Drawing Graphs A: Lec 10

8.1 The Topology of Low Dimensions
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8.1 TOPOLOGY OF LOW DIMENSIONS

The formal definition of a drawing as a continuous function depends on

- a spatial model for a graph (the domain) and
- a spatial model for a surface (the codomain),

in which the edges of a graph have length, just as they do in drawings. Surfaces are defined intrinsically, starting here and continuing into §8.2, with the aid of topological equivalence, and they need not be attached to some solid that they bound.

Some Subsets of Eucl 2-Space and 3-Space

DEF: Euclidean $n$-space $\mathbb{R}^n$ is the set of $n$-tuples $(x_1, \ldots, x_n)$ with the usual Euclidean distance metric

$$\delta((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

DEF: A Euclidean set is a subset of a Euclidean space.

DEF: The plane is another name for Euclidean 2-space $\mathbb{R}^2$. (This is consistent with the definition in §7.1 of a plane in $\mathbb{R}^3$.)
DEF: The **open unit disk** is the plane Euclidean set containing the points \( \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\} \), i.e., the points inside the unit circle.

DEF: The **closed unit disk** is the plane Euclidean set containing the points \( \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\} \), i.e., the points inside and on the unit circle.

DEF: The **standard half-disk** is the plane Euclidean set containing the points \( \{(x_1, x_2) \mid x_1 \geq 0 \land x_1^2 + x_2^2 < 1\} \).

**Example 8.1.1:** As illustrated in Figure 8.1.1, the open unit disk and the half-disk are subsets of the closed unit disk. The half-disk contains the segment of its frontier that is an open interval on the \( x_2 \)-axis, but it does not contain the semi-circle that is its other frontier segment in \( \mathbb{R}^2 \).

![Fig 8.1.1 Open disk, closed disk, and standard half-disk.](image-url)
DEF: The \textit{unit sphere} is the 3-dimensional Euclidean set containing the points \( \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\} \).

DEF: The \textit{unit cylinder} is the 3-dimensional Euclidean set containing the points \( \{(x_1, x_2, x_3) \mid 0 \leq x_1 \leq 1 \land x_2^2 + x_3^2 = 1\} \).

\begin{center}
\includegraphics[width=0.2\textwidth]{cylinder.png}
\end{center}

\textbf{Fig 8.1.2} The unit cylinder.
Topological Equivalences of Euclidean Sets

DEF: A continuous function $f : X \to Y$ between two Euclidean sets is called a \textit{topological equivalence} if it is one-to-one and onto, and if the inverse function $f^{-1} : Y \to X$ is continuous.

Example 8.1.2: The Riemann stereographic projection (see §7.1) is a topological equivalence between a sphere minus its north pole and the plane.

\begin{center}
\[ N \quad \bullet \quad w \quad \bullet \quad f(w) \quad S \]
\end{center}

Example 8.1.3: A topological equivalence between the open unit disk and the entire plane, which illustrates how topological equivalence permits spaces to be stretched, is given by the function

\[ f((x, y)) = \left( \frac{x}{1 - x^2 - y^2}, \frac{y}{1 - x^2 - y^2} \right) \]
Example 8.1.4: The interior region $R$ in the plane enclosed by any polygon but not including the polygon itself is topologically equivalent to the open unit disk. Writing out the continuous bijection with precise formulas is tedious. Instead, we describe the equivalence informally.

Case 1: a convex polygon $P$. Place the geometric center (or any other interior point) of the polygon at the origin. Next observe that the ray at angle $\alpha$ from the origin to infinity intersects the open disk in a half-open segment $I_D^\alpha$ and intersects the region $R$ in a half-open segment $I_R^\alpha$, as illustrated in Figure 8.1.3. Let the function $f$ map $I_D^\alpha$ to $I_R^\alpha$ by linear stretching or compression. Then $f$ is a topological equivalence from the unit open disk to the polygonal region $R$.

