Chapter 7

Planarity A: Lec 8

7.1 Planar Drawings and Some Basic Surfaces
7.2 Subdivision and Homeomorphism
7.3 Extending Planar Drawings
7.1 PLANAR DRAWINGS; SURFACES

DEF: A *planar drawing* of a graph is a drawing of the graph in the plane without edge-crossings.

DEF: A graph is said to be *planar* if there exists a planar drawing of it.

Example 7.1.1: Two drawings of $K_4$ are shown in Fig 7.1.1. At the right we see that $K_4$ is a planar graph.

Fig 7.1.1 Nonplanar drawing and planar drawing of $K_4$. 
TERMINOLOGY: Intuitively, we see that a planar drawing of a graph has exactly one exterior (or outer) region whose area is infinite.

Example 7.1.3: The exterior region, $R_e$, and the three finite regions of a planar drawing of a graph are shown in Fig 7.1.2.

![Fig 7.1.2](image)

The four regions of a planar drawing of a graph.
Three Basic Surfaces

**DEF:** A *plane* in Euclidean 3-space $\mathbb{R}^3$ is a set of points $(x, y, z)$ such that there are numbers $a$, $b$, $c$, and $d$ with

$$ax + by + cz = d$$

**DEF:** A *sphere* is a set of points in $\mathbb{R}^3$ equidistant from a fixed point.

**DEF:** The *standard torus* is the surface of revolution obtained by revolving a circle of radius 1 centered at $(2,0)$ in the $xy$-plane disk around the $y$-axis in 3-space, as depicted in Figure 7.1.3.

![Creating a torus.](image)

**DEF:** The solid inside a torus is called the *standard donut.*
DEF: The circle of intersection, as in Figure 7.1.4(a), of the standard torus with the half-plane

$$\{(x, y, z) \mid x = 0, y \leq 0\}$$

is called the **standard meridian**. We observe that the standard meridian bounds a disk inside the standard donut.

DEF: The circle of tangent intersection of the standard torus with the plane $y = 1$ is called the **standard longitude**. Figure 7.1.4(b) illustrates the standard longitude. We observe that the standard longitude bounds a disk in the plane $y = 1$ that lies outside the standard donut.

![Diagram](image)

**Fig 7.1.4** (a) Std meridian and (b) std longitude.

**Remark:** The three surfaces described above are all relatively uncomplicated. In §8.2, the *Möbius band* and the *Klein bottle* are defined, and it is described how the surfaces generally fall into two infinite sequences.

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Riemann Stereographic Projection

DEF: The Riemann stereographic projection is the function $\rho$ that maps each point $w$ of the unit-diameter sphere (tangent at the origin $(0,0,0)$ to the $xz$-plane in Euclidean 3-space) to the point $\rho(x)$ where the ray from the north pole $(0,1,0)$ through point $w$ intersects the $xz$-plane.

Under the Riemann projection, the “southern hemisphere” of the sphere is mapped continuously onto the unit disk. The “northern hemisphere” (minus the north pole) is mapped continuously onto the rest of the plane. The points nearest to the north pole are mapped to the points farthest from the origin. Figure 7.1.5 illustrates the construction.

**Fig 7.1.5** The Riemann stereographic projection.

**Prop 7.1.1.** A graph is planar if and only if it can be drawn without edge-crossings on the sphere.

**Pf:** This is an immediate consequence of the Riemann stereographic projection.  

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Jordan Separation Property

**DEF:** A *Euclidean set* is a subset of Euclidean space \( \mathbb{R}^n \).

**DEF:** An *open path from s to t* in a Euclidean set \( X \) is the image of a continuous bijection \( f \) from the unit interval \([0, 1]\) to a subset of \( X \) such that \( f(0) = s \) and \( f(1) = t \).

**DEF:** A *closed path* or *closed curve* in a Euclidean set is the image of a continuous function \( f \) from the unit interval \([0, 1]\) to a subset of that space such that \( f(0) = f(1) \), but which is otherwise a bijection.

**Example 7.1.4:** Fig 7.1.6 shows an open path and a closed curve in the plane.

![Open path and closed curve](image)

**Fig 7.1.6** Open path and closed curve (= closed path).

**Example 7.1.5:** In a crossing-free drawing of a graph on a surface, a cycle of the graph is a closed curve.
DEF: A Euclidean set $X$ is **connected**\(\dagger\) if for every pair of points $s, t \in X$, there exists a path within $X$ from $s$ to $t$.

