Chapter 8

Drawing Graphs B: Lec 11

8.4 Regular Maps on a Sphere
8.5 Imbeddings on Higher-Order Surfaces
8.4 REGULAR MAPS ON A SPHERE

Whereas the *imbedding theory of a given graph* studies all the various ways that graph can be imbedded in its range of possible closed surfaces, the *map theory of a given surface* studies all the various graphs that can be imbedded on that surface and all the ways they can be imbedded.

**Map Theory**

**DEF:** A *map on a surface* is an imbedding of a graph on that surface.

**TERMINOLOGY NOTE:** Whereas the term *mapping* is a generic synonym for function, the term *map* refers to a function from a graph to a surface.

**REVIEW FROM §7.5:** The *(Poincaré) dual graph* and the *dual imbedding* are derived, starting with a cellular imbedding of a graph, by inserting a new vertex into each existing face, and by then drawing through each existing edge a new edge that joins the new vertex in the region on one side to the new vertex in the region on the other side.
**DEF:** A *simple map* on a surface is an imbedding of a simple graph on that surface, whose dual graph is also a simple graph.

**DEF:** A *regular map* on a surface is an imbedding of a regular graph such that the dual graph is also a regular graph.

**Example 8.4.1:** The graphs for the *sphere maps* (i.e., maps on a sphere) in Figure 8.4.1 are both simple and regular. However, the dual graph of map (i) is the dipole $D_3$, which is regular but not simple. The dual graph of map (ii) is the complete bipartite graph $K_{2,3}$, which is simple but not regular.

![Figure 8.4.1](image)

**Figure 8.4.1** Two sphere maps: (i) one non-simple, (ii) the other non-regular.
Example 8.4.2: The maps in Figure 8.4.2 are both regular and simple. The first map is of the 3-regular simple graph $K_4$ on the sphere $S_0$, with all four faces 3-sided; the dual graph is $K_4$. The second map is of the 3-regular simple graph $CL_4$ on the sphere $S_0$, with all six faces 4-sided; the dual graph is the octahedron graph $O_3$ (just draw it!).

\[ \begin{array}{cc}
\text{Fig 8.4.2} & \text{Two regular simple maps on the sphere.} \\
K_4 \rightarrow S_0 & CL_4 \rightarrow S_0
\end{array} \]

Remark: These definitions of regular and simple also apply to maps on surfaces other than the sphere.
**Proposition 8.4.1.** A map \( \iota : G \to S \) is regular if and only if the graph \( G \) is regular and every face has the same number of sides.

**Pf:** \((\Rightarrow)\) Assume that the map \( \iota : G \to S \) is regular. Then graph \( G \) is regular. Moreover, since the dual graph \( G^* \) is regular, each dual vertex has the same degree. Since the degree of a dual vertex equals the number of sides of a primal face, each primal face has the same number of sides.

\((\Leftarrow)\) Assume that graph \( G \) is regular and that every face has the same number of sides. The latter condition implies that the dual graph \( G^* \) is regular. Thus, the map \( \iota : G \to S \) is regular. \( \square \)

**DEF:** A **self-dual map** on a surface is a map \( \iota : G \to S \) such that there is a topological equivalence of the surface \( S \) to itself that takes the graph \( G \) to the dual graph \( G^* \).

**Example 8.4.3:** The map \( K_4 \to S_0 \) in Figure 8.4.2 is self-dual, but the map \( CL_4 \to S_0 \) is not.
Degrees and Face-Sizes of Regular Maps

The following straightforward application of the numerical relations in the previous section yields an upper bound on the average degree $\delta_{\text{avg}}(G)$ of a graph $G$ that can be imbedded on the sphere.

**Theorem 8.4.2.** Let $\iota : G \to S_0$ be an imbedding of a simple graph on the sphere. Then $\delta_{\text{avg}}(G) < 6$.