Figure 8.1.3  Equivalence of a convex polygonal region to a unit disk.
Case 2: a non-convex polygon $P$. First fill polygon $P$ with “mathematical helium” that “inflates” it into a convex polygon with the same cycle of side-lengths, as illustrated in Figure 8.1.4. This inflation process is a topological equivalence. Then apply the method of the convex case.

Fig 8.1.4 Mapping a non-convex region to a convex region.
DEF: A *sphere* is any Euclidean set $S$ that is topologically equivalent to the unit sphere.

**Example 8.1.5:** The surface of any convex 3-dimensional solid is a sphere. This follows by a construction analogous to the convex-polygon example above. For instance, Figure 8.1.5 illustrates a ball, an ovoid, and a cube, each a 3-dimensional convex solid. Thus, the surface of each of these solids is a sphere.

![Fig 8.1.5](image)

**Fig 8.1.5** The surfaces of these convex solids are spheres.

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Topological Model of a Graph

To draw graphs on surfaces, except for simple graphs in the plane, we need a model of an edge.

DEF: A space curve between two points \( x \) and \( y \) in 3-space is the image of a continuous function \( f : [0, 1] \to \mathbb{R}^3 \) from the unit interval into 3-space such that \( f(0) = x \) and \( f(1) = y \), which is one-to-one, except that possibly \( x = y \). That is, the space curve is the set \( \{ f(t) \mid t \in [0,1] \} \). The interior of the space curve is the subset \( \{ f(t) \mid t \in (0,1) \} \).

DEF: In a topological model (or carrier) of a graph \( G = (V, E) \), each vertex \( v \in V \) is represented by a point \( p_v \in \mathbb{R}^3 \) called a space vertex, and each proper edge \( e \in E \) is represented by a space curve \( q_e \) joining its endpoints, called a space edge. The interior of each space edge \( q_e \) of the carrier is disjoint from all the other spaces edges and also from all the space vertices. Except when confusion might result, the topological model of a graph \( G \) is also denoted \( G \).

DEF: The 0-end of a space edge of a graph is the part near the image of the 0-end of the unit interval \([0, 1]\) in the space curve.

DEF: The 1-end of a space edge of a graph is the part near the image of the 1-end of the unit interval \([0, 1]\) in the space curve.
Remark: In this topological sense, every edge has two distinct ends, even if it is a self-loop with only one endpoint. If a large segment \( \{ f(t) \mid t \in (\epsilon, 1 - \epsilon) \} \) of the interior of a space edge is discarded, what remains is the two whisker-like ends, as illustrated in Figure 8.1.6.

![Figure 8.1.6](image-url)  

**Figure 8.1.6** The 0-end and the 1-end of a proper edge and of a self-loop.
8.2 HIGHER-ORDER SURFACES

Topology generalizes the concept of a surface, from the elementary meaning, in which it surrounds some solid, to a more powerful meaning, in which it need not enclose anything.

DEF: An open $\epsilon$-neighborhood of a point in a Euclidean set, is the set of all points whose distance from that point is less than $\epsilon$, where $\epsilon > 0$.

DEF: A surface is a Euclidean set in which every point has an $\epsilon$-neighborhood that is topologically equivalent either to the open unit disk or to the standard half-disk.

REVIEW FROM §7.1: The standard torus is the surface of revolution obtained by revolving a circle of radius 1 centered at (2,0) in the $xy$-plane disk around the $y$-axis in 3-space.

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Torus and Möbius Band

DEF: A torus is any Euclidean set that is topologically equivalent to the standard torus.

Example 8.2.1: The surface of a knotted solid donut is a torus.

![Fig 8.2.1 A knotted donut in 3-space.](image)

There is a way to represent toroidal drawings on a flat piece of paper. The rectangle in Figure 8.2.2 represents a torus.

![Fig 8.2.2 Representing a torus as a rectangle.](image)
The top edge and the bottom edge of the rectangle in Figure 8.2.2 are both marked with the same letter, namely “a”. This indicates that these two edges are to be identified with each other, that is, pasted together. This pasting of edges creates a cylinder, exactly as would occur with a paper model, as illustrated in Figure 8.2.3 below.

