cdots-r_{k-1})!}{r_k!\,0!} \\
&= \frac{n!}{r_1!\, r_2! \,\cdots\, r_k!}
\end{aligned}$$

by repeated application of the factorial formula (0.4.2) for binomial coefficients. ◇

Example 0.4.3: The distributions of $\{A, B, C, D\}$ into boxes of sizes $r_1 = 2$, $r_2 = 1$, and $r_3 = 1$ are given by this array

$$\begin{array}{ccc} AB|C|D & AC|B|D & AD|B|C \\ BC|A|D & BD|A|C & CD|A|B \\ AB|D|C & AC|D|B & AD|C|B \\ BC|D|A & BD|C|A & CD|B|A \end{array}$$

We could calculate the total number of distributions with a single multicombination coefficient

$$\binom{4}{2\;1\;1} = \frac{4!}{2!\,1!\,1!} = 12$$

TERMINOLOGY: Another name for the multicombination coefficient

$$\binom{n}{r_1 \quad r_2 \quad \cdots \quad r_k}$$

is the ***multinomial coefficient***, since it is provably the coefficient of the term

$$x_1^{r_1}\, x_2^{r_2} \;\cdots\; x_k^{r_k}$$

in the expansion of the exponentiated multinomial

$$(x_1 + x_2 + \cdots + x_k)^n$$

Distributions into Unlabeled Cells

The difference between distributions into labeled and into unlabeled cells is best explained with concrete examples. The main idea is the cells of the same size are regarded as interchangeable.

Example 0.4.4: Of four faculty in a small department, two will be advisors to the juniors and two to the seniors. According to Prop 0.4.6, the number of such distributions is

$$\frac{4!}{2!\,2!} = 6$$

If these faculty are designated A, B, C, and D, the six possible distributions are

	juniors	seniors
1.	AB	CD
2.	AC	BD
3.	AD	BC
4.	CD	AB
5.	BD	AC
6.	BC	AD

However, if we discard the labels *juniors* and *seniors* then there are only three ways that the four faculty are grouped into pairs. The distributions 1 and 4 would be indistinguishable, as would distributions 2 and 5 and distributions 3 and 6.

The following proposition gives the formula for counting distributions into unlabeled cells.

Proposition 0.4.7. *Let S be a set of n objects. Suppose that these objects are to be distributed into b_j boxes of size r_j, for $j = 1, \ldots, k$, with*

$$\sum_{j=1}^{k} b_j \, r_j \;=\; n$$

The number of ways to do this is

$$\frac{n!}{(r_1!)^{b_1} \, (r_2!)^{b_2} \, \cdots \, (r_k!)^{b_k}} \cdot \frac{1}{b_1! \, b_2! \, \cdots \, b_k!} \tag{0.4.4}$$

Proof: This follows from Proposition 0.4.6 and the Rule of Quotient. ◊

Partitions of a Set

PREVIEW OF §1.6:

- A *partition of a set* into k cells can be characterized as a distribution of that set into k unlabeled boxes with none left empty.
- The ***Stirling subset number*** $\left\{ {n \atop k} \right\}$ is the number of ways to partition a set with n objects into k cells.

Formula (0.4.4) enables us to calculate the number of partitions of a set of n objects into cells of prespecified sizes.

Example 0.4.4, cont.: The number of partitions of a set of four objects into two cells, both of size two, is

$$\frac{4!}{2!\,2!} \cdot \frac{1}{2!} \;=\; 3$$

Example 0.4.5: A set with four objects may be partitioned into two cells either with sizes 3 and 1 or with cells of sizes 2 and 2. Thus,

$$\left\{ {4 \atop 2} \right\} \;=\; \binom{4}{3\ 1} + \binom{4}{2\ 2} \cdot \frac{1}{2!} \;=\; 4+3 \;=\; 7$$

0.5 PERMUTATIONS

Counting configurations with symmetries, like the tic-tac-toe boards of Example 0.3.4, requires some algebra involving *permutations*, as seen in Chapter 9.

DEF: A ***permutation*** of a set S is a bijection (a one-to-one, onto function) from S to itself.

