**DISCRETE MATH**\(^1\) **W3203 Quiz 2**

open book

**SOLUTIONS**

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Total 100

**SUGGESTION:** Do the EASIEST problems first!

**HINT:** Some of the solution methods involve pre-college math as well as new methods from this class.

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\(^1\) An example of the Reasonable Person Principle: A reasonable student expects to lose a lot of credit for neglecting to EXPLAIN an answer. Omit explanations at your own risk.
1 (18 pts). Prove for every positive integer $n$ that

$$\sum_{j=n}^{2n-1} (2j + 1) = 3n^2$$

Proof: by induction

BASIS ($n = 1$): $\sum_{j=1}^{2 \cdot 1 - 1} (2j + 1) = (2 \cdot 1 + 1) = 3$

IND HYP: Assume that $\sum_{j=n}^{2n-1} (2j + 1) = 3n^2$

IND STEP: Then $\sum_{j=n+1}^{2n+1} (2j + 1) =$

$$\sum_{j=n}^{2n-1} (2j + 1) - [2n + 1] + [2 \cdot 2n + 1] + [2(2n + 1) + 1]$$

$$= 3n^2 - [2n + 1] + [2 \cdot 2n + 1] + [2(2n + 1) + 1] \text{ by ind hyp}$$

$$= 3n^2 + 6n + 3 \text{ by algebra}$$

$$= 3(n + 1)^2 \text{ by more algebra}$$
2 (14 pts). Prove that $2n\$ postage for any even number $2n \geq 8$ can be formed from stamps of the denominations $4\$ and $10\$.

Proof: by induction

BASIS : $8\$ = $2 \cdot 4\$

IND HYP : Assume that $2n = 4r + 10s$, with $n \geq 4$, $r \geq 0$, and $s \geq 0$

IND STEP: in two cases

Case 1. $s \geq 1$.
Replace a $10\$-stamp by three $4\$-stamps.

Case 2. $s = 0$. Then $r \geq 2$.
Replace two $4\$-stamp by one $10\$-stamp.
3a (6 pts). How many ways are there to rearrange the letters of the word WOOLLOOMOOLOO?

SOL. There are 13 letters, with 8 O’s, 3 L’s, 1 M, and 1 W.

\[
\frac{13!}{8!3!1!1!}
\]

3b (10 pts). What is the coefficient of \( x^7 y^9 \) in the expansion of \((3x + 2y)^{16}\)?

SOL. \( \binom{16}{7} 3^7 2^9 \)
4 (14 pts). Lollipops are colored red, blue, and yellow. In how many different ways can 10 lollipops be selected, so that there is at least one lollypop of each color? (Two lollipops of the same color are regarded as identical.)

SOL. There is only 1 way to choose the required one of each color. Then choose 7 more lollipops from 3 types with repetitions permitted.

\[
\binom{7 + 3 - 1}{7} = \binom{9}{2} = 36
\]
5a (6). Two 52-card decks are shuffled together, and a 5-card hand is dealt. What is the probability of getting four-of-a-kind – i.e., 4 cards of one denomination and 1 card of a different denomination?

\[
\text{SOL: } \frac{13 \binom{8}{4} \cdot 96}{\binom{104}{5}}
\]

5b (6). Two 52-card decks are shuffled together, and a 5-card hand is dealt. What is the probability of getting a flush – i.e., 5 cards of the same suit?

\[
\text{SOL: } \frac{4 \binom{26}{5}}{\binom{104}{5}}
\]

5c (8). Infinitely many 52-card decks are shuffled together, and a 5-card hand is dealt. What is the probability of getting a flush – i.e., 5 cards of the same suit?

\[
\text{SOL: } 4 \left( \frac{1}{4} \right)^5 = \left( \frac{1}{4} \right)^4
\]
6 (16 pts). Solve the following recurrence.

\[ b_0 = -1; \quad b_1 = 1; \]
\[ b_n = 10b_{n-1} - 21b_{n-2} \]

SOL:
\[ r^2 - 10r + 21 = (r - 3)(r - 7) \] write char eq and factor it
\[ b_n = A \cdot 3^n + B \cdot 7^n \] general solution
\[ b_0 = -1 = A + B \]
\[ b_1 = 1 = 3A + 7B \] use initial conditions
\[ B = 1 \quad A = -2 \] solve simult lin eq
\[ b_n = -2 \cdot 3^n + 7^n \] solution to recurrence