Chapter 16

Nonplanar Layouts B: Lec 24

16.3 Voltage-Graph Specification of Graph Layouts
16.4 Non-KVL Imbedded Voltage Graphs
16.5 The Heawood Map-Coloring Problem
16.3 VOLTAGE SPECIFIED LAYOUTS

The application that inspired the invention [Gr74] of voltage graphs was the specification of graph imbeddings.

Signed Walks and Net Voltage

DEF: A signed walk in a digraph is an alternating sequence of vertices and edge-ends

\[ v_0, e_1^{\sigma_1}, v_1, e_2^{\sigma_2}, v_2, \ldots, v_{n-1}, e_n^{\sigma_n}, v_n \]

such that the result of deleting signs and ignoring directions is a walk (i.e., in the underlying graph)

\[ v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n \]

and such that reversing the direction on every minus-signed edge would yield a directed walk.

The edge-directions along a signed walk are not traversal restrictions. Rather, a minus-signed edge-end represents an edge traversed against its designated direction.

DEF: Let \( W = \langle v_0, e_1^{\sigma_1}, v_1, e_2^{\sigma_2}, v_2, \ldots, v_{n-1}, e_n^{\sigma_n}, v_n \rangle \) be a signed walk in a voltage graph \( \langle G, \alpha : E_G \to \mathcal{B} \rangle \). The net voltage on the walk \( W \) is the signed sum

\[ \alpha(W) = \sum_{j=1}^{n} \sigma_j \alpha(e_j) \]
That is, the net voltage is the sum of the voltages on \((+)-signed\) edges minus the sum of the voltages on \((-)-signed\) edges.

**Example 16.3.1:** Figure 16.3.1 shows a voltage-graph specification of the Petersen graph. The signed walk

\[
W = \langle v, d^+, v, c^+, u, e^-, u \rangle
\]

starts at vertex \(v\) of the voltage graph, traverses self-loop \(d\) in forward direction back to vertex \(v\), next traverses edge \(c\) to vertex \(u\), and then traverses self-loop \(e\) in reverse direction back to vertex \(u\). Thus, its net voltage is

\[
\alpha(W) = \alpha(d) + \alpha(c) - \alpha(e) = 1 + 0 - 2 = -1 \equiv 4 \mod 5
\]

![Voltage graph specifying the Petersen graph.](image-url)
UNIQUE WALK LIFTING

Thm 16.3.1 (Unique Walk Lifting). Let \( \langle G, \alpha : E_G \rightarrow \mathcal{A} \rangle \) be a voltage graph in which there is a signed walk
\[
W = v_0, e_1^{\sigma_1}, v_1, \ldots, v_{n-1}, e_n^{\sigma_n}, v_n
\]
and let \( a \in \mathcal{A} \). Then there is a unique alternating sequence
\[
W_a = v_{0,a}, e_{1,a}^{\sigma_1}, v_{1,k_1}, \ldots, v_{n-1,k_{n-1}}, e_{n,k_n}^{\sigma_n}, v_{n,k_n}
\]
that forms a signed walk in the covering digraph \( G^{\alpha} \).

Pf: For \( j = 1, \ldots, n \), choose
\[
k_j = a + \sum_{\ell=1}^{j-1} \alpha(e_{\ell})
\]
This formula supports an inductive argument that each successive edge choice \( e_{j,k_j} \) is the only possible choice of an edge in the fiber of \( e_j \) that extends the signed walk.

\[\diamond\]

**Figure 16.3.1** Reprise: Lifting walk \( W \) to walk \( W_2 \).

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DEF: Let $W$ be a signed walk in a voltage graph with initial vertex $v$. The unique signed walk $W_a$ in the covering graph that starts at vertex $v_a$ and projects onto walk $W$ (as in Thm 16.3.1) is called the lift starting at $v_a$ of the walk $W$.

Proposition 16.3.2. Let $W$ be a signed walk from $u$ to $v$ in the voltage graph $(G, \alpha: E_G \to \mathcal{A})$. Then each lift $W_b$ is a signed walk in the covering graph $G^\alpha$ from the vertex $u_b$ to the vertex $v_{b+\alpha(W)}$.