**Pf:**

1. $\text{girth}(G) \cdot |F| \leq 2|E|$ \hspace{1cm} edge-face ineq.
2. $3 \leq \text{girth}(G)$ \hspace{1cm} since $G$ is simple
3. $3|F| \leq 2|E|$ \hspace{1cm} combine (1) and (2)
4. $|F| \leq \frac{2|E|}{3}$
5. $|V| - |E| + |F| = 2$ \hspace{1cm} Euler poly. eq.
6. $|V| - \frac{|E|}{3} \geq 2$ \hspace{1cm} subst. (4) into (5)
7. $2|E| \leq 6|V| - 12$
8. $\frac{2|E|}{|V|} \leq 6 - \frac{12}{|V|}$

CONTINUE TO NEXT PAGE
(9) \[ \frac{2|E|}{|V|} \leq 6 - \frac{12}{|V|} \]

(10) \[ \delta_{\text{avg}}(G) = \frac{\sum_{v \in V} \text{deg}(v)}{|V|} \] by def. of avg.

(11) \[ \delta_{\text{avg}}(G) = \frac{2|E|}{|V|} \] by Theorem 1.1.2

(12) \[ \delta_{\text{avg}}(G) \leq 6 - \frac{12}{|V|} \] combine (9) and (11)

(13) \[ \delta_{\text{avg}}(G) < 6 \] from (12), since \[ \frac{12}{|V|} > 0 \]

**Corollary 8.4.3.** The only possible degrees of the graph with a regular simple map on the sphere are 3, 4, and 5.

**Pf:** Theorem 8.4.2 precludes the possibility of any degree greater than 5.

**Corollary 8.4.4.** The only possible face-sizes of a regular simple map on the sphere are 3, 4, and 5.

**Pf:** Apply Corollary 8.4.3 to the dual map.
Construction of the Regular Simple Maps

In view of Corollaries 8.4.3 and 8.4.4, a regular simple map on the sphere must have constant degree 3, 4, or 5 and constant face-size 3, 4, or 5. That seems to permit nine combinations. Theorem 8.4.5 provides further information, based on the numerical relations.

**Theorem 8.4.5.** If a regular map on the sphere has \( d \) for its constant degree and \( r \) for its constant face-size, then

\[
|V| = \frac{4r}{2(d+r) - dr} \quad |E| = \frac{2dr}{2(d+r) - dr} \quad |F| = \frac{4d}{2(d+r) - dr}
\]

**Pf:** Solving the following three linear equations for the “unknowns” \( |V|, |E|, \) and \( |F| \) in terms of \( d \) and \( r \) yields the result.

\[
|V| - |E| + |F| = 2 \quad \text{Euler polyhedral equation}
\]
\[
2|E| = r|F| \quad \text{face-size equation}
\]
\[
2|E| = d|V| \quad \text{degree-sum equation}
\]

\[\Box\]
The following table gives the numbers of vertices, edges, and faces that would correspond to each of the nine possible combinations of degree $d$ and face-size $r$ in a regular simple map on the sphere. The common denominator in the conclusion of Theorem 8.4.5 is denoted $Y$. In the rightmost column, the table names the polyhedron whose 1-skeleton and surface realize the map if it exists.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$r$</th>
<th>$Y$</th>
<th>$\frac{4r}{Y}$</th>
<th>$\frac{2dr}{Y}$</th>
<th>$\frac{4d}{Y}$</th>
<th>Name of polyhedron</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>Tetrahedron</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>2</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>Cube</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>1</td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>Dodecahedron</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>Octahedron</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
<td>undef.</td>
<td>undef.</td>
<td>undef.</td>
<td>no solution</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>$-2$</td>
<td>$-10$</td>
<td>$-20$</td>
<td>$-8$</td>
<td>no solution</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>1</td>
<td>12</td>
<td>30</td>
<td>20</td>
<td>Icosahedron</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>$-2$</td>
<td>$-8$</td>
<td>$-20$</td>
<td>$-10$</td>
<td>no solution</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>$-5$</td>
<td>$-4$</td>
<td>$-10$</td>
<td>$-4$</td>
<td>no solution</td>
</tr>
</tbody>
</table>
DEF: A **platonic solid** is a *geometrically regular* 3-dimensional polyhedron. That is, its 1-skeleton is a regular graph and also that each of the faces is geometrically a regular polygon.

