19:37 11/12/2011

Chapter 10

Graph Theory

coursenotes by Prof. J. L. Gross for Rosen 7th Edition

10.1 INTRODUCTION TO GRAPHS

DEF: A ***graph*** $G = (V, E)$ has two sets as its domains.

- The elements of the set V are called ***vertices***.
- The elements of the set E are called ***edges***.
- For each edge e there is a set of one or two vertices, called the ***endpoints*** of e.

A ***vertex*** is typically conceptualized as a point in $\mathbb{R}^n$, most often in 2-space or 3-space.

An ***edge*** is conceptualized as a space curve (without self-intersections) joining its endpoints.

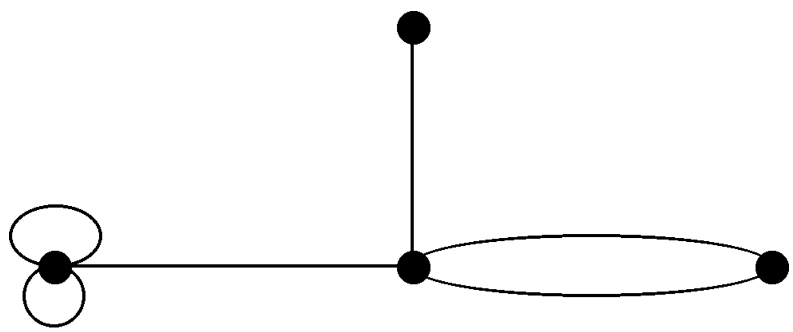


Fig 10.1.1 A general graph.

DEF: Two vertices are ***adjacent*** if there is an edge joining them.

DEF: A graph is ***simple*** if
(1) there are no self-loops, and
(2) there is at most one edge between any pair of vertices. (better etymology: "simplicial")

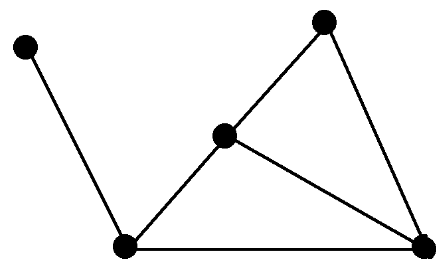


Fig 10.1.2 A simple graph.

OPTIONAL FEATURES of GRAPHS

DEF: A ***direction*** on an edge is a designation of a foward sense.

DEF: An ***arc*** is an edge with a direction.

DEF: A ***digraph*** is a collection of vertices and arcs.

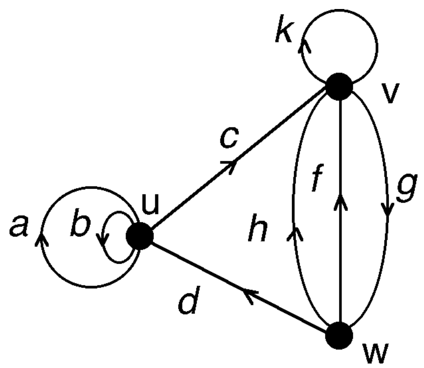
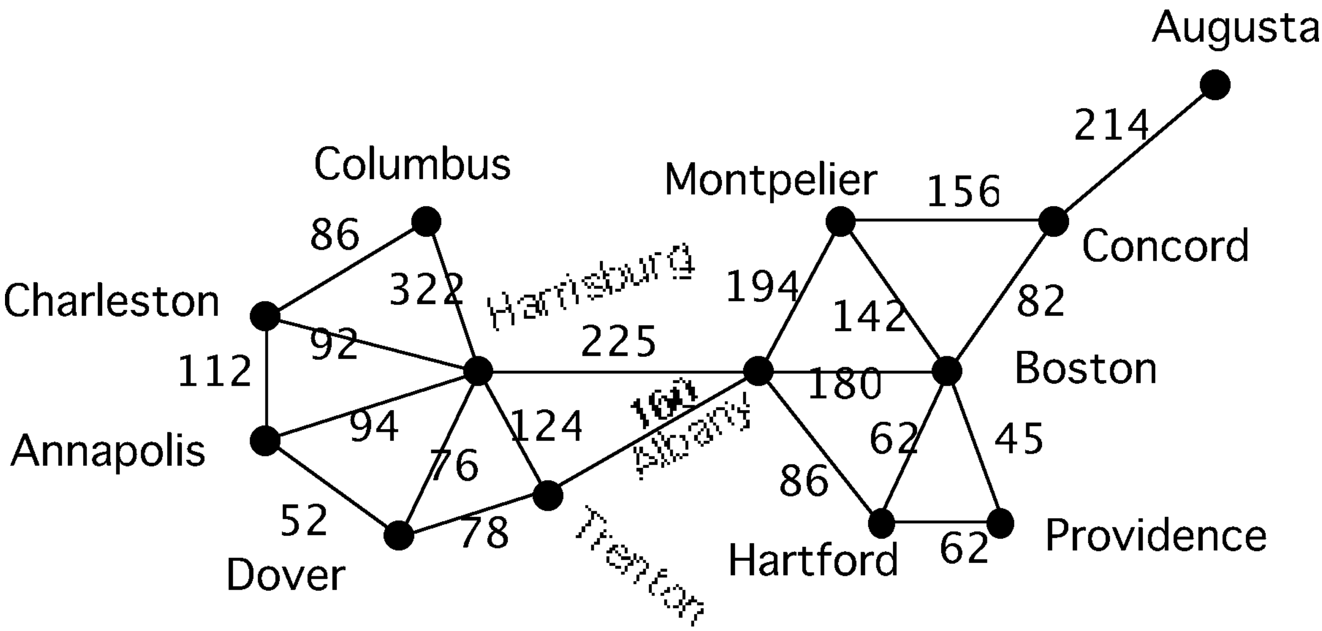