Pf: This follows by induction on the length of the walk. ◊
Example 16.3.2: The walk \( W = \langle v, d^+, v, c^+, u, e^-, u \rangle \) in the voltage graph of Figure 16.3.1 has these five lifts:

\[
\begin{align*}
W_0 &= \langle v_0, d_0^+, v_1, c_1^+, u_1, e_4^-, u_4 \rangle \\
W_1 &= \langle v_1, d_1^+, v_2, c_2^+, u_2, e_0^-, u_0 \rangle \\
W_2 &= \langle v_2, d_2^+, v_3, c_3^+, u_3, e_1^-, u_1 \rangle \\
W_3 &= \langle v_3, d_3^+, v_4, c_4^+, u_4, e_2^-, u_2 \rangle \\
W_4 &= \langle v_4, d_4^+, v_0, c_0^+, u_0, e_3^-, u_3 \rangle
\end{align*}
\]

The net voltage on walk \( W \) is 4 mod 5. Observe that each walk \( W_j \) begins at \( v_j \) and ends at \( u_{j+4} \). For instance, in the reprise of Figure 16.3.1, observe that the walk \( W_2 \) starts at \( v_2 \) and ends at \( u_{2+4} = u_1 \) mod 5.

![Figure 16.3.1](image-url)  **Figure 16.3.1**  Reprise: Lifting walk \( W \) to walk \( W_2 \).
Kirchoff Voltage Law

DEF: A closed walk of an imbedded voltage graph satisfies the **Kirchoff voltage law (abbr. KVL)** if its net voltage is the identity element of the voltage group.

**Theorem 16.3.3.** Let $W$ be a signed closed walk in a voltage graph that satisfies KVL. Then every lift $W_b$ is a closed walk in the covering graph $G^\alpha$.

**Pf:** This is an immediate consequence of Prop 16.3.2. \(\diamondsuit\)

**Theorem 16.3.4.** Let $\langle G, \alpha : E_G \to A \rangle$ be a voltage graph, and let $W$ be a signed closed walk in $G$, such that $\alpha(W) = b$ is of order $m$ in the voltage group $A$. Then the concatenation

$$W_a W_{a+b} \cdots W_{a+(m-1)b}$$

is a closed walk in $G^\alpha$.

**Pf:** Suppose walk $W$ starts at vertex $v$. By Prop 16.3.2, the lift $W_{a+\ell b}$ is a walk from $v_{a+\ell b}$ to $v_{a+(\ell+1)b}$, for $\ell = 0, \ldots, m-1$. Therefore, the iterated concatenation $W_a W_{a+b} \cdots W_{a+(m-1)b}$ is a walk from $v_a$ to $v_{a+mb} = v_a$, that is, a closed walk. \(\diamondsuit\)

**Remark:** A cyclic permutation of the constituents of the concatenation

$$W_a W_{a+b} \cdots W_{a+(m-1)b}$$

may start and stop at some vertex $v_{a+\ell b}$ different from $v_a$, but it traverses exactly the same cyclic sequence of vertices and edges. In the context of imbedded voltage graphs, it is called a **cyclically equivalent concatenation.**

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KVL Imbedded Voltage Graphs

DEF: An imbedded voltage graph is a pair \( \langle \iota : G \to S, \alpha \rangle \). The first component is a cellular imbedding of a graph \( G \) in a surface \( S \), called the base imbedding. The second component is a voltage assignment on graph \( G \).

DEF: A face of an imbedded voltage graph is said to satisfy the Kirchoff voltage law (abbr. KVL) if the net voltage on every face boundary walk is 0.

DEF: A KVL imbedded voltage graph is an imbedded voltage graph in which every face satisfies KVL.

Remark: In the preceding definition, notice the difference between the classical Kirchoff voltage law of physics and its topological namesake. Whereas the physical law is that a net voltage gain of 0 occurs in the traversal of every closed walk, the topological law imposes this requirement only on face-boundary walks. (See Exercises.)

