Abstract. Chapter 1 introduces some basic terminology. §1.1 is concerned with the existence and construction of a graph with a given degree sequence. §1.2 presents some families of graphs to which frequent reference occurs throughout the course. §1.4 introduces the notion of distance, which is fundamental to many applications. §1.5 introduces paths, trees, and cycles, which are critical concepts to much of the theory.

Outline

1.1 Graphs and Digraphs
1.2 Common Families of Graphs
1.4 Walks and Distance
1.5 Paths, Cycles, and Trees
1. **Graphs and Digraphs**

terminology for graphical objects

![Graph A](image1)

![Graph B](image2)

Figure 1.1: *Simple graph A; graph B.*

![Digraph D](image3)

![Underlying Graph G](image4)

Figure 1.3: *Digraph D; its underlying graph G.*
Degree

Figure 1.9: A graph with degree sequence 6, 6, 4, 1, 1, 0.

Figure 1.10: Both degree sequences are ⟨3, 3, 2, 2, 2, 2⟩.
Proposition 1.1. A non-trivial simple graph $G$ must have at least one pair of vertices whose degrees are equal.

Proof. pigeonhole principle \hfill \Box

Theorem 1.2 (Euler’s Degree-Sum Thm). The sum of the degrees of the vertices of a graph is twice the number of edges.

Corollary 1.3. In a graph, the number of vertices having odd degree is an even number.

Corollary 1.4. The degree sequence of a graph is a finite, non-increasing sequence of nonnegative integers whose sum is even.
**General Graph with Given Degree Sequence**

![Graph with degree sequence](image)

Figure 1.11: General graph with deg seq $\langle 5, 4, 3, 3, 2, 1, 0 \rangle$.

**Simple Graph with Given Degree Sequence**

![Simple graph with degree sequence](image)

Figure 1.13: Simple graph with deg seq $\langle 3, 3, 2, 2, 1, 1 \rangle$. 
Havel-Hakimi Theorem

**Theorem 1.6.** Let \( \langle d_1, d_2, \ldots, d_n \rangle \) be a graphic sequence, with \( d_1 \geq d_2 \geq \ldots \geq d_n \). Then there is a simple graph with vertex-set \( \{v_1, \ldots, v_n\} \) s.t.
\[
\deg(v_i) = d_i \quad \text{for} \quad i = 1, 2, \ldots, n
\]
with \( v_1 \) adjacent to vertices \( v_2, \ldots, v_{d_1+1} \).

**Proof.** Among all simple graphs with vertex-set \( V = \{v_1, v_2, \ldots, v_n\} \) and \( \deg(v_i) = d_i : i = 1, 2, \ldots, n \) let \( G \) be a graph for which the number
\[
r = |N_G(v_1) \cap \{v_2, \ldots, v_{d_1+1}\}|
\]
is maximum. If \( r = d_1 \), then the conclusion follows.

Alternatively, if \( r < d_1 \), then there is a vertex
\[
v_s : \quad 2 \leq s \leq d_1 + 1
\]
such that \( v_1 \) is not adjacent to \( v_s \), and \( \exists \) vertex
\[
v_t : \quad t > d_1 + 1
\]
such that \( v_1 \) is adjacent to \( v_t \) (since \( \deg(v_1) = d_1 \)).
Moreover, since \( \text{deg}(v_s) \geq \text{deg}(v_t) \), \( \exists \) vertex \( v_k \) such that \( v_k \) is adj to \( v_s \) but not to \( v_t \), as on the left of Fig 1.14. Let \( \tilde{G} \) be the graph obtained from \( G \) by replacing edges \( v_1v_t \) and \( v_sv_k \) with edges \( v_1v_s \) and \( v_tv_k \), as on the right of Fig 1.14, so all degrees are all preserved.