Fig 10.1.3 A digraph.

Remark: Other optional features include ***vertex labels***, ***edge labels***, ***vertex weights***, and ***edge weights***.



CLASSROOM QUESTION

Can the numerical edge labels shown above be correct distances, for any collection of cities?

GRAPH-THEORETIC SOFTWARE

Graph theory software should **always** be designed for all graphs, not just for simple graphs. It should **always** be designed to permit (but not require) directions, vertex labels, edge labels, and the capacity for adding unforseen features at a later time.

It takes only a few additional minutes of design effort to plan for **reusability**.

Retrofitting tends to be formidable, and often infeasible.

10.2 GRAPH TERMINOLOGY

DEGREE

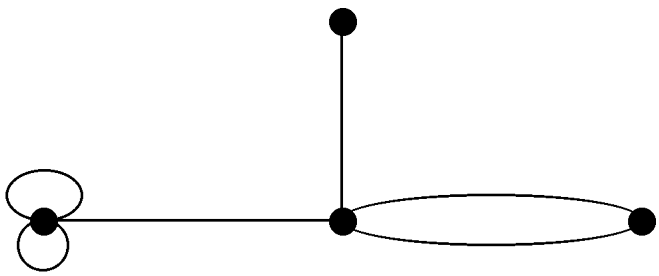
DEF: The ***degree*** or ***valence*** of a vertex v is the number of edge-ends incident on v.

Remark: A self-loop contributes 2 edge-ends to the degree of a vertex.

DEF: The ***degree sequence*** of a graph G is a list of all degrees in ascending order.

Example 10.2.1: The degree sequence for this graph is

$$1, \quad 2, \quad 4, \quad 5$$



Thm 10.2.1. *(Euler) The sum of the degrees of a graph equals twice the number of edges.*

Pf: Every edge contributes two to the degree sum. $\diamondsuit$

Cor 10.2.2. *A graph has evenly many vertices of odd degree.*

Pf: Parity. $\diamondsuit$

Thm 10.2.3. *Let G be a simple graph with at least two vertices. Then G has two vertices with the same degree.*

Pf: By pigeonholing and induction. $\diamondsuit$

CLASSROOM EXERCISE

Construct a non-simple graph whose vertices all have different degrees.

SOCIOLOGICAL APPLICATIONS

Represent the students in a discrete math class as vertices, with an edge joining each pair of students who were acquainted before the course began. This is a simple graph.

Cor 9.2.2 implies that the number of students who knew an odd number of other students is an even number.

Thm 9.2.3 implies that there must be two students who know the exact same number of other students.

Remark: Thm 9.2.3, Cor 9.2.2, and Thm 9.2.1 are all sometimes given the same cutesy sociologically inspired name.

GRAPH THEORETIC DEFINITIONS

⇒ Graph theory terminology and notations differ from one textbook to another.