DEF: In a KVL imbedded voltage graph with voltage group \( \mathcal{A} \), let \( f \) be a \( k \)-sided face with boundary walk \( W \). Then the fiber over \( f \), denoted \( \tilde{f} \), is a set \( \{ f_a \mid a \in \mathcal{A} \} \) of \( k \)-sided polygons, called the covering faces. The boundary walk of the covering face \( f_a \) is labeled by the lifted walk \( W_a \) of the covering graph.
DEF: Let \( \langle \nu : G \rightarrow S, \alpha : E_G \rightarrow A \rangle \) be a KVL imbedded voltage graph. Fitting each covering face \( f_a \), for \( f \in F_t \) and \( a \in A \), in accordance with its labeling to the corresponding lift of a boundary walk of \( f \) forms the covering surface \( S^\alpha \) and the covering imbedding \( \nu^\alpha : G^\alpha \rightarrow S^\alpha \).

**Example 16.3.3:** Figure 16.3.2 illustrates a one-face KVL imbedding \( B_2 \rightarrow S_1 \) with voltages in \( \mathbb{Z}_5 \) and the covering imbedding \( K_5 \rightarrow S_1 \) that it specifies.

![Diagram](image)

**Fig 16.3.2** A KVL voltage graph and its covering imbedding.

**TERMINOLOGY:** The imbedded-voltage-graph construction extends the natural projection \( p : G^\alpha \rightarrow G \), so that it maps each face \( f_a \) in the fiber \( \tilde{f} \) to the face \( f \), and thereby becomes a mapping \( p : S^\alpha \rightarrow S \) of surfaces.

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Example 16.3.4: Often, it helps to construct the covering graph alone before constructing the covering imbedding. For instance, Figure 16.3.3 shows how to assign voltages from $\mathbb{Z}_7$ to the bouquet $B_3$ so that the covering graph is the complete graph $K_7$, with 7 vertices and 21 edges.

Fig 16.3.3 Specifying $K_7$ by $\mathbb{Z}_7$-voltages on $B_3$.

Figure 16.3.4 illustrates a two-face KVL imbedding of that voltage graph on $S_1$ and the covering imbedding.

Fig 16.3.4 Specification of an imbedding $K_7 \rightarrow S_1$.

Since the covering imbedding has 14 faces, the covering surface must be the surface $S_g$ whose Euler characteristic is $7 - 21 + 14 = 0 = 2 - 2g$, that is, the surface $S_1$. Shading is used to group the faces of the covering imbedding into fibers.

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16.4 NON-KVL IMBEDDED GRAPHS

When an imbedded voltage graph \( \langle \nu : G \to S, \alpha : E_G \to A \rangle \) does not satisfy KVL, constructing the covering faces is somewhat more complicated, because a single lift of a non-KVL face-boundary walk \( W \) in the voltage graph is not a closed walk in the covering graph. However, it is possible to concatenate several lifts of \( W \) together into a closed walk and to fit a polygon to that closed walk.

**NOTATION:** In a non-KVL imbedded voltage graph with voltage group \( A \), let \( f \) be a \( k \)-sided face with boundary walk \( W \) having net voltage \( b \) of order \( m \). Then the set containing the walk

\[
W_a W_{a+b} \cdots W_{a+(m-1)b}
\]

and all cyclically equivalent walks is denoted \( W_{a+<b>} \).

**Non-KVL Covering Faces**

**DEF:** In a non-KVL imbedded voltage graph with voltage group \( A \), let \( f \) be a \( k \)-sided face with boundary walk \( W \) having net voltage \( b \) of order \( m \). Then the **fiber over** \( f \), denoted \( \tilde{f} \), is a set of \( mk \)-sided polygons \( f_{a+<b>} \), called the **covering faces**, one for each equivalence class \( W_{a+<b>} \) of closed walks. The boundary walk of the covering face \( f_{a+<b>} \) is labeled by any closed walk in \( W_{a+<b>} \).