2-Line Representation of Permutations

DEF: The ***2-line representation of a permutation*** π of a set S is a 2-line array that lists the objects of S in its top row. Below each object x is its image $\pi(x)$ under the permutation.

Example 0.5.1: The permutation π of $\{1, 2, \ldots, 9\}$ s.t.

$$\begin{array}{c} 1 \mapsto 7 \quad 2 \mapsto 4 \quad 3 \mapsto 1 \quad 4 \mapsto 8 \\ 5 \mapsto 5 \quad 6 \mapsto 2 \quad 7 \mapsto 9 \quad 8 \mapsto 6 \quad 9 \mapsto 3 \end{array}$$

is represented by the 2-line array

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix}$$

which is illustrated by Figure 0.5.1.

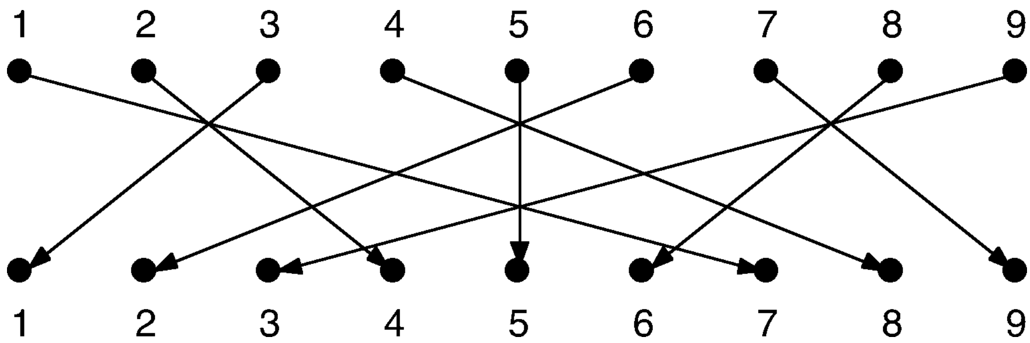

Fig 0.5.1 A permutation of the set $\{1, 2, \ldots, 9\}$.

DEF: The ***inverse of a permutation*** π on a set S is the permutation π^{-1} that restores each object of S to its position before the application of π.

The 2-line representation of the inverse of a permutation can be obtained by transposing the rows, possibly sorting the columns according to the entry in the resulting first row.

Example 0.5.1, cont.:

$$\pi^{-1} = \begin{pmatrix} 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} \quad \text{or}$$

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 9 & 2 & 5 & 8 & 1 & 4 & 7 \end{pmatrix}$$

Composition of Permutations

DEF: The ***composition of permutations*** π and τ is the permutation $\pi \circ \tau$ resulting from first applying π and then applying τ. Thus, $(\pi \circ \tau)(x) = \tau(\pi(x))$.

Obtaining the 2-line representation of the composition $\pi \circ \tau$ is a 2-step process.

1. Rearrange the columns of τ (the perm to be applied second) so that in each column, the top entry is the same as the bottom entry of π (the perm to be applied first).
2. The top line of the 2-line array for the composition $\pi \circ \tau$ is the top line of the array for π. The bottom line for $\pi \circ \tau$ is the bottom line for the rearranged representation of τ.

Example 0.5.1, cont.: Suppose that

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 5 & 3 & 1 & 9 & 2 & 8 & 7 & 4 \end{pmatrix}$$

Transposing the columns of τ facilitates the computation

$$\begin{aligned} \pi &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix} \\ \tau &= \begin{pmatrix} 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \\ 8 & 1 & 6 & 7 & 9 & 5 & 4 & 2 & 3 \end{pmatrix} \\ \pi \circ \tau &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 1 & 6 & 7 & 9 & 5 & 4 & 2 & 3 \end{pmatrix} \end{aligned}$$

For instance, since π maps whatever is in position 1 to position 7 and τ maps whatever is in position 7 to position 8, the composition $\pi \circ \tau$ maps whatever is in position 1 to position 8. This composition is illustrated in Figure 0.5.2.