- Some graph theorists say "degree" and others say "valence".
- Some graph theorists stigmatize graphs with self-loops by calling them "pseudographs".

How did this happen?

1. Thousands of different researchers have published journal articles on graph theory.

   Explore www.graphtheory.com.

   Click on Graph Theory Resources.

2. There are hundreds of books about graph theory.

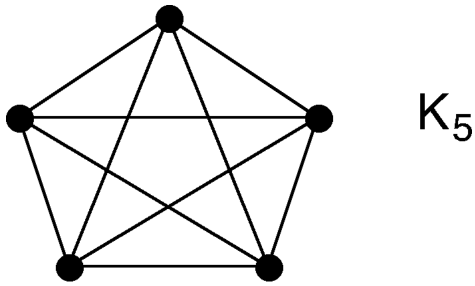
   See www.amazon.com

   or www.bn.com.

SPECIAL GRAPHS

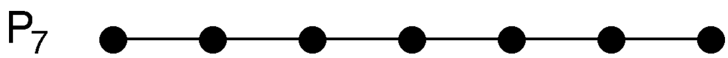
DEF: A ***complete graph*** is a simple graph such that every pair of vertices is joined by an edge.

NOTATION: The complete graph on n vertices is denoted K_n.



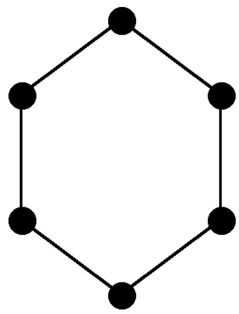
DEF: A ***path graph*** has vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_{n-1}$, such that edge e_k joins vertices v_k and v_{k+1}.

NOTATION: The path graph on n vertices is denoted P_n. (Elsewhere, P_n may denote the graph with n edges and $n+1$ vertices.)



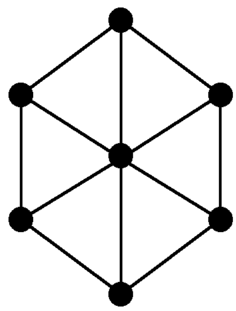
DEF: A ***cycle graph*** has vertices $v_0, v_1, \ldots, v_{n-1}$ and edges $e_0, e_1, \ldots, e_{n-1}$, such that edge e_k joins vertices v_k and $v_{k+1 \pmod{n}}$.

NOTATION: The cycle graph on n vertices is denoted C_n.



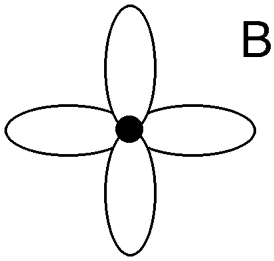
DEF: A ***wheel graph*** has a ***hub vertex*** joined to every other vertex and a cycle through all the other vertices.

NOTATION: The wheel graph whose rim is an n-cycle is denoted W_n. (Elsewhere, W_n may denote the n-vertex graph with an $(n-1)$-cycle on its rim.)



DEF: A ***bouquet*** is a graph with only one vertex.

NOTATION: The bouquet on n edges is denoted B_n.

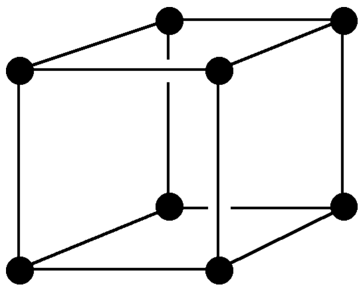


B4

DEF: The ***1-skeleton*** of a polyhedron is the graph comprising all the vertices and edges of the polyhedron.

DEF: The ***cube graph*** of dimension n is the 1-skeleton of the n-dimensional cube.

NOTATION: The n-dimensional cube graph is denoted Q_n.



Q3

REGULAR GRAPHS

DEF: A graph (not just a simple graph) is ***regular*** if every vertex has the same degree.

Example 10.2.2: The following graphs are regular.

- The complete graph K_n is regular of degree $n - 1$.
- A cycle graph is regular of degree 2.
- The cube graph Q_n is regular of degree n.
- The bouquet B_n is regular of degree $2n$.

Example 10.2.3: The only regular wheel graph is W_3, which is isomorphic to K_4.