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DEF: Let \( \langle \iota: G \to S, \alpha:E_G \to \mathcal{A} \rangle \) be a non-KVL imbedded voltage graph. Fitting each covering face \( f_{a^+<b^+} \), for \( f \in F_k \) and \( a \in \mathcal{A} \), in accordance with its labeling to the corresponding closed lift of a boundary walk of \( f \) forms the covering surface \( S^\alpha \) and the covering imbedding \( \iota^\alpha : G^\alpha \to S^\alpha \).

**Example 16.4.1:** The imbedded \((Z_2 \times Z_2)\)-voltage graph in Figure 16.4.1 has three faces, all 2-sided. The net voltage on each face boundary has order 2. Thus, the 12 boundary walk lifts combine to form 6 face boundaries in the covering graph. They fit together, as shown, to form the surface of a cube. In particular, the upper digon of the imbedded voltage graph specifies the top and bottom faces of the cube; the lower digon specifies the left and right faces; and the exterior digon specifies the front and back faces.

**Fig 16.4.1  Specification of an imbedding** \( Q_3 \to S_0 \).
Minimum-Genus of Hypercube Graphs

A standard approach to deriving a minimum-genus formula for a family of graphs is to use algebraic methods to establish a lower-bound formula and a voltage-graph construction of imbeddings that realize that lower bound. The family of hypercubes $Q_n$ illustrates the doubly fortunate circumstance in which no surface surgery is needed, and one simple pattern of imbedded-voltage-graph drawing is enough to specify all the imbeddings.

**Proposition 16.4.1.** $\gamma_{\text{min}}(Q_n) \geq (n - 4) \cdot 2^{n-3} + 1$, for $n \geq 2$.

**Pf:** For $n = 2$ or $n = 3$, the right side is 0. For $n \geq 4$, Theorem 16.2.1 gives the generic lower bound

$$\gamma_{\text{min}}(G) \geq \left\lceil \frac{|E|(\text{girth}(G) - 2)}{2\text{girth}(G)} - \frac{|V|}{2} + 1 \right\rceil$$

Since $|V(Q_n)| = 2^n$, $|E(Q_n)| = n \cdot 2^{n-1}$, and girth$(Q_n) = 4$, this particularizes to

$$\gamma_{\text{min}}(Q_n) \geq \left\lceil \frac{n \cdot 2^{n-1}(4 - 2)}{2 \cdot 4} - \frac{2^n}{2} + 1 \right\rceil$$

$$= \frac{n \cdot 2^{n-1}(4 - 2)}{2 \cdot 4} - \frac{2^n}{2} + 1$$

$$= n \cdot 2^{n-3} - 2^{n-1} + 1$$

$$= (n - 4) \cdot 2^{n-3} + 1$$

$\diamond$

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Proposition 16.4.2. $\gamma_{\min}(Q_n) \leq (n - 4) \cdot 2^{n-3} + 1$, for $n \geq 2$.

Pf: It suffices to specify an imbedding of $Q_n$ on the surface of genus $(n - 4) \cdot 2^{n-3} + 1$. The voltage graph of Figure 16.4.1 for $Q_3$ generalizes to dimension $n$, as shown in Figure 16.4.2.

![Figure 16.4.2 Voltage graph specifying $Q_n$.](image)

Since $girth(Q_n) = 4$, every face of an imbedding of $Q_n$ must have at least four sides. The formula provided by Theorem 16.2.1 depends on having as many faces as possible, which implies that the number of sides of almost every face of the imbedding must equal the girth.

With the voltage graph of Figure 16.4.2, it is easy to construct an imbedding in which all the faces are 2-sided and have net voltage of order 2 on their boundary walks. In fact, if that drawing is interpreted as an imbedding in $S_0$, then each face is a digon with net voltage of order 2 on its boundary walk. 

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16.5 HEAWOOD PROBLEM

The question of sufficiency of 4 colors (see §9.2) needed for a proper coloring of an arbitrary sphere map was generalized by Percy Heawood. Heawood derived an upper bound for the number of colors needed for an arbitrary map on any closed surface, given as a formula in its Euler characteristic. Heawood’s upper bound was ultimately proved to be the exact maximum for all surfaces except the Klein bottle $N_2$.