**Example 8.4.4:** The five platonic solids, graphs, and maps are illustrated in Figure 8.4.3.

Fig 8.4.3 The five platonic graphs.

DEF: A **platonic graph** is the 1-skeleton of a platonic solid.
DEF: A \textit{platonic map} is the imbedding of the 1-skeleton of a platonic solid into its surface.

\textbf{Remark:} The five maps in Figure 8.4.3 are the only regular simple maps on the sphere. The illustration proves their existence.

\textbf{Remark:} Since the three numerical equations used in the proof of Theorem 8.4.5 are true for non-simple graphs, the resulting formulas can be used to determine possibilities for non-simple regular maps on the sphere.
8.5 DRAWINGS ON HIGHER SURFACES

To draw a graph on a higher surface, it sometimes helps to cut the surface open along a few strategic closed curves, so that the surface can be flattened out. Some algebraic relations previously derived, including the Euler polyhedral equation, can be generalized to drawings on higher surfaces.

Cellular Imbeddings

When a graph is drawn on a closed surface other than the sphere, perhaps there are handles or crosscaps in the interiors of one or more regions, but usually not.

**DEF:** A region of a graph imbedding \( \imath : G \rightarrow S \) on a surface \( S \) is a *cellular region* if it is topologically equivalent to an open disk. (Topologists sometimes call a disk a “2-cell”.) It is a *strongly cellular region* if, moreover, the boundary walk is a cycle in the graph.

**DEF:** A graph imbedding \( \imath : G \rightarrow S \) on a surface \( S \) is a *cellular imbedding* if every region is cellular. It is a *strongly cellular imbedding* if, moreover, every region is strongly cellular.
Example 8.5.1: In Figure 8.5.1, the imbeddings of the bouquets $B_1$ and $B_2$ are both non-cellular.

![Fig 8.5.1](image)

Fig 8.5.1 Two non-cellular imbeddings on the torus.

Any non-cellular imbedding of a graph on a closed surface can be obtained by adding handles and crosscaps to a cellular imbedding. Largely for this reason, we concentrate on cellular imbeddings of graphs.

Proposition 8.5.1. The boundary of a face $f$ of a connected graph $G$ imbedded on the sphere is connected.

Pf: A closed loop drawn just inside one of the boundary components of face $f$ would separate the sphere, by the Jordan curve theorem. Since the graph $G$ is connected, all of it lies on the same side of that closed loop as that boundary component. Thus, there is no other boundary component. ☐

Proposition 8.5.2. Every imbedding of a connected graph on the sphere is a cellular imbedding.

Pf: This follows from Proposition 8.5.1 and the Schoenflies theorem, that every closed curve on a sphere bounds a disk. ☐

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Example 8.5.2: In the planar imbedding of $K_4$ in Figure 8.5.2(a), there are four regions, all strongly cellular. In the toroidal imbedding of $K_4$ in Figure 8.5.2(b), there are two regions. Centered in drawing (b) is a 4-sided strongly cellular region. The other region is cellular, but it "meets itself" along the lines used to paste the rectangle into a torus, so its boundary is not a cycle. Accordingly, that region is not strongly cellular.

Observe in drawing (b) that the partial edge incident on the topmost vertex meets the partial edge incident on the bottommost vertex. These two parts of the same edge are joined when the top broken edge of the rectangle is identified with the bottom broken edge. Similarly, the partial edge incident on the leftmost vertex is part of the same edge as the partial edge incident on the rightmost vertex.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{example.png}
\caption{Two imbeddings of $K_4$: (a) on the plane; (b) on the torus.}
\end{figure}
Flat Polygon Drawings

DEF: A flat polygon representation of a surface $S$ is a drawing of a polygon with markings to match its sides in pairs, such that when the sides are pasted together as the markings indicate, the resulting surface obtained is topologically equivalent to $S$.