CLASSROOM EXERCISES

Construct all the (isomorphism types of) regular simple n-vertex graphs, for

$$n = 2, 3, 4, 5$$

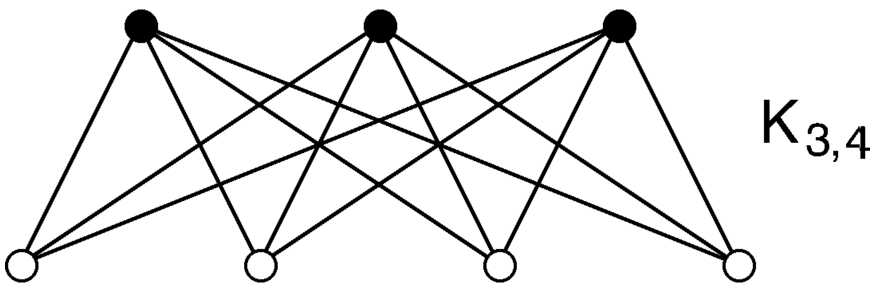
BIPARTITE GRAPHS

DEF: A graph is ***bipartite*** if its vertex set can be partitioned into two cells such that every edge joins a vertex in one cell to a vertex in the other cell.

Example 10.2.4: A path graph is bipartite.

Example 10.2.5: An even cycle graph is bipartite.

DEF: A simple graph is ***complete bipartite*** if it is bipartite so that every vertex in once cell is joined to every vertex in the other cell.

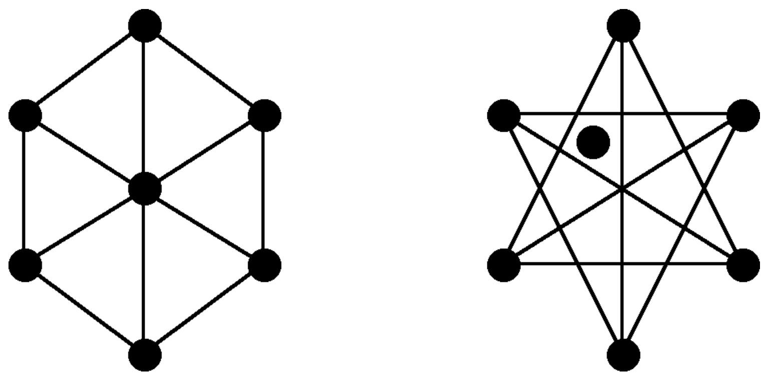


NEW GRAPHS FROM OLD

DEF: A ***subgraph*** of a graph $G = \langle V, E\rangle$ is a graph $H = \langle U, D\rangle$ such that $U \subseteq V$ and $D \subseteq E$.

Remark: Since the subgraph $H = \langle U, D\rangle$ is a graph, it follows that U must contain all the endpoints of the edges in D.

DEF: The ***edge-complement*** of a simple graph G is the graph $\overline{G}$ on the same vertex set as G, such that two vertices of $\overline{G}$ are joined by an edge if and only if they are *not* adjacent in G.

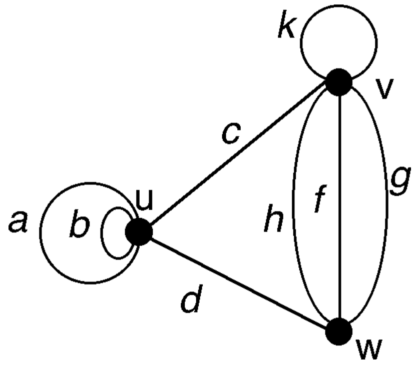


10.3 REPRESENTATIONS & ISOMORPHISM

INCIDENCE TABLE REPRESENTATION

DEF: An ***incidence table*** for a graph has a column indexed by each edge. The entries in the column for an edge are its endpoints. If the edge is a self-loop, then the endpoint appears twice.

Example 10.3.1:

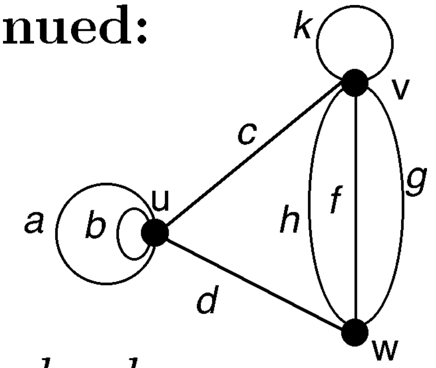


$$V = \{u,\ v,\ w\}\ and\ E = \{a,\ b,\ c,\ d,\ f,\ g,\ h,\ k\}$$

edge	a	b	c	d	f	g	h	k
endpts	u	u	u	w	v	v	w	v
	u	u	v	u	w	w	v	v

INCIDENCE MATRIX REPRESENTATION

Example 10.3.1, continued:



$$\begin{array}{ccccccccc} & a & b & c & d & f & g & h & k \\ u. & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ v. & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \\ w. & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}$$

Incidence matrices waste space on all the zeroes. However, they are sometimes useful in conceptualization.