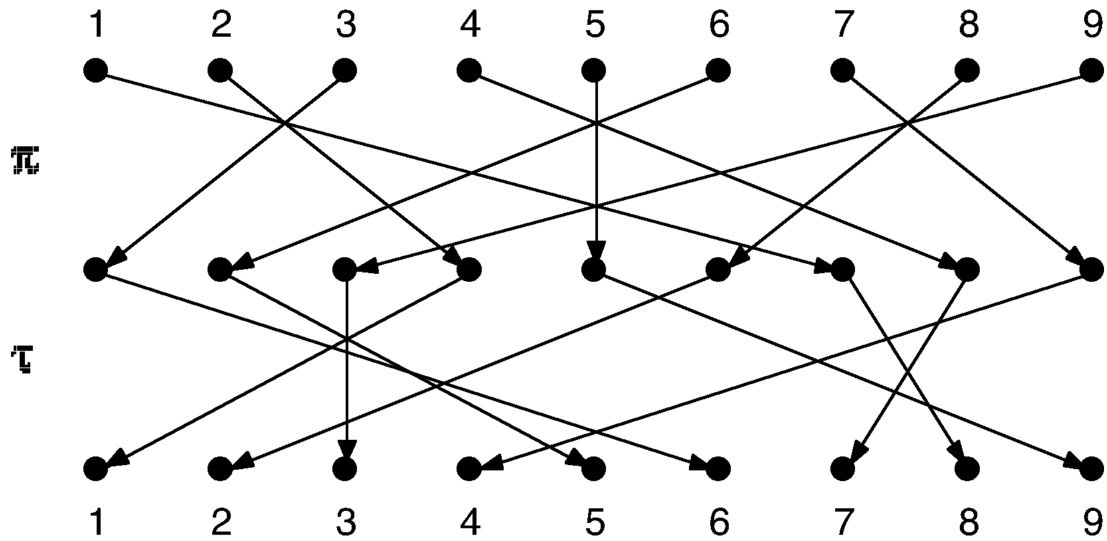

Fig 0.5.2 A composition $\pi \circ \tau$ of permutations.

Cyclic Permutations

A *cyclic permutation* takes each object of the permuted set successively through the positions of all the other objects.

DEF: A permutation of the form

$$\begin{pmatrix} x & \pi(x) & \pi^2(x) & \cdots & \pi^{p-2}(x) & \pi^{p-1}(x) \\ \pi(x) & \pi^2(x) & \pi^3(3) & \cdots & \pi^{p-1}(x) & x \end{pmatrix}$$

is said to be ***cyclic*** of ***period*** p.

NOTATION: A cyclic permutation is commonly represented in the ***cyclic form***

$$\begin{pmatrix} x & \pi(x) & \pi^2(x) & \cdots & \pi^{p-2}(x) & \pi^{p-1}(x) \end{pmatrix}$$

Example 0.5.2: The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{pmatrix}$$
$$= \quad (1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7)$$

is cyclic of period 7. Its cyclic form is depicted by Figure 0.5.3 as a directed cycle.

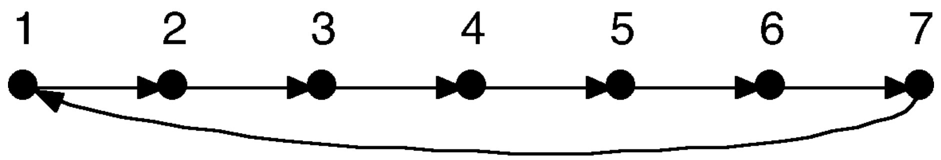

Fig 0.5.3 Cyclic perm as directed cycle.

Example 0.5.3: The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 2 & 5 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 & 6 & 4 & 5 \\ 3 & 2 & 6 & 4 & 5 & 1 \end{pmatrix}$$
$$= \quad (1 \quad 3 \quad 2 \quad 6 \quad 4 \quad 5)$$

is cyclic of period 6. It is depicted as a directed cycle in Figure 0.5.4.