![Figure 1.14: Switching adjacencies while preserving all degrees.](image)

Thus, \( |N_{\tilde{G}}(v_1) \cap \{v_2, \ldots, v_{d_1+1}\}| = r + 1 \), which contradicts the choice of graph \( G \). \( \square \)

**Corollary 1.7** (Havel (1955) and Hakimi (1961)). A sequence \( \langle d_1, d_2, \ldots, d_n \rangle \) of nonneg ints, such that \( d_1 \geq d_2 \geq \ldots \geq d_n \), is graphic if and only if the sequence

\[
\langle d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n \rangle
\]

is graphic. (See Exercises for proof.)
Remark 1.1. Cor 1.7 yields a recursive algorithm that decides whether a non-increasing sequence is graphic.

**Algorithm:** Recursive GraphicSequence($\langle d_1, d_2, \ldots, d_n \rangle$)

*Input:* a non-increasing sequence $\langle d_1, d_2, \ldots, d_n \rangle$.

*Output:* TRUE if the sequence is graphic; FALSE if it is not.

If $d_1 = 0$
- Return TRUE

Else
  If $d_n < 0$
    Return FALSE
  Else
    Let $\langle a_1, a_2, \ldots, a_{n-1} \rangle$ be a non-incr permutation of $\langle d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n \rangle$.
    Return GraphicSequence($\langle a_1, a_2, \ldots, a_{n-1} \rangle$)
2. Families of Graphs

Figure 2.1: The first five *complete graphs*.

Figure 2.2: Two *bipartite graphs*.

Figure 2.4: The *complete bipartite graph* $K_{3,4}$.
Figure 2.5: The five *platonic graphs*.

Figure 2.6: The *Petersen graph*.
Figure 2.8: **Bouquets** $B_2$ and $B_4$.

Figure 2.9: The **Dipoles** $D_3$ and $D_4$.

Figure 2.10: **Path graphs** $P_2$ and $P_4$. 
Figure 2.11: **Cycle graphs** $C_1$, $C_2$, and $C_4$.

Figure 2.12: **Circular ladder graph** $CL_4$. 
**Circulant Graphs**

**Def 2.1.** To the group of integers

\[ \mathbb{Z}_n = \{0, 1, \ldots, n - 1\} \]

under addition modulo \( n \) and a set

\[ S \subseteq \{1, \ldots, n - 1\} \]

we associate the **circulant graph**

\[ \text{circ}(n : S) \]

whose vertex set is \( \mathbb{Z}_n \), such that two vertices \( i \) and \( j \) are adjacent if and only if there is a number \( s \in S \) such that \( i + s = j \mod n \) or \( j + s = i \mod n \). In this regard, the elements of the set \( S \) are called **connections**.

\[ \text{circ}(5 : 1, 2) \quad \text{circ}(6 : 1, 2) \quad \text{circ}(8 : 1, 4) \]

Figure 2.13: Three circulant graphs.
**Intersection and Interval Graphs**

**Def 2.2.** A simple graph $G$ with vertex set

$$V_G = \{v_1, v_2, \ldots, v_n\}$$

is an **intersection graph** if there exists a family of sets

$$\mathcal{F} = \{S_1, S_2, \ldots, S_n\}$$

s. t. vertex $v_i$ is adjacent to $v_j$ if and only if $i \neq j$ and $S_i \cap S_j \neq \emptyset$.

**Def 2.3.** A simple graph $G$ is an **interval graph** if it is an intersection graph corresponding to a family of intervals on the real line.

**Example 2.1.** The graph $G$ in Figure 2.14 is an interval graph for the following family of intervals:

$$a \leftrightarrow (1, 3) \quad b \leftrightarrow (2, 6) \quad c \leftrightarrow (5, 8) \quad d \leftrightarrow (4, 7)$$

![Figure 2.14: An interval graph.](image-url)
Line Graphs

*Line graphs* are a special case of intersection graphs.

**Def 2.4.** The *line graph* $L(G)$ of a graph $G$ has a vertex for each edge of $G$, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in $G$ have a vertex in common.