DEF: The Euclidean set $X$ **separates** the connected Euclidean set $Y$ if there exist a pair of points $s$ and $t$ in $Y - X$, such that every path in $Y$ from $s$ to $t$ intersects the set $X$.

**Example 7.1.6:** Fig 7.1.7 shows a meandering closed curve in the plane. It separates the white part from the shaded part.

\[\text{Fig 7.1.7  A closed curve separating the plane.}\]

\(\dagger\) A topologist would say **path-connected**.
What makes the plane and the sphere the simplest surfaces for drawing graphs is the *Jordan separation property*.

**DEF:** A Euclidean set $X$ has the *Jordan separation property* if every closed curve in $X$ separates $X$.

**Theorem 7.1.2 (Jordan Curve Thm).** Every closed curve in the sphere (plane) has the Jordan separation property, that is, it separates the sphere (plane) into two regions, one of which contains the north pole (contains “infinity”). ◇ (pf omitted)

**Corollary 7.1.3.** A path from one point on the boundary of a disk through the interior to another point on the boundary separates the disk. ◇ (proof omitted)
Applying JCT to $K_5$ and $K_{3,3}$

It is possible to prove that a particular graph is nonplanar directly from the Jordan Curve Theorem.

**Thm 7.1.4.** Every drawing of the complete graph $K_5$ in the sphere (or plane) contains at least one edge-crossing.

**Pf:** Label the vertices $0, \ldots, 4$. By the JCT, any drawing of the cycle

$$\langle 1, 2, 3, 4, 1 \rangle$$

separates the sphere into two regions. Consider the region with vertex 0 in its interior as the “inside” of the cycle. By the JCT, the edges joining vertex 0 to each of the vertices 1, 2, 3, and 4 must also lie entirely inside the cycle, as illustrated in Figure 7.1.8.

![Figure 7.1.8](image)

**Fig 7.1.8** Drawing most of $K_5$ in the sphere.
Moreover, each of the 3-cycles

$$\langle 0, 1, 2, 0 \rangle, \langle 0, 2, 3, 0 \rangle, \langle 0, 3, 4, 0 \rangle, \text{ and } \langle 0, 4, 1, 0 \rangle$$

also separates the sphere, and, hence, edge 24 must lie to the exterior of the cycle $$\langle 1, 2, 3, 4, 1 \rangle$$, as shown.

It follows that the cycle formed by edges 24, 40, and 02 separates vertices 1 and 3, again by the JCT. Thus, it is impossible to draw edge 13 without crossing an edge of that cycle. \(\diamond\)
Thm 7.1.5. Every drawing of the complete bipartite graph $K_{3,3}$ in the sphere (or plane) contains at least one edge-crossing.

Pf: Label the vertices of one partite set $0, 2, 4,$ and of the other $1, 3, 5$. By the JCT, cycle

$$\langle 2, 3, 4, 5, 2 \rangle$$

separates the sphere into two regions, and, as in the previous proof, we regard the region containing vertex $0$ as the "inside" of the cycle. By the JCT, the edges joining vertex $0$ to each of the vertices $3$ and $5$ lie entirely inside that cycle, and each of the cycles $\langle 0, 3, 2, 5, 0 \rangle$ and $\langle 0, 3, 4, 5, 0 \rangle$ separates the sphere, as illustrated in Figure 7.1.9.

![Figure 7.1.9](image_url)

Fig 7.1.9  Drawing most of $K_{3,3}$ in the sphere.

Thus, there are three regions: the exterior of cycle $\langle 2, 3, 4, 5, 2 \rangle$, and the inside of each of the other two cycles. It follows that no matter which region contains vertex $1$, there must be some even-numbered vertex that is not in that region, and, hence, the edge from vertex $1$ to that even-numbered vertex would have to cross some cycle edge.  

\[\diamond\]
Cor 7.1.6. If either $K_5$ or $K_{3,3}$ is a subgraph of a graph $G$ then every drawing of $G$ in the sphere (or plane) contains at least one edge-crossing.  

DEF: The complete graph $K_5$ and the complete bipartite graph $K_{3,3}$ are called the Kuratowski graphs.

