Computer Graphics (Spring 2008)

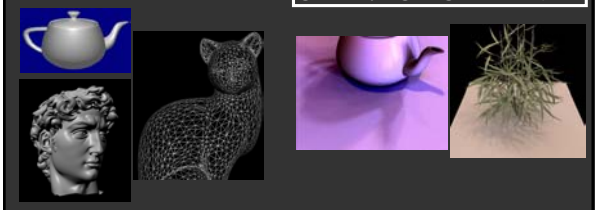
COMS 4160, Lecture 6: Curves 1

http://www.cs.columbia.edu/~cs4160

Course Outline

- 3D Graphics Pipeline

Modeling
(Creating 3D Geometry) → **Rendering**
(Creating, shading images from geometry, lighting, materials)



Course Outline

- 3D Graphics Pipeline

Modeling
(Creating 3D Geometry) → **Rendering**
(Creating, shading images from geometry, lighting, materials)

Unit 1: Transformations
Weeks 1,2. Ass 1 due Feb 14

Unit 2: Spline Curves
Modeling geometric objects
Weeks 3,4 hw2.exe
Ass 2 due Feb 26 (demo)

Motivation

- How do we model complex shapes?
  - In this course, only 2D curves, but can be used to create interesting 3D shapes by surface of revolution, lofting etc
- Techniques known as spline curves
- This unit is about mathematics required to draw these spline curves, as in HW 2
- History: From using computer modeling to define car bodies in auto-manufacturing. Pioneers are Pierre Bezier (Renault), de Casteljau (Citroen)

Outline of Unit

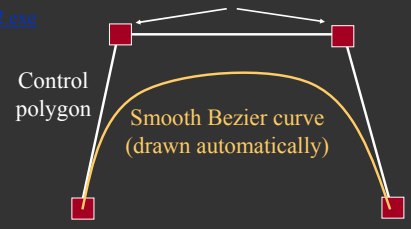
- Bezier curves
- deCasteljau algorithm, explicit form, matrix form
- Polar form labeling (next time)
- B-spline curves (next time)
- Not well covered in textbooks (especially as taught here). Main reference will be lecture notes. If you do want a printed ref, handouts from CAGD, Seidel

Bezier Curve (with HW2 demo)

- *Motivation: Draw a smooth intuitive curve (or surface) given a few key user-specified control points*

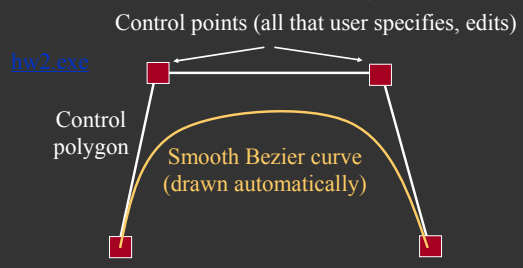
Control points (all that user specifies, edits)

hw2.exe



Bezier Curve: (Desirable) properties

- Interpolates, is tangent to end points
- Curve within convex hull of control polygon



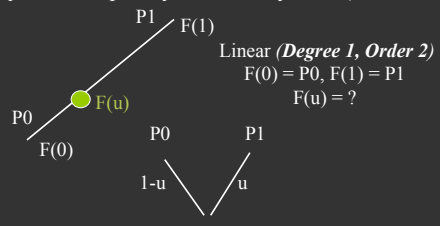
Issues for Bezier Curves

Main question: Given control points and constraints (interpolation, tangent), how to construct curve?

- *Algorithmic: deCasteljau algorithm*
- Explicit: Bernstein-Bezier polynomial basis
- 4x4 matrix for cubics
- Properties: Advantages and Disadvantages

deCasteljau: Linear Bezier Curve

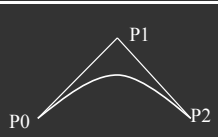
- Just a simple linear combination or interpolation (easy to code up, very numerically stable)



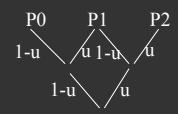
Linear ***(Degree 1, Order 2)***
F(0) = P0, F(1) = P1
F(u) = ?

$$F(u) = (1-u)\,P0 + u\,P1$$

deCasteljau: Quadratic Bezier Curve

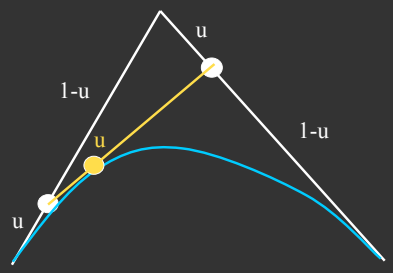


Quadratic
Degree 2, Order 3
F(0) = P0, F(1) = P2
F(u) = ?

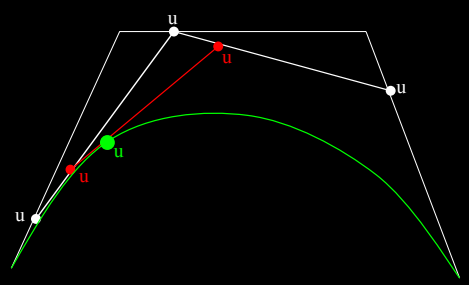


$$F(u) = (1-u)^2\,P0 + 2u(1-u)\,P1 + u^2\,P2$$

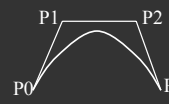
Geometric interpretation: Quadratic



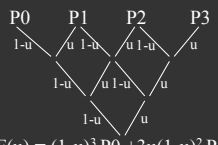
Geometric Interpretation: Cubic



deCasteljau: Cubic Bezier Curve

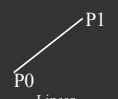


Cubic
Degree 3, Order 4
F(0) = P0, F(1) = P3



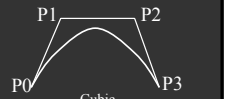
$F(u) = (1\text{-}u)^3 P0 + 3u(1\text{-}u)^2 P1 + 3u^2(1\text{-}u) P2 + u^3 P3$