**DEF:** The *chromatic number of a surface* $S$ is the maximum of the chromatic numbers of the simple graphs that can be imbedded on $S$.

**REVIEW FROM §8.5:** The *Euler characteristic* of the surface $S_g$ with $g$ handles is the number $2 - 2g$.

**NOTATION:** As explained in §8.5, when colorings arise in topological graph theory, we need to avoid confusion between conflicting uses of the Greek letter $\chi$. Here we denote the chromatic number of a graph or of a surface by $\text{chr}(G)$ or $\text{chr}(S)$, respectively. The Euler characteristic of a surface is denoted by $\text{ec}(S)$.
Heawood Number of a Surface

The following lemmas provide basic upper bounds for the chromatic number.

Lemma 16.5.1. Let $G$ be a $\text{chr}(S)$-colorable chromatically critical graph imbedded on a closed surface $S$ of Euler characteristic $\text{ec}(S)$. Then $\text{chr}(S) \leq \left\lfloor 7 - \frac{6 \cdot \text{ec}(S)}{|V_G|} \right\rfloor$.

Pf: The notations $\delta_{\text{avg}}$ and $\delta_{\text{min}}$ indicate average degree and minimum degree.

\begin{align*}
(1) \quad \delta_{\text{avg}}(G) & \leq 6 - \frac{6 \cdot \text{ec}(S)}{|V_G|} \quad \text{(Thm 8.5.6)} \\
(2) \quad \delta_{\text{min}}(G) & \leq \delta_{\text{avg}}(G) \\
(3) \quad \text{chr}(G) - 1 & \leq \delta_{\text{min}}(G) \quad \text{(Thm 9.1.18)} \\
(4) \quad \text{chr}(G) - 1 & \leq 6 - \frac{6 \cdot \text{ec}(S)}{|V_G|} \quad \text{(from (1), (2), (3))} \\
(5) \quad \text{chr}(S) & \leq 7 - \frac{6 \cdot \text{ec}(S)}{|V_G|} \quad \text{(from (4))} \\
(6) \quad \text{chr}(S) & \leq \left\lfloor 7 - \frac{6 \cdot \text{ec}(S)}{|V_G|} \right\rfloor \quad \text{(from (5), since $\text{chr}(S)$ is integer)}
\end{align*}

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Lemma 16.5.2. Let $S$ be a closed surface of Euler characteristic $ec(S) \leq 0$. Then

$$chr(S) \leq 7 - \frac{6 \cdot ec(S)}{chr(S)}$$

Pf: Let $G$ be a $chr(S)$-colorable chromatically critical graph imbedded on a closed surface $S$ of Euler characteristic $ec(S)$. Then

1. $chr(S) \leq 7 - \frac{6 \cdot ec(S)}{|V_G|}$ (Lemma 16.5.1)
2. $chr(S) \leq |V_G|$ ($chr(S) = chr(G) \leq |V_G|$)
3. $\frac{6 \cdot ec(S)}{|V_G|} \geq \frac{6 \cdot ec(S)}{chr(S)}$ (by (2), since $-6c \geq 0$)
4. $chr(S) \leq 7 - \frac{6 \cdot ec(S)}{chr(S)}$ (from (1) and (3)) ◊

DEF: The Heawood number of a closed surface of Euler characteristic $ec(S)$ is the number

$$H(ec(S)) = \left\lfloor \frac{7 + \sqrt{49 - 24 \cdot ec(S)}}{2} \right\rfloor$$
Theorem 16.5.3 (Heawood, 1890). Let $S$ be a closed surface, orientable or non-orientable, with Euler characteristic $ec(S) \leq 1$. Then $chr(S) \leq H(ec(S))$.

Proof: By Poincaré duality, it suffices to show that an arbitrary graph $G$ imbeddable on a surface $S$ has chromatic number less than or equal to the Heawood number of the surface. Moreover, it suffices to assume that $G$ is chromatically critical.