![Graphs K5 and K3,3](image_url)

Fig 7.1.10 The Kuratowski graphs.
**Example 7.1.7:** Figure 7.1.11 illustrates that $K_5$ can be drawn without edge-crossings on the torus, even though this is impossible on the sphere. It becomes clear in Chapter 8 that for every graph there is a surface on which a crossing-free drawing is possible.

**Fig 7.1.11** A crossing-free drawing of $K_5$ on the torus.
7.2 SUBDIVISION; HOMEO MorPHISM

Fig 7.2.2 Subdividing an edge.

Fig 7.2.2 Smoothing away a vertex.

Fig 7.2.3 Subdividing an edge of $C_3$ yields $C_4$.

**DEF:** *Subdividing a graph* $G$ means performing a sequence of edge-subdivision operations. The resulting graph is called a ***subdivision of the graph*** $G$.

**Prop 7.2.1.** A subdivision of a graph can be drawn without edge-crossings on a surface iff the graph itself can be drawn without edge-crossings on that surface.

**Pf:** When the operations of subdivision and smoothing are performed on a copy of the graph already drawn on the surface, they neither introduce nor remove edge-crossings.

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Barycentric Subdivision

DEF: The (first) barycentric subdivision of a graph is the subdivision in which one new vertex is inserted in the interior of each edge.

Fig 7.2.4 A graph and its barycentric subdivision.

Prop 7.2.2. The bary subdiv of any graph is bipartite.

Prop 7.2.3. The bary subdiv of any graph is loopless.

Prop 7.2.4. The bary subdiv of any loopless graph is simple.
DEF: The $n^{th}$ **barycentric subdivision** of a graph is the first barycentric subdivision of the $(n-1)^{st}$ barycentric subdivision.

**Prop 7.2.5.** The 2nd bary subdiv of any graph is a simple graph.

![Diagram](image)

**Fig 7.2.5** A graph and its 2nd barycentric subdivision.
Graph Homeomorphism

**Def:** Graphs $G$ and $H$ are **homeomorphic graphs** if there is an isomorphism from a subdivision of $G$ to a subdivision of $H$.

**Example 7.2.1:** Graphs $G$ and $H$ in Fig 7.2.6 cannot be isom, since $G$ is bipartite and $H$ contains a 3-cycle. However, they are homeom, because if edge $d$ in $G$ and edge $e$ in $H$ are both subdivided, then the resulting graphs are isomorphic.

![Fig 7.2.6](image)

**Fig 7.2.6** Two non-isom graphs that are homeomorphic.

**Remark:** Notice in Example 7.2.1 that no subdivision of graph $G$ is isomorphic to graph $H$, and that no subdivision of graph $H$ is isomorphic to graph $G$.

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Prop 7.2.6. Let $G$ and $H$ be homeomorphic graphs. Then $G$ can be drawn without edge-crossings on a surface $S$ if and only if $H$ can be drawn on $S$ without edge-crossings.

Pf: This follows from iterated application of Prop 7.2.1. ◊

Prop 7.2.7. Every graph is homeom to a bipartite graph.

Pf: By Proposition 7.2.2, the barycentric subdivision of a graph is bipartite. Of course, a graph is homeomorphic to a subdivision of itself. ◊
Subgraph Homeomorphism Problem

Deciding whether a graph $G$ contains a subgraph that is homeomorphic to a target graph $H$ is a common problem in graph theory.

**Example 7.2.2:** Fig 7.2.7 shows that $K_{3,3}$ contains a subgraph that is homeom to $K_4$. Four of the edges of $K_4$ are represented by the cycle 0-1-2-3-0, and the other two are represented by the two paths of length 2 indicated by broken lines.

![Figure 7.2.7](image)

**Fig 7.2.7** A homeomorphic copy of $K_4$ in $K_{3,3}$.

Notice in Fig 7.2.7 that each side of the bipartition of $K_{3,3}$ contains two of the images of vertices of $K_4$. A homeomorphic copy of $K_4$ in $K_{3,3}$ cannot have three vertex images on one side of the bipartition.
To see this, suppose (without loss of generality) that vertex images 0, 1, and 2 are on one side, and that vertex 3 is on the other side, as illustrated in Fig 7.2.8 below. A homeomorphic copy of $K_4$ would require three internally disjoint paths joining the three possible pairs of vertices 0, 1, and 2. But this is impossible, since there are only two remaining vertices on the other side that can be used as internal vertices of these paths.