Summary: deCasteljau Algorithm



Linear
Degree 1, Order 2
F(0) = P0, F(1) = P1

Quadratic
Degree 2, Order 3
F(0) = P0, F(1) = P2



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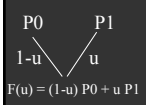

$F(u) = (1\text{-}u) P0 + u P1$

$F(u) = (1\text{-}u)^2 P0 + 2u(1\text{-}u) P1 + u^2 P2$

$F(u) = (1\text{-}u)^3 P0 + 3u(1\text{-}u)^2 P1 + 3u^2(1\text{-}u) P2 + u^3 P3$

DeCasteljau Implementation

Input: Control points C_i with $0 < i < n$ where n is the degree.
Output: $f(u)$ where u is the parameter for evaluation

```
1 for (level = n ; level ≥ 0 ; level − −) {
2    if (level == n) { // Initial control points
3       ∀i : 0 ≤ i ≤ n : p_i^level = C_i ; continue ; }
4    for (i = 0 ; i < level ; i++)
5       p_i^level = (1 − u) * p_i^level+1 + u * p_i+1^level+1 ;
6 }
7 f(u) = p_0^0
```

- Can be optimized to do without auxiliary storage

Summary of HW2 Implementation

Bezier (Bezier2 and Bspline discussed next time)
- Arbitrary degree curve (number of control points)
- Break curve into detail segments. Line segments for these
- Evaluate curve at locations 0, 1/detail, 2/detail, … , 1
- Evaluation done using deCasteljau

- Key implementation: deCasteljau for arbitrary degree
  - Is anyone confused? About handling arbitrary degree?
- Can also use alternative formula if you want
  - Explicit Bernstein-Bezier polynomial form (next)
- Questions?

Issues for Bezier Curves

Main question: Given control points and constraints (interpolation, tangent), how to construct curve?

- Algorithmic: deCasteljau algorithm
- *Explicit: Bernstein-Bezier polynomial basis*
- 4x4 matrix for cubics
- Properties: Advantages and Disadvantages

Recap formulae

- Linear combination of basis functions

Linear: $F(u) = P_0(1-u) + P_1 u$

Quadratic: $F(u) = P_0(1-u)^2 + P_1[2u(1-u)] + P_2 u^2$

Cubic: $F(u) = P_0(1-u)^3 + P_1[3u(1-u)^2] + P_2[3u^2(1-u)] + P_3 u^3$

Degree n: $F(u) = \sum_k P_k B_k^n(u)$ $B_k^n(u)$ are Bernstein-Bezier polynomials

- Explicit form for basis functions? Guess it?

Recap formulae

- Linear combination of basis functions

Linear: $F(u) = P_0(1-u) + P_1 u$

Quadratic: $F(u) = P_0(1-u)^2 + P_1[2u(1-u)] + P_2 u^2$

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Degree n: $F(u) = \sum_k P_k B_k^n(u)$ $B_k^n(u)$ are Bernstein-Bezier polynomials

- Explicit form for basis functions? Guess it?
- ***Binomial coefficients in [(1-u)+u]^n***

Summary of Explicit Form

Linear: $F(u) = P_0(1-u) + P_1 u$

Quadratic: $F(u) = P_0(1-u)^2 + P_1[2u(1-u)] + P_2 u^2$

Cubic: $F(u) = P_0(1-u)^3 + P_1[3u(1-u)^2] + P_2[3u^2(1-u)] + P_3 u^3$

Degree n: $F(u) = \sum_k P_k B_k^n(u)$ $B_k^n(u)$ are Bernstein-Bezier polynomials

$$B_k^n(u) = \frac{n!}{k!(n-k)!}(1-u)^{n-k} u^k$$

Issues for Bezier Curves

Main question: Given control points and constraints (interpolation, tangent), how to construct curve?

- Algorithmic: deCasteljau algorithm
- Explicit: Bernstein-Bezier polynomial basis
- *4x4 matrix for cubics*
- Properties: Advantages and Disadvantages

Cubic 4x4 Matrix (derive)

$$F(u) = P_0(1-u)^3 + P_1[3u(1-u)^2] + P_2[3u^2(1-u)] + P_3 u^3$$

$$= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} & M = ? & \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Cubic 4x4 Matrix (derive)

$$F(u) = P_0(1-u)^3 + P_1[3u(1-u)^2] + P_2[3u^2(1-u)] + P_3 u^3$$

$$= \begin{pmatrix} u^3 & u^2 & u & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Issues for Bezier Curves

Main question: Given control points and constraints (interpolation, tangent), how to construct curve?

- Algorithmic: deCasteljau algorithm
- Explicit: Bernstein-Bezier polynomial basis
- 4x4 matrix for cubics
- *Properties: Advantages and Disadvantages*

Properties (brief discussion)

- Demo: hw2.exe
- Interpolation: End-points, but approximates others
- Single piece, moving one point affects whole curve (no local control as in B-splines later)
- Invariant to translations, rotations, scales etc. That is, translating all control points translates entire curve
- Easily subdivided into parts for drawing (next lecture): Hence, Bezier curves easiest for drawing