Thus, the line graph $L(G)$ is the intersection graph corresponding to the endpoint sets of the edges of $G$.

**Example 2.2.** Figure 2.15 shows a graph $G$ and its line graph $L(G)$.

![Figure 2.15: A graph and its line graph.](image-url)
4. **Walks and Distance**

**Def 4.1.** A *walk* from $v_0$ to $v_n$ is an alternating sequence

$$W = \langle v_0, e_1, v_1, e_2, ..., v_{n-1}, e_n, v_n \rangle$$

of vertices and edges, such that

$$\text{endpts}(e_i) = \{v_{i-1}, v_i\}, \quad \text{for } i = 1, ..., n$$

In a simple graph, there is only one edge between two consecutive vertices of a walk, so one could abbreviate the walk as

$$W = \langle v_0, v_1, ..., v_n \rangle$$

In a general graph, one might abbreviate as

$$W = \langle v_0, e_1, e_2, ..., e_n, v_n \rangle$$

**Def 4.2.** The *length* of a walk or directed walk is the number of edge-steps in the walk sequence.

**Def 4.3.** A walk of length zero, i.e., with one vertex and no edges, is called a *trivial walk*.

**Def 4.4.** A *closed walk* (or *closed directed walk*) is a nontrivial walk (or directed walk) that begins and ends at the same vertex. An *open walk* (or *open directed walk*) begins and ends at different vertices.

**Def 4.5.** The *distance* $d(s, t)$ from a vertex $s$ to a vertex $t$ in a graph $G$ is the length of a shortest $s$-$t$ walk if one exists; otherwise, $d(s, t) = \infty$. 
Eccentricity, Diameter, and Radius

**Def 4.6.** The *eccentricity* of a vertex $v$, denoted $ecc(v)$, is the distance from $v$ to a vertex farthest from $v$. That is,

$$ecc(v) = \max_{x \in V_G} \{d(v, x)\}$$

**Def 4.7.** The *diameter* of a graph is the max of its eccentricities, or, equivalently, the max distance between two vertices. i.e.,

$$diam(G) = \max_{x \in V_G} \{ecc(x)\} = \max_{x,y \in V_G} \{d(x, y)\}$$

**Def 4.8.** The *radius* of a graph $G$, denoted $rad(G)$, is the min of the vertex eccentricities. That is,

$$rad(G) = \min_{x \in V_G} \{ecc(x)\}$$

**Def 4.9.** A *central vertex* $v$ of a graph $G$ is a vertex with min eccentricity. Thus, $ecc(v) = rad(G)$.

**Example 4.7.** The graph of Fig 4.7 below has diameter 4, achieved by the vertex pairs $u, v$ and $u, w$. Vertices $x$ and $y$ have eccentricity 2 and all other vertices have greater eccentricity. Thus, the graph has radius 2 and central vertices $x$ and $y$.

![Figure 4.7: A graph with diameter 4 and radius 2.](image-url)
CONNECTEDNESS

Def 4.10. Vertex \( v \) is \textit{reachable from} vertex \( u \) if there is a walk from \( u \) to \( v \).

Def 4.11. A graph is \textit{connected} if for every pair of vertices \( u \) and \( v \), there is a walk from \( u \) to \( v \).

Def 4.12. A digraph is \textit{connected} if its underlying graph is connected.

Example 4.8. The non-connected graph in Figure 4.8 is made up of connected pieces called \textit{components}. See §2.3.

Figure 4.8: Non-connected graph with three components.
5. Paths, Cycles, and Trees

Def 5.1. A trail is a walk with no repeated edges.

Def 5.2. A path is a trail with no repeated vertices (except possibly the initial and final vertices).

Def 5.3. A walk, trail, or path is trivial if it has only one vertex and no edges.

Example 5.1. In Fig 5.1, \( W = \langle v, a, e, f, a, d, z \rangle \) is the edge sequence of a walk but not a trail, because edge \( a \) is repeated, and \( T = \langle v, a, b, c, d, e, u \rangle \) is a trail but not a path, because vertex \( x \) is repeated.