Fig 7.2.8  An impossible way to place $K_4$ into $K_{3,3}$. 
Example 7.2.3: Fig 7.2.9 shows that the hypercube graph $Q_4$ contains a subgraph that is homeomorphic to the complete graph $K_5$. The five labeled white vertices represent vertices of $K_5$. The ten internally disjoint paths joining the ten different pairs of these vertices are given bold edges. Gray vertices are internal vertices along such paths.

Fig 7.2.9 A homeomorphic copy of $K_5$ in $Q_4$. 

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7.3 EXTENDING PLANAR DRAWINGS

DEF: An imbedding of a graph $G$ on a surface $S$ is a drawing without any edge-crossings. Notation: $\iota : G \rightarrow S$.

DEF: A region of an imbedding $G \rightarrow S$ is a component of the Euclidean set $S - \iota(G)$.

DEF: The boundary of a region $r$ of a graph imbedding $\iota : G \rightarrow S$ is the subgraph of $G$ that comprises all vertices that abut the region and all edges whose interiors abut the region.

DEF: A face of a graph imbedding $\iota : G \rightarrow S$ is the union of a region and its boundary.
Planar Extensions of a Planar Subgraph

Prop 7.3.1. A planar graph $G$ remains planar if a multiple edge or self-loop is added to it. 

\[
\begin{array}{c}
\text{Fig 7.3.1} \quad \text{self-loop or multi-edge preserves planarity.}
\end{array}
\]

Corollary 7.3.2. A nonplanar graph remains nonplanar if a self-loop is deleted or if one edge of a multi-edge is deleted. 

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Prop 7.3.3. A planar graph $G$ remains planar if any edge is subdivided or if a new edge $e$ is attached to a vertex $v \in V_G$ (with the other endpoint of $e$ added as a new vertex).  

Fig 7.3.2 Subdividing an edge; adding a spike.
The following proposition applies to imbeddings in all surfaces, not just to imbeddings in the sphere.

**Prop 7.3.4.** Let \( \iota : G \to S \) be a graph imbedding on a sphere or on any other surface. Let \( d \) be an edge of \( G \) with endpoints \( u \) and \( v \). Then the imbedding of the graph \( G - d \) obtained by deleting edge \( d \) from the imbedding \( \iota : G \to S \) has a face whose boundary contains both of the vertices \( u \) and \( v \). ◊

![Fig 7.3.3 Merging two regions by deleting an edge.](image)
Amalgamating Planar Graphs

Prop 7.3.5. Let $f$ be a face of a planar drawing of a connected graph $G$. Then there is a planar drawing of $G$ in which the boundary walk of face $f$ bounds a disk in the plane that contains the entire graph $G$. That is, in the new drawing, face $f$ is the “outer” face.

Pf: Copy the planar drawing of $G$ onto the sphere so that the north pole lies in the interior of face $f$. Then apply the Riemann stereographic projection.

Prop 7.3.6. Let $f$ be a face of a planar drawing of a graph $H$, and let $u_1, \ldots, u_n$ be a subseq of vertices in the bdry walk of $f$. Let $f'$ be a face of a planar drawing of a graph $J$, and let $w_1, \ldots, w_n$ be a subseq of vertices in the boundary walk of $f'$. Then the amalgamated graph $(H \cup J)/\{u_1 = w_1, \ldots, u_n = w_n\}$ is planar.

Pf: Planar drawings of graphs $H$ and $J$ with $n = 3$ are illustrated in Figure 7.3.4.

![Figure 7.3.4](image-url) Two planar graph drawings.
First redraw the plane imbedding of $H$ s.t. the unit disk lies wholly inside face $f$. Next redraw graph $J$ s.t. the bdry walk of face $f'$ surrounds the rest of graph $J$, which is possible — by Prop 7.3.5. Fig 7.3.5 shows these redrawings of $H$ and $J$.

**Fig 7.3.5  Redrawings of the planar imbeddings of $H$ & $J$.**

We assume the cyclic orderings of the vtx seqs $\{u_1, \ldots, u_n\}$ and $\{w_1, \ldots, w_n\}$ are consistent with each other — reflect the drawing of graph $J$, if nec. Now shrink the drawing of graph $J$ so that it fits inside the unit disk in face $f$, as in Fig 7.3.6(left). Then stretch the small copy of $J$ outward, as in Fig 7.3.6(right), thereby obtaining a crossing-free drawing of the amalgamation $(H \cup J)/\{a = x, b = y, c = z\}$.