Example 8.5.3: Figure 8.5.3 recalls how the torus was represented in §8.2 as a rectangle with its sides identified in pairs. Another perspective on this representation is that cutting the torus open on its meridian and longitude permits it to be flattened into a rectangle.

![Diagram of torus as a rectangle](image)

Fig 8.5.3 Representing a torus as a rectangle.
Every orientable surface $S_g$ can be represented as a flat polygon with $4g$ sides (for a detailed proof, see [Ma67]). As one traverses the boundary of the polygon, each handle is represented by a sequence of four consecutive sides, which are marked with the pasting pattern $sts^{-1}t^{-1}$.

**Example 8.5.4:** Figure 8.5.4 shows a representation of the double torus $S_2$ as an octagon, from which one handle is formed by the four sides marked $aba^{-1}b^{-1}$ and the other by the four sided marked $cdc^{-1}d^{-1}$.

![Diagram](image)

**Fig 8.5.4** Representing a double torus as an octagon.
Similarly, every non-orientable surface $N_k$ can be represented as a flat polygon with $2k$ sides (for further details, see [Ma67]). As one traverses the boundary of the polygon, each crosscap is represented by a sequence of two sides marked with the pasting pattern $ss$. Figure 8.5.5 shows a representation of the surface $N_3$ as a hexagon.

**Fig 8.5.5** Representing the surface $N_3$ as an hexagon.
In a drawing of a graph on the flat polygon representation of a surface, an edge may cross through one copy of a matched side to the other copy. For instance, Figure 8.5.6 shows an imbedding of $K_{3,3}$ on the torus with three hexagonal regions. One edge crosses through the meridian cut $a$, two edges cross through the longitudinal cut $b$, and one edge cuts through the point where the meridian and longitude intersect.

Fig 8.5.6  Toroidal imbedding of $K_{3,3}$. 
Surgery on Imbeddings

Trying to draw graph imbeddings on flat polygon representations of surfaces more complicated than a torus (the surface $S_1$) or a Klein bottle (the surface $N_2$) is often quite frustrating. It is frequently easier to draw most of the graph on a simpler surface and to complete the imbedding by attaching extra handles and drawing edges across the extra handles. Informally, such modifications are called surgery.

Example 8.5.5: Drawing $K_{4,5}$ on a flat polygon representation of $S_2$ is no easy task. (Try it and see!) A surgical approach to this imbedding problem is to start by drawing $K_{4,4}$ on $S_1$, as shown in Figure 8.5.7.

Figure 8.5.7  Toroidal imbedding of $K_{4,4}$, with holes punched in two regions.
Two regions $f$ and $f'$ of this imbedding are selected so that the union of their face-boundary walks contains all the black vertices. Next, a hole is punched in regions $f$ and $f'$, shown by shaded disks in Figure 8.5.7. Then a tube is attached from one hole to the other, thereby reclosing the surface, so that a single new non-cellular region is formed from the two regions with the holes plus the tube. The resulting surface is $S_2$ and the new region has two boundary components, at opposite ends of the tube. Finally, a fifth white vertex is drawn, as shown in Figure 8.5.8, on the tube (depicted as shaded) and joined to both black vertices on the boundary components. The final result of this surgery is a cellular imbedding of $K_{4,5}$ on $S_2$.

![Figure 8.5.8](image)

**Fig 8.5.8** Joining the fifth white vertex.
Euler Equations for All Closed Surfaces

We observe that the **Face-Size Equation**

\[ 2|E_G| = \sum_{f \in F} \text{size}(f) \]

holds for every graph imbedding \( \iota : G \to S \), cellular or not, in every surface. Similarly, the **Edge-Face Inequality**

\[ 2|E| \geq \text{girth}(G) \cdot |F| \]

holds whenever the graph is connected but not a tree (for which girth is undefined), in every surface, even if the imbedding is non-cellular. Moreover, for any cellular graph imbedding \( \iota : G \to S \), the following Poincaré duality relations hold.