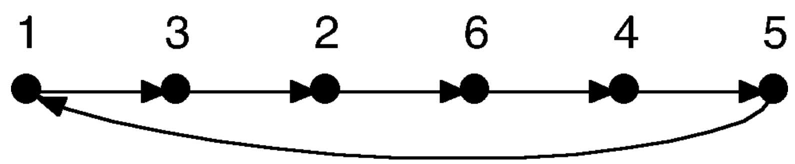

Fig 0.5.4 Another cyclic permutation.

Disjoint Cycle Representation

A permutation π of a finite set S is a composition of cyclic permutations on various disjoint subsets of S. Its structure is understood in terms of the lengths of these cycles of objects.

Prop 0.5.1. *Let π be a permutation on a finite set S and let $x \in S$. Then the sequence*

$$x \quad \pi(x) \quad \pi^2(x) \quad \pi^3(x) \quad \ldots$$

has an entry $\pi^j(x)$ s.t. $\pi^j(x) = x$, and the seq is periodic with period j, for the 1^{st} such j.

Proof: Since the set S is finite, the sequence must eventually contain some entry $\pi^j(x)$ that matches a previous entry. Suppose that $\pi^i(x)$ is the previous entry such that

$$\pi^j(x) = \pi^i(x)$$

Then

$$\begin{aligned} \pi^{j-i}(x) &= \pi^{-i}(\pi^j(x)) \\ &= \pi^{-i}(\pi^i(x)) \\ &= \pi^0(x) \;=\; x \end{aligned}$$

Since $j > i \geq 0$, since $\pi^j(x)$ is the first duplicate of a previous entry, and since π^{j-i} duplicates the initial entry x, it follows that $j - i \geq j$, which implies that $i = 0$. Since $\pi^j(x) = x$, it follows that the subsequence

$$x \quad \pi(x) \quad \pi^2(x) \quad \pi^3(x) \quad \ldots \quad \pi^{j-1}(x)$$

is endlessly reiterated. $\diamondsuit$

What now follows is a somewhat informal description of a method for representing an arbitrary permutation π on a finite set S as a composition of cyclic permutations.

Step 1: Choose an arbitrary element $x_1 \in S$. Let k_1 be the smallest integer s.t. $\pi^{k_1}(x_1) = x_1$. Let T_1 be the subset

$$T_1 = \{x_1,\ \pi(x_1),\ \pi^2(x_1),\ \ldots,\ \pi^{k_1-1}(x_1)\}$$

Then the restriction $\pi|_{T_1}$ of the permutation π to the subset T_1 is the cyclic permutation

$$\pi|_{T_1} = \big(x_1 \quad \pi(x_1) \quad \pi^2(x_1) \quad \ldots \quad \pi^{k_1-1}(x_1)\big)$$

Example 0.5.1, cont.: For the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix}$$

consider the choice $x_1 = 1$. This leads to the subset

$$T_1 = \{1,\ 7,\ 9,\ 3\}$$

and to the restricted permutation

$$\pi|_{T_1} = (1 \quad 7 \quad 9 \quad 3)$$

Step 2: In general, if $T_1 = S$, then π is cyclic on S, and $\pi = \pi|_{T_1}$. Otherwise, choose an arbitrary element

$$x_2 \in S - T_1$$

Let k_2 be the smallest integer s.t. $\pi^{k_2}(x_2) = x_2$. Let T_2 be the subset

$$T_2 \;=\; \{x_2,\ \pi(x_2),\ \pi^2(x_2),\ \ldots,\ \pi^{k_2-1}(x_2)\}$$

Then the restriction $\pi|_{T_2}$ of the permutation π to the subset T_2 is the cyclic permutation

$$\begin{pmatrix} x_2 & \pi(x_2) & \pi^2(x_2) & \ldots & \pi^{k_2-1}(x_2) \end{pmatrix}$$

Example 0.5.1, cont.: Choosing the second element $x_2 = 2$ for the permutation

$$\pi \;=\; \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix}$$

leads to the subset

$$T_2 \;=\; \{2,\ 4,\ 8,\ 6\}$$

and to the restricted permutation

$$\pi|_{T_2} \;=\; \begin{pmatrix} 2 & 4 & 8 & 6 \end{pmatrix}$$

We observe that subsets T_1 and T_2 are disjoint.