For $ec(S) = 1$, Lemma 16.5.1 yields $chr(N_1) \leq 6 = H(1)$. For $ec(S) \leq 0$, Lemma 16.5.2 implies that

$$\text{chr}(S)^2 - 7\text{chr}(S) + 6 \cdot ec(S) \leq 0$$

Factoring the quadratic on the left side yields the inequality

$$\left(\text{chr}(S) - \frac{7 - \sqrt{49 - 24 \cdot ec(S)}}{2}\right) \times\left(\text{chr}(S) - \frac{7 + \sqrt{49 - 24 \cdot ec(S)}}{2}\right) \leq 0$$

Since $ec(S) \leq 0$, the value of the radical is larger than 7, from which it follows that the first factor is positive. This implies that the second factor is nonpositive. Since $chr(S)$ is an integer, the conclusion follows. \hfill\Box

Remark: Of course, the sphere has Euler char $ec(S) = 2$. Although $H(2) = 4$, the proof of Theorem 16.5.3 does not cover this case.
Heawood Conjecture and Minimum Genus

Heawood omitted proof that, for each surface, there is a map that realizes the Heawood number. When the gap was noticed, Heawood’s assertion was reformulated as a conjecture.

**DEF:** The *Heawood conjecture* is that a surface \( S \) of Euler characteristic \( ec(S) \) has chromatic number \( H(ec(S)) \).

**Prop 16.5.4.** If \( \gamma_{\text{min}}(K_n) \leq \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor \) for all \( n \geq 4 \), then the Heawood conjecture holds for all orientable surfaces.

**Pf:** The surface \( S_g \) has Heawood number

\[
H(2-2g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor
\]

If

\[
\gamma_{\text{min}}(K_n) \leq \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor \text{ for all } n \geq 4
\]

and if

\[
\left\lfloor \frac{(H(2-2g) - 3)(H(2-2g) - 4)}{12} \right\rfloor \leq g
\]

then

\[
\gamma_{\text{min}}(K_{H(2-2g)}) \leq g
\]

Proving inequality (♣) requires routine computation and is left as an exercise. \( \diamond \)

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Philip Franklin proved in 1934 that \( chr(N_2) = 6 \). Since the Euler characteristic of the Klein bottle (the surface \( N_2 \)) is 0, and since \( H(0) = 7 \), Franklin’s result is a counterexample to the Heawood conjecture. However, the Klein bottle is the only surface whose chromatic number is less than its Heawood number.

**Prop 16.5.5.** If \( \bar{\gamma}_{\text{min}}(K_n) \leq \left\lfloor \frac{(n-3)(n-4)}{6} \right\rfloor \quad \forall n \geq 8 \), then the Heawood conjecture holds for all non-orientable surfaces except the Klein bottle.

**Pf:** Follows the same lines as for the orientable case. \( \diamond \)
Min Genus of Complete Graphs, Case 7

The construction of a minimum imbedding for the complete graph $K_n$ has many details. The least complicated case is for $n = 7 \mod 12$. We recall that Example 16.3.4 gives an imbedded voltage graph that specifies a minimum imbedding of $K_7$. This smallest instance starts a pattern that continues with minimum imbeddings for $K_{19}$ and for $K_{31}$.

**Example 16.5.1:** The imbedded voltage graph in Fig 16.5.1 specifies a minimum imbedding of $K_{19}$.

![Figure 16.5.1](image_url) voltages in $\mathbb{Z}_{19}$

**Fig 16.5.2** KVL voltage graph for imbedding $K_{19} \rightarrow S_{20}$.

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Example 16.5.2: The imbedded voltage graph in Fig 16.5.3 specifies a minimum imbedding of $K_{31}$.

![Diagram of voltage graph](image)

**Fig 16.5.3** KVL voltage graph for imbedding $K_{31} \rightarrow S_{63}$.

Notice that KVL holds globally. Assigning voltages so that KVL holds globally is the difficult part.
16.6 SUPPLEMENTARY EXERCISES

16.6.13  a. For the imbedded voltage graph on the torus of Figure 16.6.1, calculate the genus of the derived surface when the voltages are in $Z_7$.  

b. Calculate the genus of the derived surface when the voltages are in $Z_8$.

![Figure 16.6.1](image-url)