(i) \( |V^*| = |F| \)

(ii) \( |E^*| = |E| \)

(iii) \( |F^*| = |V| \)

However, the Euler polyhedral equation is restricted to cellular imbeddings. As the following theorems indicate, the value of this formula is unique on each orientable surface, and it is unique on each non-orientable surface.
Lemma 8.5.3. Let $G \to S$ be a cellular imbedding of a graph in a surface. Then the result of subdividing an edge of the imbedded copy of $G$, or of joining two vertices on the boundary of a face by a new edge is an imbedding whose Euler formula has the same value as for the imbedding $G \to S$.

Proof: Subdividing an edge adds one new vertex and increases the number of edges by one. Drawing a new edge across a cellular face separates an edge into two faces. In either case the changes have offsetting effects on the value of the Euler formula. 

\[ |V| - |E| + |F| = 2 - 2g \]

Theorem 8.5.4. Let a graph $G$ be cellularly imbedded in the orientable surface $S_g$. Then

\[ |V| - |E| + |F| = 2 - 2g \]

Proof: For imbeddings on the sphere $S_0$, we proved in Chapter 7 that the value of the Euler formula $|V| - |E| + |F|$ must be equal to 2. This serves as the base case for an induction. As an inductive hypothesis, assume that the equation is correct for every cellular imbedding in the surface $S_g$. We consider a cellular imbedding of a graph $G$ in the surface $S_{g+1}$. 

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Draw a meridian on the rightmost handle of $S_{g+1}$ so that it meets the graph $G$ in finitely many points, each in the interior of some edge of $G$. If necessary, we subdivide edges of $G$ so that no edge crosses the meridian more than once. By Lemma 8.5.3, the value of the Euler formula in the resulting graph imbedding is unchanged from the original.

Next we thicken the meridian to an annulus $A$, as shown at the left of Figure 8.5.9, and we subdivide each edge of $G$ that crosses the annulus at both of its intersection points with a boundary component of the annulus, which, by Lemma 8.5.3, also preserves the value of the Euler formula. Then we augment the edge-set of the graph by adding to it each segment of annulus boundary that joins two vertices on it. By Lemma 8.5.3, the resulting graph imbedding has the same value of the formula $|V| - |E| + |F|$ as the original imbedding.

![Fig 8.5.9 Thickening a meridian and subsequent surgery.](image_url)
In the interior of annulus $A$, edges and faces alternate, so there are equally many. Thus, excising the interior of the annulus preserves the value of the formula

$$|V| - |E| + |F|$$

Finally, by capping the two resulting holes in the surface with disks, as shown at the right of Figure 8.5.9, we obtain a surface that is topologically equivalent to $S_g$ and a cellular imbedding with vertex set $V'$, edge-set $E'$, and face set $F'$, such that

$$|V'| - |E'| + |F'| = |V| - |E| + |F| + 2$$

By the induction hypothesis,

$$|V'| - |E'| + |F'| = 2 - 2g$$

It follows that

$$|V| - |E| + |F| = -2g = 2 - 2(g + 1)$$

\textbf{Theorem 8.5.5.} Let a graph $G$ be cellularly imbedded in the non-orientable surface $N_k$. Then

$$|V| - |E| + |F| = 2 - k$$

\textbf{Pf:} This proof follows the same pattern at that of Theorem 8.5.4, except that the role of the meridian is filled by the central cycle on a Möbius band.

