**SOLUTIONS**

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Total 100

SUGGESTION: Do the EASIEST problems first!

HINT: Some of the solution methods involve pre-college math as well as new methods from this class.

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1 An example of the Reasonable Person Principle: A reasonable student expects to lose a lot of credit for neglecting to EXPLAIN an answer. Omit explanations at your own risk.
1 (16 pts). Consider the following recurrence.

\[
g_0 = 1; \quad g_1 = 1; \quad g_2 = 2; \\
g_n = g_{n-1} + g_{n-2} + g_{n-3}
\]

Prove for all \( n \geq 0 \) that \( g_n \geq 1.8^{n-1} \).

PROOF by induction.

BASIS. \( g_0 = 1 > 1.8^{-1}; \quad g_1 = 1 = 1.8^0; \quad g_2 = 2 > 1.8^1 \).

IND HYP. Assume that

\[
g_{n-3} \geq 1.8^{n-4}; \quad g_{n-2} \geq 1.8^{n-3}; \quad g_{n-1} \geq 1.8^{n-2}.
\]

IND STEP. Then

\[
g_n = g_{n-1} + g_{n-2} + g_{n-3}
\]

\[
\geq 1.8^{n-2} + 1.8^{n-3} + 1.8^{n-4}
\]

\[
= (1.8^2 + 1.8 + 1) \cdot 1.8^{n-4}
\]

\[
= 6.04 \cdot 1.8^{n-4}
\]

\[
> 1.8^3 \cdot 1.8^{n-4} = 1.8^{n-1}
\]

since \( 1.8^3 = 5.832 < 6.04 \).
2 (16 pts). Prove that \( n \)¢ postage for any \( n \geq 14 \) can be formed from stamps of the denominations 5¢, 7¢, and 11¢.

PROOF by induction.
BASIS. 14 = 2 \( \cdot \) 7

IND HYP. Assume that \( n \geq 14 \) and that

\[
n = 5r + 7s + 11t
\]

for some non-negative integers \( r, s, t \).

IND STEP.
Case 1. \( t > 0 \).
Replace one 11¢ stamp by a 5¢ stamp and a 7¢ stamp.

Case 2. \( s \geq 2 \).
Replace two 7¢ stamps by three 5¢ stamps.

Case 3. \( t = 0 \) and \( s \leq 1 \).
Since \( 5r + 7s \geq 14 \) and \( s \leq 1 \), it follows that \( r \geq 2 \).
Replace two 5¢ stamps by one 11¢ stamp.
3 (16 pts). Prove the following inequality for $1 \leq k < n$.

$$\binom{n}{k}^2 - \binom{n-1}{k-1}^2 \geq \binom{n-1}{k}^2 + \binom{n-1}{k-1}\binom{n-1}{k}$$

**PROOF.**

$$\binom{n}{k}^2 = \left(\binom{n-1}{k} + \binom{n-1}{k-1}\right)^2$$

$$= \binom{n-1}{k}^2 + 2\binom{n-1}{k}\binom{n-1}{k-1} + \binom{n-1}{k-1}^2$$

$$\binom{n}{k}^2 - \binom{n-1}{k-1}^2 = \binom{n-1}{k}^2 + 2\binom{n-1}{k-1}\binom{n-1}{k}$$

$$\geq \binom{n-1}{k}^2 + \binom{n-1}{k-1}\binom{n-1}{k}$$
4 (16 pts). Count the number of solutions of the equation

\[ x + y + z = 24 \]

with integers \( x \geq 2, \ y \geq 3, \) and \( z \geq 4 \). Hint: this is equivalent to counting the number of ways to choose 24 pieces of fruit from large supplies of apples, bananas, and coconuts with at least 2 apples, at least 3 bananas, and at least 4 coconuts.

SOLUTION. This is simply the number of ways to choose

\[ 24 - (2 + 3 + 4) = 15 \]

objects from a set of 3 with repetitions allowed.

\[ \binom{17}{2} = 136 \]
5 (18 pts). Five fair dice are rolled.

5a (4). What is the probability of no fours?

SOL \[ \frac{5^5}{6^5} \]

5b (4). What is the probability of exactly one four.

\[ \frac{\binom{5}{1} \cdot 1 \cdot 5^4}{6^5} = \frac{5^5}{6^5} \]

5c (6). What is the probability of exactly two fours?

SOL \[ \frac{\binom{5}{2} \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right)^3}{6^5} \]

5d(4). What is the probability of at least two fours?

SOL \[ 1 - \left( \frac{5}{6} \right)^5 - \binom{5}{2} \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right)^3 \]
6 (16 pts). Solve the following recurrence.

\[ b_0 = 0; \quad b_1 = 1; \]
\[ b_n = 8b_{n-1} - 15b_{n-2} \]

**SOLUTION**

\[ r^2 - 8r + 15 = 0 \quad \text{characteristic eq} \]
\[ (r - 3)(r - 5) = 0 \quad r = 3, 5 \quad \text{roots} \]

\[ b_n = C3^n + D5^n \quad \text{general solution} \]

\[ 0 = b_0 = C3^0 + D5^0 = C + D \]
\[ 1 = b_1 = C3^1 + D5^1 = 3C + 5D \]

solve: \( C = -\frac{1}{2} \quad D = \frac{1}{2} \)

\[ b_n = \frac{-1}{2} \cdot 3^n + \frac{1}{2} \cdot 5^n \]