Proposition 0.5.2. *Let π be a permutation on a finite set S and let $x \in S$. Let*

$$T = \{\pi^i(x) \mid i \in \mathbb{N}\}$$

Let $y \in S - T$ and let

$$T' = \{\pi^j(y) \mid j \in \mathbb{N}\}$$

Then the subsets T and T' are disjoint.

Proof: If not, then there are nonnegative numbers i and j such that

$$\pi^i(x) = \pi^j(y) \tag{0.5.1}$$

Without loss of generality, assume that $j \leq i$. Then

$$\begin{aligned} \pi^{i-j}(x) &= \pi^{-j}(\pi^i(x)) \\ &= \pi^{-j}(\pi^j(y)) \qquad \text{by (0.5.1)} \\ &= y \end{aligned}$$

which contradicts the premise that $y \notin T$. $\diamondsuit$

Step 3: Having selected the mutually disjoint subsets $T_1,\ T_2, \ldots,\ T_k$ in this manner, if

$$T_1 \cup T_2 \cup \cdots \cup T_k = S$$

then go to Step 4, since the decomposition of π is complete. Otherwise, choose $x_{k+1} \in S - (T_1 \cup T_2 \cup \cdots \cup T_k)$ and continue as in Step 2.

Example 0.5.1, cont.: The only remaining element in the set $\{1, 2, \ldots, 9\}$, on which the perm

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix}$$

acts, is the element $x_3 = 5$, which leads to the subset

$$T_3 = \{5\}$$

and to the restricted permutation

$$\pi|_{T_3} = (\,5\,)$$

We observe that the subsets T_1, T_2, and T_3 form a partition of the set $[1:9]$.

Step 4: This step occurs after the set S has been partitioned into subsets T_1, T_2, $\ldots$, T_k. Represent the permutation π in the form

$$\pi = \pi|_{T_1}\,\pi|_{T_2}\;\cdots\;\pi|_{T_k}$$

Example 0.5.1, continued: The net result of applying these steps to the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix}$$

is the representation

$$\pi = (1 \quad 7 \quad 9 \quad 3)\,(2 \quad 4 \quad 8 \quad 6)\,(\,5\,)$$

DEF: A ***disjoint cycle representation*** of a perm π on a set S is as a composition of cyclic perms on subsets of S that constitute a partition of S, one cyclic perm for each subset in the partition.

The decomposition process described just above serves as a constructive proof of the following theorem.

Thm 0.5.3. *Let π be a perm of a finite set S. Then π has a disjoint cycle representation.* $\diamondsuit$

We conclude this subsection with an illustration that it is straightforward to compute the disjoint cycle representation of a composition of two permutations π and τ from the disjoint cycle representations of the factors π and τ.

Example 0.5.1, cont: The disjoint cycle forms of the permutations

$$\begin{aligned}
\pi &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{pmatrix} \\
\tau &= \begin{pmatrix} 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \\ 8 & 1 & 6 & 7 & 9 & 5 & 4 & 2 & 3 \end{pmatrix} \\
\pi \circ \tau &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 1 & 6 & 7 & 9 & 5 & 4 & 2 & 3 \end{pmatrix}
\end{aligned}$$

are

$$\begin{aligned}
\pi &= (1 \quad 7 \quad 9 \quad 3)(2 \quad 4 \quad 8 \quad 6)(5) \\
\tau &= (1 \quad 6 \quad 2 \quad 5 \quad 9 \quad 4)(3)(7 \quad 8) \\
\pi \circ \tau &= (1 \quad 8 \quad 2)(3 \quad 6 \quad 5 \quad 9)(4 \quad 7)
\end{aligned}$$

Starting with the disjoint cycle forms

$$\begin{aligned}
\pi &= (1 \quad 7 \quad 9 \quad 3)(2 \quad 4 \quad 8 \quad 6)(5) \\
\text{and} \quad \tau &= (1 \quad 6 \quad 2 \quad 5 \quad 9 \quad 4)(3)(7 \quad 8)
\end{aligned}$$

the first cycle of $\pi \circ \tau$ is computed as follows:

$$\begin{aligned}
1 \overbrace{\mapsto}^{\pi} 7 \overbrace{\mapsto}^{\tau} 8 \\
8 \overbrace{\mapsto}^{\pi} 6 \overbrace{\mapsto}^{\tau} 2 \\
2 \overbrace{\mapsto}^{\pi} 4 \overbrace{\mapsto}^{\tau} 1
\end{aligned}$$