Thm 10.2.1. *(Euler's Thm, revisited)*
The sum of the degrees of a graph equals $2|E|$.

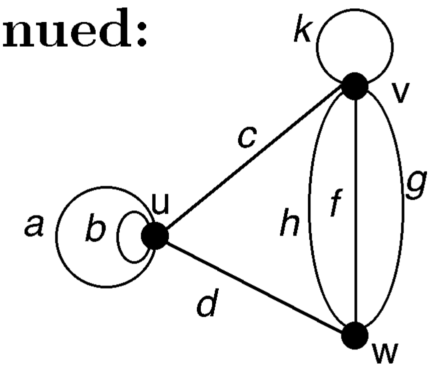
Pf: The degrees of a graph are the row sums of its incidence matrix. Thus, the sum of the degrees equals the sum of the row sums. There is a column for each edge, and every column sum is 2. Thus, $2\,|E|$ equals the sum of the column sums. Therefore, the sum of the row sums equals the sum of the column sums. ◊

ADJACENCY LIST REPRESENTATION

DEF: An ***adjacency list for a vertex*** v of a graph G is a list containing each vertex w of G once for each edge between v and w.

DEF: An ***adjacency list representation*** of a graph is a table of all the adjacency lists.

Example 10.3.1, continued:



$u.$	u	u	v	w	
$v.$	u	v	w	w	w
$w.$	u	v	v	v	

ADJACENCY MATRIX REPRESENTATION

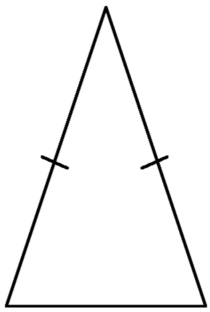
	u	v	w
$u.$	2	1	1
$v.$	1	1	3
$w.$	1	3	0

Remark: Lots of wasted space. Clumsy for self-loops.

GRAPH ISOMORPHISM

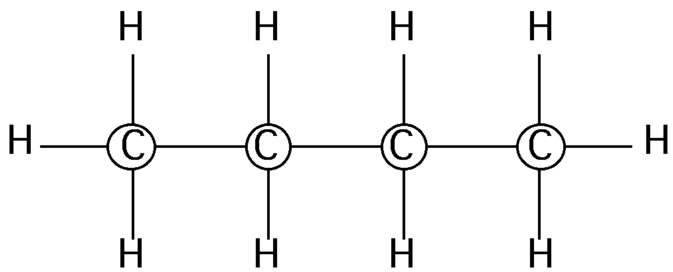
The Greek root "iso" means "same". The Greek root "morphism" means "form".

Example 10.3.2: An *isosceles triangle* has two edges that are the same length.

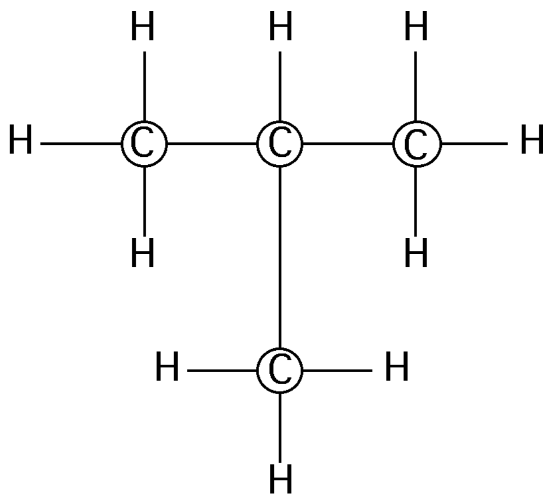


isosceles triangle

Example 10.3.3: Two molecules with the same chemical formula are called *isomers*.