![Diagram showing a cylinder with edges identified](image)

**Fig 8.2.3** Identifying top and bottom edges yields a cylinder.

An effect of pasting the top edge to the bottom edge was to turn the left edge (marked “b”), and likewise the right edge (also marked “b”), into a closed curve. When the left end of the cylinder is pasted to the right end, the resulting surface is a torus. A paper cylinder would crumple, of course, but a flexible plastic tube could be so converted into a torus.

**Remark:** Pasting the left edge to the right edge converts line “a” into a closed curve. Indeed, on the resulting torus, what was once edge “a” has become a longitude, and what was once edge “b” has become a meridian.
DEF: A *Möbius band* is a space formed from a rectangular strip with a half-twist, as illustrated in Figure 8.2.4, by identifying the left edge with the right edge, to form a continuous band, as shown in Figure 8.2.5.

![Fig 8.2.4 A rectangular strip with a half-twist.](image)

ends pasted together

![Fig 8.2.5 A Möbius band.](image)
DEF: The **central circuit of a Möbius band** is the result of matching one end of the line halfway between the top and bottom of the rectangular strip to the other end, as the right and left ends are identified.

![Diagram of a Möbius band](image)

**Fig 8.2.6** A Möbius band.

**Theorem 8.2.1.** *The Möbius band does not have the Jordan separation property.*

**Pf:** The central circuit does not separate a Möbius band into two parts. 

**Remark:** To illustrate that the Jordan curve theorem does not hold for the Möbius band, make a paper model of a Möbius band. Cut the Möbius band open along the central circuit. Notice that result is a single connected surface, not two surfaces.

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Bounded and Boundaryless Surfaces

DEF: An **interior point of a surface** is a point that has a $\varepsilon$-neighborhood topologically equivalent to an open disk.

DEF: A **boundary point of a surface** is a point that is not an interior point.

DEF: A surface is **boundaryless** if every point is an interior point.

**Example 8.2.2:** The Euclidean plane is a boundaryless surface.

**Example 8.2.3:** A sphere is a boundaryless surface. Although a sphere may bound a solid ball, the sphere itself is boundaryless.

**Example 8.2.4:** The closed unit disk is not boundaryless. The points on the unit circle are its boundary points.

**Example 8.2.5:** The unit cylinder is not boundaryless. The points on the circles at either end are its boundary points.

**Example 8.2.6:** The Möbius band is not a boundaryless surface.
Closed Surfaces

**Def:** A Euclidean set is **finite** if there is a real number $M$ such that the maximum distance of any point from the origin is at most $M$.

**Def:** A connected surface is **closed** if it is a Euclidean set that satisfies these three conditions:

(i). The surface $S$ is finite.

(ii). The surface $S$ is boundaryless.

(iii). The endpoints of every open arc in $S$ are in the surface $S$ itself.

or if it is topologically equivalent to a Euclidean surface satisfying all three conditions.

**Example 8.2.7:** The sphere and the torus are closed surfaces.

**Example 8.2.8:** The open unit disk is not closed, because the endpoints of the open line segment \[ \{(x_1, x_2) \mid x_1 = 0 \text{ and } -1 < x_2 < 1\} \] are on the unit circle, not in the open disk itself.
Example 8.2.9: The plane is not closed, because it is not finite.

Example 8.2.10: The Möbius band is not closed, because it has boundary points.

TERMINOLOGY NOTE: The present usage of “closed” in the geometric-topology sense of a closed surface differs from the usage of “closed” in point-set topology.

DEF: A surface is **non-orientable** if it contains a subspace that is topologically equivalent to a Möbius band, and **orientable** otherwise.

Remark: Suppose that a small square is placed with its bottom edge on the central circuit of a Möbius band in 3-space. Suppose that the square is slid forward around the band exactly *once* (not twice). Then the top of the square will be positioned below the bottom. This reversability of top and bottom is the intuitive idea of “non-orientability”.