That is, the first cycle of the disjoint cycle representation of $\pi \circ \tau$ may be written as

$$(1 \quad 8 \quad 2)$$

The computation then continues

$$\begin{gathered}
3 \overbrace{\mapsto}^{\pi} 1 \overbrace{\mapsto}^{\tau} 6 \\
6 \overbrace{\mapsto}^{\pi} 2 \overbrace{\mapsto}^{\tau} 5 \\
5 \overbrace{\mapsto}^{\pi} 5 \overbrace{\mapsto}^{\tau} 9 \\
9 \overbrace{\mapsto}^{\pi} 3 \overbrace{\mapsto}^{\tau} 3
\end{gathered}$$

which yields

$$(3 \quad 6 \quad 5 \quad 9)$$

as the second cycle of the permutation $\pi \circ \tau$. It concludes with

$$\begin{gathered}
4 \overbrace{\mapsto}^{\pi} 8 \overbrace{\mapsto}^{\tau} 7 \\
7 \overbrace{\mapsto}^{\pi} 9 \overbrace{\mapsto}^{\tau} 4
\end{gathered}$$

which yields as the third cycle

$$(4 \quad 7)$$

0.6 GRAPHS

One widely studied combinatorial structure is called a *graph*.

DEF: A ***graph*** $G = (V, E)$ has two finite sets V and E, called ***vertices*** and ***edges***, respectively. Each edge has a set of one or two vertices associated to it, which are called its ***endpoints***.

Example 0.6.1: Fig 0.6.1 illustrates a graph.

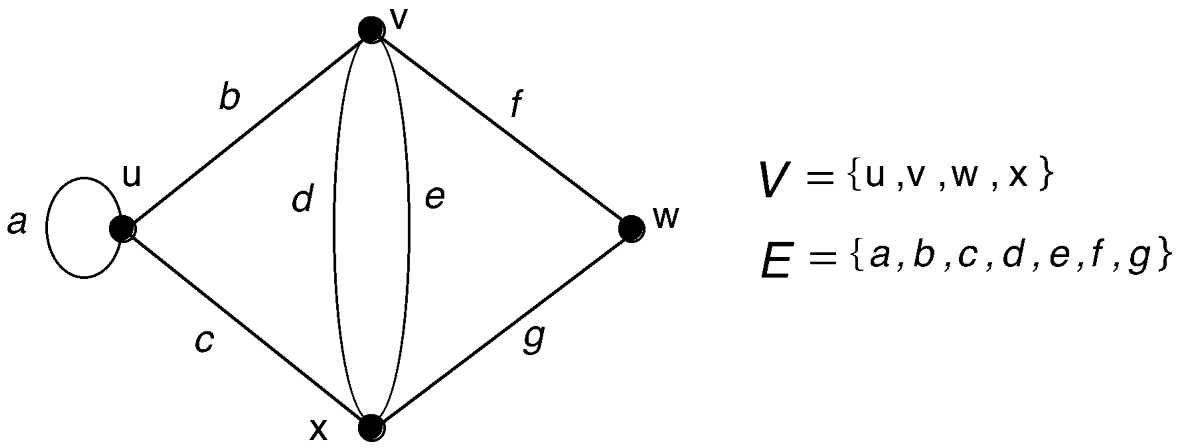

Fig 0.6.1 A graph.

TERMINOLOGY: An edge is said to ***join*** its endpoints. A vertex joined by an edge to a vertex v is said to be a ***neighbor*** of v. Two neighboring vertices are said to be ***adjacent***.

Simple and General Graphs

DEF: A ***proper edge*** is an edge that joins two distinct vertices. A ***self-loop*** is an edge that joins a single endpoint to itself.*

DEF: A ***multi-edge*** is a collection of two or more edges having identical endpoints. The ***multiplicity of a multi-edge*** is the number of edges within the multi-edge.