![Graph with vertices and edges labeled v, u, e, x, d, z, a, b, c, f, y and edge sequences W and T.]

Figure 5.1: Walk \( W \) is not a trail; trail \( T \) is not a path.
Cycles

Def 5.4. A nontrivial closed path is called a cycle. It is called an odd cycle or an even cycle, depending on the parity of its length.

Def 5.5. An acyclic graph is a graph that has no cycles.

Eulerian Graphs

Def 5.6. An eulerian trail in a graph is a trail that contains every edge of that graph.

Def 5.7. An eulerian tour is a closed eulerian trail.

Def 5.8. An eulerian graph is a graph that has an eulerian tour.

Figure 5.6: An eulerian graph.
**Hamiltonian Graphs**

**Def 5.9.** A cycle that includes every vertex of a graph is called a *hamiltonian cycle*.

**Def 5.10.** A *hamiltonian graph* is a graph that has a hamiltonian cycle. (§6.3 elaborates on hamiltonian graphs).

![Figure 5.3: An hamiltonian graph.](image-url)
Girth

Def 5.11. The *girth* of a graph with at least one cycle is the length of a shortest cycle. The girth of an acyclic graph is undefined.

Example 5.2. The girth of the graph in Figure 5.7 is 3 since there is a 3-cycle but no 2-cycle or 1-cycle.

Figure 5.7: A graph with girth 3.
**TREES**

**Def 5.12.** A *tree* is a connected graph that has no cycles.

Figure 5.8: A tree and two non-trees.
**Theorem 5.4.** A graph $G$ is bipartite iff it has no odd cycles.

**Proof.** Nec ($\Rightarrow$): Suppose $G$ is bipartite. Since traversing each edge in a walk switches sides of the bipartition, it requires an even number of steps for a walk to return to the side from which it started. Thus, a cycle must have even length.

Suff ($\Leftarrow$): Let $G$ be a graph with $n \geq 2$ vertices and no odd cycles. W.l.o.g., assume that $G$ is connected. Pick any vertex $u$ of $G$, and define a partition $(X, Y)$ of $V$ as follows:

$$X = \{x \mid d(u, x) \text{ is even}\}; \quad Y = \{y \mid d(u, y) \text{ is odd}\}$$

Suppose two vertices $v$ and $w$ in one of the sets are joined by an edge $e$. Let $P_1$ be a shortest $u$-$v$ path, and let $P_2$ be a shortest $u$-$w$ path. By definition of the sets $X$ and $Y$, the lengths of these paths are both even or both odd. Starting from vertex $u$, let $x$ be the last vertex common to both paths (see Fig 5.9).

![Figure 5.9: Figure for suff part of Thm 5.4 proof.](image)

Since $P_1$ and $P_2$ are both shortest paths, their $u \rightarrow x$ sections have equal length. Thus, the lengths of the $x \rightarrow v$ section of $P_1$ and the $x \rightarrow w$ section of $P_2$ are either both even or both odd. But then the concatenation of those two sections with the edge $e$ forms an odd cycle, contradicting the hypothesis. Hence, $(X, Y)$ is a bipartition of $G$. $\square$
7. Supplementary Exercises

Exercise 1  A 20-vertex graph has 62 edges. Every vertex has degree 3 or 7. How many vertices have degree 3?

Exercise 8  How many edges are in the hypercube graph $Q_4$?

Exercise 11  In the circulant graph $circ(24 : 1, 5)$, what vertices are at distance 2 from vertex 3?

Def 7.1. The edge-complement of a simple graph $G$ is the simple graph $\overline{G}$ on the same vertex set such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in $G$.

Exercise 20  Let $G$ be a simple bipartite graph with at least 5 vertices. Prove that $\overline{G}$ is not bipartite. (See §2.4.)