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Classification of Surfaces

DEF: The *sequence of orientable surfaces* $S_0, S_1, S_2, \ldots$ is defined recursively. $S_0$ is the sphere, and $S_1$ is the torus. The surface $S_{n+1}$ is obtained by attaching a handle to the surface $S_n$. This sequence is illustrated in Figure 8.2.7 below.

![Sequence of closed orientable surfaces](image)

*Fig 8.2.7* The sequence of closed orientable surfaces.

**Theorem 8.2.2.** Classification of Closed Orientable Surfaces. Every closed orientable surface is topologically equivalent to exactly one of the surfaces in the infinite sequence $S_0, S_1, S_2, \ldots$. ◊ (proof omitted here; see [GrTu87])

DEF: The *genus of a closed orientable surface* is the subscript of the surface $S_g$ to which it is topologically equivalent. That is, the genus is the “number of handles” on the surface.
DEF: By **adding a crosscap to a surface**, we mean punching a hole in the surface and reclosing it by attaching a Möbius band. That is, the boundary of the Möbius band is identified with the boundary of the hole by a topological equivalence, which specifies exactly how to make the attachment.

DEF: The **seq of non-orientable surfaces** $N_1, N_2, N_3, \ldots$ is defined recursively, starting from the sphere, which in this context is denoted $N_0$, even though it is orientable. The surface $N_{n+1}$ is obtained by adding a crosscap to the surface $N_n$. This sequence of surfaces is depicted in Figure 8.2.8.

\begin{center}
\begin{tikzpicture}
  \node at (0,0) [circle, draw] (n1) {$\otimes$};
  \node at (2,0) [circle, draw] (n2) {$\otimes \otimes$};
  \node at (4,0) [circle, draw] (n3) {$\otimes \otimes \otimes$};

  \node at (0,-1) [below] (n1text) {$N_1$};
  \node at (2,-1) [below] (n2text) {$N_2$};
  \node at (4,-1) [below] (n3text) {$N_3$};

  \node at (6,0) [above] (ellipses) {$\cdots$};
\end{tikzpicture}
\end{center}

**Fig 8.2.8** The sequence of closed non-orientable surfaces.

**Remark:** The topological operation of replacing a disk on a sphere or other surface by a Möbius band can be readily performed in 4-space, but not in 3-space. Fortunately, there is a way to conceptualize the operation without thinking 4-dimensionally. The result of the operation is simply that whenever we cross through from the “main part” of the surface through the boundary of a removed disk, we cross into the Möbius band that replaced the disk.
**Remark:** Trying to draw a surface from 4-space in the plane would be a drop of two dimensions. (In other words, it would be a loss of detail comparable to representing a 3-dimensional figure by a subset of a single line.) The circle with the $\times$ is the graphic device by which the result of the crosscap operation is represented.

**Thm 8.2.3.** *Classification of Closed Non-Orientable Surfaces.* Every closed non-orientable surface is topologically equivalent to exactly one surface in the infinite sequence $N_1, N_2, N_3, \ldots$. $\Diamond$ (proof omitted here; see [GrTu87])

**DEF:** The *crosscap number of a closed non-orientable surface* (or *non-orientable genus*) is the subscript of the surface $N_k$ to which it is topologically equivalent. That is, it is the number of (disjoint) Möbius bands on the surface.
8.3 MATH MODEL OF GRAPH DRAWING

REVIEW FROM ELEMENTARY SET THEORY: The image of a point or subset of the domain of a function is the point or subset in the codomain onto which it is mapped.

REVIEW FROM §7.1: An open path from s to t in a Euclidean set X is the image of a continuous bijection f from the unit interval [0, 1] to a subset of X such that f(0) = s and f(1) = t. (One may visualize a path as the trace of a particle traveling through space for a fixed length of time.)

TERMINOLOGY NOTE: A singularity of a continuous function is a point of the image where the function is not one-to-one.