DEF: A ***simple graph*** is a graph with no self-loops or multi-edges. A ***general graph*** may have self-loops and/or multi-edges. (Thus, the graph in Figure 0.6.1 is a general graph.)

Degree of a Vertex

DEF: The ***degree*** (or ***valence***) of a vertex v in a graph G, denoted $deg(v)$, is the number of proper edges incident on v plus twice the number of self-loops.

Example 0.6.1, cont.: The caption of Figure 0.6.2 lists the degrees of the graph from Figure 0.6.1.

---

* We use the more precise term "self-loop" instead of the more commonly used term "loop".

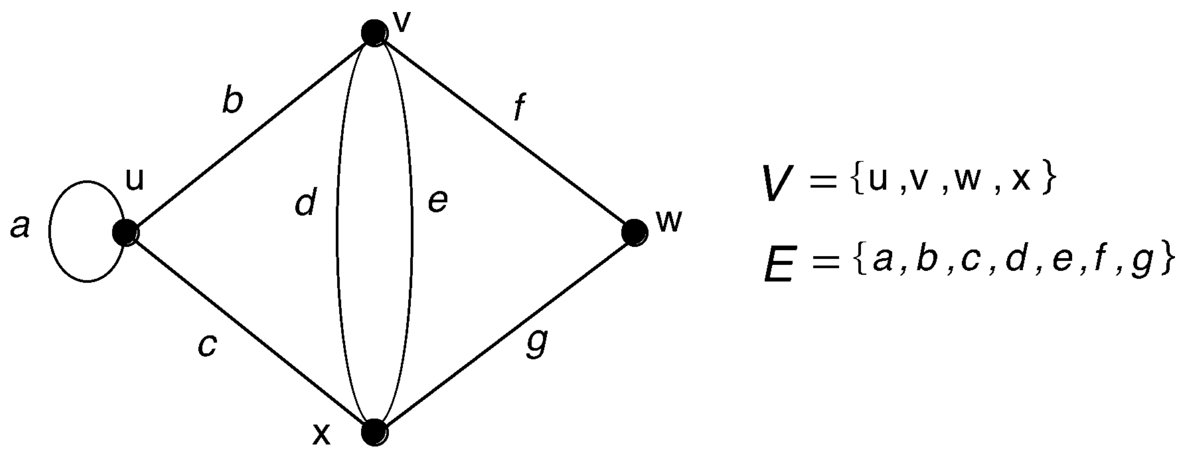

Fig 0.6.2 $\mathbf{deg(u) = deg(v) = deg(x) = 4}$**, and** $\mathbf{deg(w) = 2}$**.**

Thm 0.6.1 [*Euler's Degree-Sum Thm*] The sum of the degrees of the vertices of a graph is twice the number of edges.

Proof: Each edge contributes two to the degree sum. ♢

N.B. Chapter 9 presumes familiarity with this section and with Chapter 7.

0.7 NUMERIC OPERATIONS

One number-theoretic operation that occurs often in combinatorics is the *greatest common divisor.* Although our textbook examples are focused on small enough problems of this type to do the calculation by hand, consider trying to calculate the greatest common divisor of larger numbers, such as

$$32582657 \quad \text{and} \quad 24036583$$

DEF: The ***greatest common divisor*** of two integers m and n, not both zero, mnemonically denoted

$$\gcd(m, n)$$

is the largest integer that divides both m and n.

DEF: The ***least common multiple*** of two integers m and n, mnemonically denoted

$$\operatorname{lcm}(m, n)$$

is the smallest non-negative integer that is a non-zero multiple of both m and n.

When the prime factors are already known or easily calculated, it is quite easy to calculate a greatest common divisor by a method commonly taught in middle schools.

It involves factoring the two numbers into products of primes. Although this might seem easy for small numbers, the factoring of large numbers may require considerable effort. A method called the

Euclidean algorithm

described in Chapter 6, avoids the need to factor, and it produces the answer in time proportional to the number of digits of the larger number.