**Fig 7.3.6  Position $H$ & $J$ and then amalg by stretching.**
Cor 7.3.7. Let $H$ and $J$ be planar graphs. Let $U$ be a set of one, two, or three vertices in the boundary of a face $f$ of the drawing of $H$, and let $W$ be a set of the same number of vertices in the boundary of a face $f'$ of the drawing of $J$. Then the amalgamated graph $(H \cup J)/\{U = W\}$ is planar.

Remark: However, Fig 7.3.7 shows how $K_5$ can be derived as the amalgamation of the planar graphs $W_4$ and $K_2$.

Fig 7.3.7  A 2-vertex amalg of two planar graphs into $K_5$.  

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**Remark:** Amalgamating two planar graphs across sets of four or more vertices per face may yield a nonplanar graph, as shown in Fig 7.3.8. The resulting graph shown is $K_{3,3}$ with two doubled edges. The bipartition of $K_{3,3}$ is shown in black and white.

**Fig 7.3.8** A 4-vertex amalg of two planar graphs into $K_{3,3}$. 

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Appendages to a Subgraph

DEF: Let $H$ be a subgraph of a connected graph $G$. Two edges $e_1$ and $e_2$ of $E_G - E_H$ are unseparated by subgraph $H$ if there exists a walk in $G$ that contains both $e_1$ and $e_2$, but whose internal vertices are not in $H$.

DEF: Let $H$ be a subgraph of a graph $G$. Then an appendage to subgraph $H$ is the induced subgraph on an equivalence class of edges of $E_G - E_H$ under the relation unseparated by $H$.

Fig 7.3.9 Subgraph $H$ and appendages $B_1, B_2, B_3, \text{ and } B_4$.

DEF: Let $H$ be a subgraph of a graph. An appendage to $H$ is called a chord if it contains only one edge. Thus, a chord joins two vertices of $H$, but does not lie in the subgraph $H$ itself.

DEF: Let $B$ be an appendage to a subgraph $H$. Then a contact point of $B$ is a vertex of $B \cap H$.

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Also notice that each non-chord appendage contains a single component of the deletion subgraph $G - V_H$. In addition to a component of $G - V_H$, a non-chord appendage also contains every edge extending from that component to a contact point, and the contact point as well.

**Remark:** Every subgraph of a graph (not just cycles) has appendages. Even a subgraph comprising a set of vertices and no edges would have appendages.
Overlapping Appendages

The construction of a planar drawing of a connected graph $G$ commonly begins with the selection of a “large” cycle to be the subgraph whose appendages constitute the rest of $G$. In this case, some special terminology describes the relationship between two appendages, in terms of their contact points.

DEF: Let $C$ be a cycle in a graph. The appendages $B_1$ and $B_2$ of $C$ overlap if either of these conditions holds:

i. Two contact points of $B_1$ alternate with two contact points of $B_2$ on cycle $C$.

ii. $B_1$ and $B_2$ have three contact points in common.

Example 7.3.1: Both possibilities (i) and (ii) for overlapping appendages are illustrated in Figure 7.3.10.

![Fig 7.3.10 The two ways that appendages can overlap.](image)

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**Prop 7.3.8.** Let $C$ be a cycle in a planar drawing of a graph, and let $B_1$ and $B_2$ be overlapping appendages of $C$. Then one appendage lies inside cycle $C$ and the other outside.

**Pf:** Overlapping appendages on the same side of cycle $C$ would cross, by Cor 7.1.3 to the Jordan Curve Theorem. ◊

**TERMINOLOGY:** Let $C$ be a cycle of a connected graph, and suppose that $C$ has been drawn in the plane. Relative to that drawing, an appendage of $C$ is said to be *inner* or *outer*, according to whether that appendage is drawn inside or outside of $C$.

**Example 7.3.2:** In both parts of Fig 7.3.10, appendage $B_1$ is an inner appendage and appendage $B_2$ is an outer appendage.
7.8 SUPPLEMENTARY EXERCISES

7.8.5 Draw a 4-regular simple 9-vertex planar graph.

7.8.11 Draw each of the four isomorphism types of simple 6-vertex graph that contains $K_5$ homeomorphically, but does not contain $K_{3,3}$.