Designing a mathematical model of a drawing of a graph on a surface starts by representing each vertex as a point on the surface and representing each edge as an open path in the surface between the images of its endpoints.
DEF: A *drawing of a graph* $G$ on a surface $S$ is the union of a set of points of $S$, one for each vertex of $G$, and a set of open paths, one for each edge of $G$, which joins the images of the endpoints of that edge. Three types of *forbidden singularities* that occasionally occur in representations of graphs on surfaces are illustrated in Figure 8.3.1, but are otherwise absent from this book.

(i) Two vertex images $p_u$ and $p_v$ occur at the same point of $S$.

(ii) An edge image $q_e$ has a self-intersection or other singularities.

(iii) The interior of an edge image intersects a vertex image.

![Figure 8.3.1 Three forbidden singularity types for graph drawings.](image)

DEF: An *edge-crossing* in a graph drawing is a singularity such that the images of two different edges meet.
Normalized Drawings and Imbeddings

DEF: An abnormality in a drawing of a graph on a surface is any of these three types of singularities:

(i) The images of two edges meet at more than one point of the surface.

(ii) The images of two different edges meet, but fail to actually cross each other (i.e., a tangency).

(iii) Images of three or more edges meet at the same point of the surface.

![Diagram](image)

Figure 8.3.2 Remedies for the three types of abnormalities in drawings.

Remedies in Fig 8.3.2 for these three kinds of abnormalities:

(i) an edge that crosses another several times can stay on the same side until it gets to the final crossing;

(ii) if one edge just touches another, then it can be pulled away from the other edge;

(iii) a triple crossing can be changed into 3 double-crossings.

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DEF: A normal drawing of a graph on a surface is a drawing that is free of all three types of abnormalities.

Unless explicitly specified otherwise in context, it is assumed throughout this book that a drawing is normalized. (We occasionally relax rule (iii).)

DEF: An imbedding of a graph \( G \) on a surface \( S \) is a drawing with no edge-crossings at all; that is, it is the image of a topological equivalence \( \iota : G \to S \) from the topological model of \( G \) to a subset of \( S \).
Eliminating Edge-Crossings

Frequently, it is desirable to keep the number of edge-crossings to a minimum, which may involve some redrawing.

**Example 8.3.1:** Sometimes a drawing of a graph can be changed into an imbedding by rerouting the image of one or more edges to different positions. For instance, in Figure 8.3.3, a drawing of the complete graph $K_4$ with one edge-crossing is changed into an imbedding by moving the image of edge $e$.

![Figure 8.3.3](image)

**Figure 8.3.3** Eliminating an edge-crossing by moving the image of an edge.
DEF: By **attaching a handle to a surface**, we mean punching two separate holes in the surface and connecting one to the other with a tube.

**Example 8.3.2:** Adding a handle to a surface is another way to eliminate an edge-crossing from a drawing of a given graph, as illustrated in Figure 8.3.4. The handle is added from one side of edge $e$ to the other, and then the image of edge $d$ is rerouted so that it lies on the new handle, instead of crossing edge $e$. Of course, the handle-adding operation changes the surface on which the graph is drawn.

![Diagram](image)

Fig 8.3.4  Eliminating an edge-crossing by adding a handle.
Remark: Every graph can be imbedded in 3-space. First draw the graph in the plane, and then eliminate the crossings by adding handles as in Example 8.3.2. Different handles can be at different heights off the plane, so they do not intersect.

Application 8.3.1 Printed Circuit Boards: A planar electronic circuit can be printed onto a board. Since the wires of a planar circuit need not cross, no insulation is needed. When a circuit is non-planar, one practical approach, described in §7.7, is to partition its edges into a few layers.
8.8 SUPPLEMENTARY EXERCISES

8.8.1 Draw an imbedding of the Möbius ladder $ML_4$ in the torus.

**DEF:** A 2-complex $K = (V_K, E_K, R_K)$ is a generalization of a graph to a 2-dimensional object. It consists of a graph $G = (V_K, E_K)$, called the 1-skeleton of the complex, and a set $R_K$ of regions, to each of which is associated a closed walk in the graph that is the boundary walk of the polygon.

8.8.7 Draw a 2-complex whose 1-skeleton is planar, but which cannot be drawn in the sphere with no overlap of edges or regions.