DISCRETE MATH: W3203 Final Exam
open book

SOLUTIONS

Your Name (2 pts for LEGIBLY PRINTING your name on this line)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>your name</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

-----------------------------
Total  200

SUGGESTION: Do the EASIEST problems first!

HINT: Some of the solution methods involve pre-college math as well as new methods from this class.

---

1An example of the Reasonable Person Principle: A reasonable student expects to lose a lot of credit for neglecting to EXPLAIN an answer. Omit explanations at your own risk.
1 (24 pts). Consider the following three relations on the integers:

\[ R = \{(x, y) \mid x \cdot y \geq 0\} \]
\[ S = \{(x, y) \mid \gcd(x, y) = 1\} \]
\[ T = \{(x, y) \mid x - y < 1\} \]

For each entry of the following matrix, write YES if the relation labeling its column has the property labeling its row. Else write NO.

**SOLUTION**

<table>
<thead>
<tr>
<th></th>
<th>reflexive</th>
<th>symmetric</th>
<th>anti – symmetric</th>
<th>transitive</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>YES</td>
<td>YES</td>
<td><strong>NO</strong></td>
<td><strong>NO</strong></td>
</tr>
<tr>
<td>S</td>
<td><strong>NO</strong></td>
<td>YES</td>
<td><strong>NO</strong></td>
<td><strong>NO</strong></td>
</tr>
<tr>
<td>T</td>
<td>YES</td>
<td><strong>NO</strong></td>
<td><strong>YES</strong></td>
<td><strong>YES</strong></td>
</tr>
</tbody>
</table>
2 (18 pts). Recopy your answers from question 1 into this matrix. For each NO, give a counter-example to prove that the given relation does not have the given property.

<table>
<thead>
<tr>
<th></th>
<th>refl</th>
<th>sym</th>
<th>anti – sym</th>
<th>trans</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R = {(x,y) \mid x \cdot y \geq 0}$</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>$S = {(x,y) \mid \gcd(x,y) = 1}$</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>$T = {(x,y) \mid x - y &lt; 1}$</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>Y</td>
</tr>
</tbody>
</table>

SOLUTION

R not anti-sym: $4 \cdot 2 \geq 0 \land 2 \cdot 4 \geq 0$, but $2 \neq 4$

R not trans: $-4 \cdot 0 \geq 0 \land 0 \cdot 2 \geq 0$, but $-4 \cdot 2 < 0$

S not refl: $\gcd(5,5) = 5 \neq 1$

S not anti-sym: $\gcd(4,5) = 1 \land \gcd(5,4) = 1$, but $5 \neq 4$

S not trans: $\gcd(4,5) = 1 \land \gcd(5,6) = 1$, but $\gcd(4,6) \neq 1$

T not sym: $3 - 5 < 1$, but $5 - 3 > 1$
3 (24 pts). Draw 4 mutually non-isomorphic simple 7-vertex graphs with the degree sequence 3332111.

SOLUTION

[Diagram of four non-isomorphic graphs with 7 vertices and the specified degree sequence]
4 (30 pts). The following four 12-vertex graphs are all 3-regular, and each has only one 3-cycle. To prove that they are mutually non-isomorphic, first let A-, B-, C-, and D- be the subgraphs that result from deleting the 3-cycles from A, B, C, and D, respectively.

4a (10). Draw a Hamiltonian cycle in each of the subgraphs A-, B-, C-, and D- that is Hamiltonian.
SOL: B and D are Hamiltonian, as indicated.

4b (10). Prove that the ones that are Hamiltonian are mutually non-isomorphic.
SOL: The three 2-valent vertices of D- are mutually non-adjacent. That is not so in B-.

4c (10). Prove that the ones that are non-Hamiltonian are mutually non-isomorphic.
SOL: The three 2-valent vertices of C- are all adjacent to the same 3-valent vertex of C-. That is not so in A-.
5 (30 pts). Decide whether the given graphs are planar or not. In each case, give a proof.

5a (5). SOL: Non-planar, has $K_{3,3}$

5b (5). SOL: Non-planar, has $K_{3,3}$

5c (10). SOL: Non-planar, has $K_{3,3}$

5d (10). SOL: Non-planar, has $K_{3,3}$
6a (6). Give a 3-coloring of the following graph.

SOL:

6b (8). Show where to insert two additional edges so that the chromatic number increases to 4.

SOL:

6c (6). Give a proof that your modification in part (b) results in a graph that has no proper 3-coloring.

SOL: Makes a $K_4$. 
7 (20 pts). Consider the following recurrence system.

\[ a_0 = 0, \quad a_1 = 1, \quad a_2 = 2; \]

\[ a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{for} \quad n \geq 3 \]

Prove that \( a_n \leq 7 \cdot 2^{n-3} \) for \( n \geq 0 \)

**PROOF by INDUCTION:**

**BASIS:**

\[ a_0 = 0 \leq 7 \cdot 2^{-3} = \frac{7}{8}; \quad a_1 = 1 \leq 7 \cdot 2^{-2} = \frac{7}{4}; \]

\[ a_2 = 2 \leq 7 \cdot 2^{-1} = \frac{7}{2} \]

**IND HYP:** Assume \( n > 2 \) and

\[ a_j \leq 7 \cdot 2^{j-3} \quad \text{for} \quad j = n - 1, \ n - 2, \ n - 3 \]

**IND STEP:** Then

\[ a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{given recursion} \]

\[ \leq 7 \cdot 2^{-4} + 7 \cdot 2^{-5} + 7 \cdot 2^{-6} \quad \text{by IND HYP} \]

\[ = 7 \left( 2^{-4} + 2^{-5} + 2^{-6} \right) \]

\[ = 7(7 \cdot 2^{-6}) = 49 \cdot 2^{-6} \]

\[ \leq 56 \cdot 2^{-6} = 7 \cdot 2^{-3} \]
8a (8 pts). From an unlimited supply of blocks that are blue, red, and yellow, in how many ways can 7 be selected.

**SOL:** \[\binom{9}{2}\]

8b (4 pts). If there must be more blue than red, and more red than yellow, in how many ways can 7 be selected.

**SOL:** Ad hoc counting is easiest. 4 ways

\[B^6R, B^5R^2, B^4R^3, B^4R^2Y\]
9 (20 pts). Solve the following recurrence.

\[ a_0 = 1, \quad a_1 = 2; \]

\[ a_n = 4a_{n-1} - 4a_{n-2} + 1 \quad \text{for } n \geq 2 \]

SOLUTION:

\[ \hat{a}_n = B \cdot 2^n + C \cdot n2^n \]

\[ \hat{a}_n = 1 \]

\[ a_n = \hat{a}_n + \dot{a}_n = B \cdot 2^n + C \cdot n2^n + 1 \]

\[ a_0 = 1 = B \cdot 2^0 + C \cdot 0 \cdot 2^0 + 1 = B + 1 \]

\[ a_1 = 2 = B \cdot 2^1 + C \cdot 1 \cdot 2^1 + 1 = 2B + 2C + 1 \]

\[ \Rightarrow B = 0, \quad C = \frac{1}{2} \]

\[ a_n = n \cdot 2^{n-1} + 1 \]