Another operation we use in trying to count or to construct all the graphs of a given kind involves listing all the ways to decompose an integer n into an iterated sum of positive integers. Such a sum is called a

partition of the integer n

Example 0.7.1: The number 8 has five partitions into exactly four summands, namely

$$\begin{array}{lll} 5+1+1+1 & 4+2+1+1 & 3+3+1+1 \\ 3+2+2+1 & 2+2+2+2 & \end{array}$$

0.8 COMBINATORIAL DESIGNS

The final type of discrete structure in this book, in Chapter 10, is called a *combinatorial design*.

DEF: A ***combinatorial design*** $\mathcal{B}$ has a non-empty domain of objects

$$X = \{x_1, x_2, \ldots, x_v\}$$

and a non-empty collection of subsets of objects from X.

$$B = \{B_1, B_2, \ldots, B_b\}$$

Sometimes these subsets may be called *blocks*.

The art of constructing combinatorial designs is in meeting various additional requirements on the subsets B_j.

DEF: In a ***regular block design*** the subsets B_j all have the same cardinality k, called the ***blocksize***. Moreover, each object x_i occurs in the same number r of blocks, which is called the ***replication number***.

The following example illustrates a possible application of such a design. In a ***round-robin tournament*** each contestant plays each other contestant. r

Example 0.8.1: Consider designing a round-robin playoff for 13 contestants in a competitive game for 4 players that ranks the players from 1^{st} to 4^{th} in each round. Such an event might plausibly have 13 rounds in which each of the players, designated as

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad A \quad B \quad C$$

plays four rounds and meets each other player exactly once, as follows:

Round	Players
1	0146
2	1257
3	2368
4	3479
5	$458A$
6	$569B$
7	$67AC$
8	$78B0$
9	$89C1$
10	$9A02$
11	$AB13$
12	$BC24$
13	$C035$

This playoff might be represented by Fig 0.8.1. Each of 12 groupings of four players is represented by a curve that goes through the corresponding four points. (The 13^{th} grouping is $67AC$.) Only four of these groupings are actually represented by straight lines in the drawing.

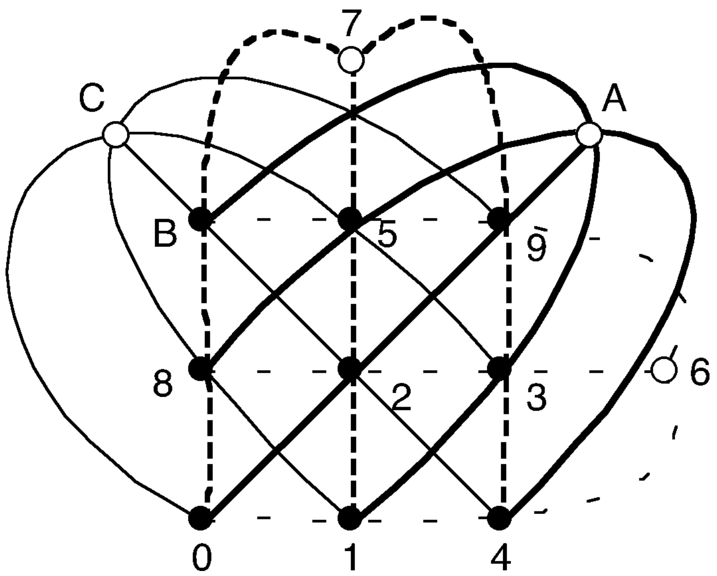

Fig 0.8.1 Geometric representation of a block design.

A ***balanced block design*** is a regular block design in which each pair of points occurs in the same number of lines. It is called incomplete if the blocksize is less than the number of points in the domain.

Example 0.8.1, cont.: The playoff described here is a balanced incomplete block design, if each grouping of four players is regarded as a line.

In a kind of combinatorial design called a ***finite geometry***, the subsets of objects are called ***lines***. There are numerous kinds of finite geometry. A general requirement is that each pair of points lies on at most one line.

Example 0.8.1, cont.: As it happens, the balanced block design of Figure 0.8.1 is also a finite geometry.