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**Example 8.5.6:** The imbedding $K_{3,3} \rightarrow S_1$ in Figure 8.5.6 above is cellular, and the value of the Euler polyhedral formula $|V| - |E| + |F|$ is

$$6 - 9 + 3 = 0 = 2 - 2g$$

**Example 8.5.5, continued:** The imbedding $K_{4,5} \rightarrow S_2$ in Figures 8.5.7 and 8.5.8 above is cellular, and the value of the Euler polyhedral formula $|V| - |E| + |F|$ is

$$9 - 20 + 9 = -2 = 2 - 2g$$

**Example 8.5.7:** The imbedding of $K_6$ into $N_2$ in Figure 8.5.10 has 9 regions, each marked with a distinguishing numeral. Thus,

$$|V| - |E| + |F| = 6 - 15 + 9 = 0 = 2 - k$$

**Fig 8.5.10** An imbedding $K_6 \rightarrow N_2$ with 9 regions.
DEF: The **Euler characteristic** for a surface equals the value of the Euler polyhedral formula $|V| - |E| + |F|$ for any cellular imbedding on that surface. It is denoted $ec(S)$. That is,

$$ec(S) = \begin{cases} 
2 - 2g & \text{for the orientable surface } S_g \\
2 - k & \text{for the non-orientable surface } N_k
\end{cases}$$

**Example 8.5.8:** Here are the Euler characteristics of some orientable surfaces:

$$ec(S_0) = 2 \quad ec(S_1) = 0 \quad ec(S_2) = -2 \quad ec(S_3) = -4 \quad \cdots$$

**Example 8.5.9:** Here are the Euler characteristics of some non-orientable surfaces:

$$ec(N_0) = 2 \quad ec(N_1) = 1 \quad ec(N_2) = 2 \quad ec(N_3) = -1 \quad \cdots$$

**Average Degree and General Surfaces**

The following generalization of Theorem 8.4.2 is often useful in proving that a given graph cannot be imbedded in some given surface.

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**Theorem 8.5.6.** Let \( \iota : G \to S \) be an imbedding of a simple graph \( G \) on a surface \( S \) of Euler characteristic \( ec(S) \). Then \( \delta_{\text{avg}}(G) \leq 6 - \frac{6 \cdot ec(S)}{|V|} \).

**Pf:** This follows the proof of Theorem 8.4.2 exactly, with details condensed.

\[
\begin{align*}
(1) \quad |F| & \leq \frac{2|E|}{3} \quad \text{(from edge-face ineq.)} \\
(2) \quad |V| - |E| + |F| & = ec(S) \quad \text{(Euler poly. eq.)} \\
(3) \quad |V| - \frac{|E|}{3} & \geq ec(S) \quad \text{(subst. (1) into (2))} \\
(4) \quad \frac{2|E|}{|V|} & \leq 6 - \frac{6 \cdot ec(S)}{|V|} \quad \text{(multiply (3) by \( \frac{6}{|V|} \))} \\
(5) \quad \delta_{\text{avg}}(G) & = \frac{2|E|}{|V|} \quad \text{(by Theorem 1.1.2)} \\
(6) \quad \delta_{\text{avg}}(G) & \leq 6 - \frac{6 \cdot ec(S)}{|V|} \quad \text{(combine (4) and (5))} \diamond
\end{align*}
\]

**Example 8.5.10:** Since \( \chi(N_1) = 1 \), the average degree of an imbedding in the non-orientable surface \( N_1 \) is less than 6. Thus, \( K_7 \) cannot be imbedded in \( N_1 \).

**Example 8.5.11:** The average degree of \( C_4 + C_5 \) is \( 6 \frac{4}{9} \), and \( \chi(N_2) = 0 \). Theorem 8.5.6 implies that \( C_4 + C_5 \) cannot be imbedded in \( N_2 \).
8.8 SUPPLEMENTARY EXERCISES

8.8.2 Consider the following specs for a graph imbedding: three 3-sided faces and four 4-sided faces. Either draw such an imbedding in the sphere or prove it is impossible.

8.8.6 Either draw a self-dual imbedding of a regular simple graph in $S_2$ or prove that such an imbedding cannot exist. (Hint: Use the Euler Polyhedral Equation and self-duality to determine the number of edges.)