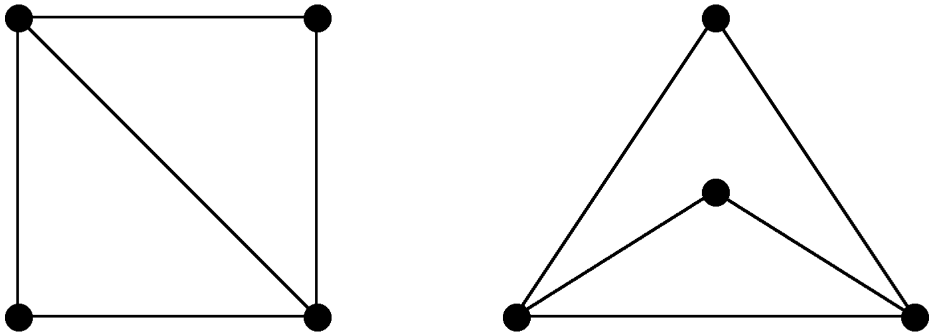


butane



isobutane

And now for graphs. How are these the same??



DEF: The graphs G and H are ***isomorphic*** if there exists a one-to-one onto function

$$f : V_G \to V_H$$

such that $\forall u, v \in V_G$, the number of edges between $f(u)$ and $f(v)$ equals the number of edges between u and v.

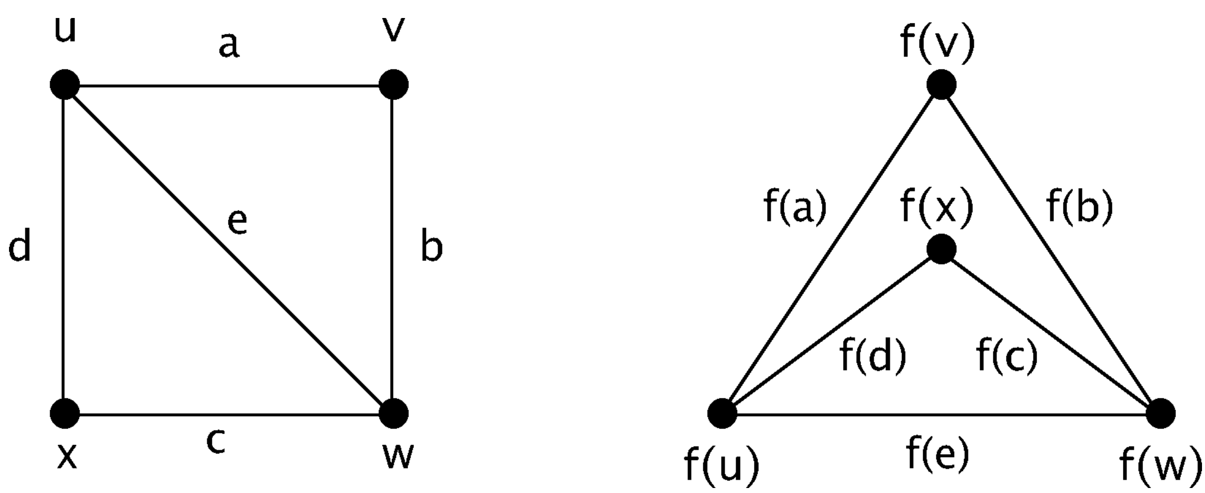
SIMPLE ISOMORPHISM

Proposition 10.3.2. *Two simple graphs G and H are isomorphic if and only if there if a bijection*

$$f : V_G \to V_H$$

such that vertices $f(u)$ and $f(v)$ are adjacent in H if and only if vertices u and v are adjacent in G.

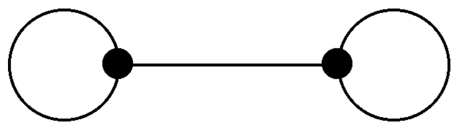
Example 10.3.4: The graph mapping f is an isomorphism.

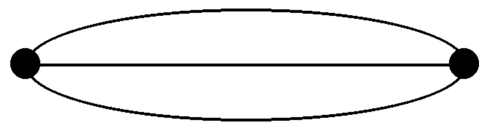


Clearly, two isomorphic graphs have
- the same number of vertices
- the same number of edges
- the same degree sequence

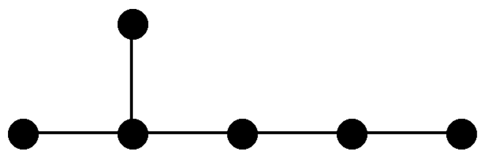
But this is not enough!

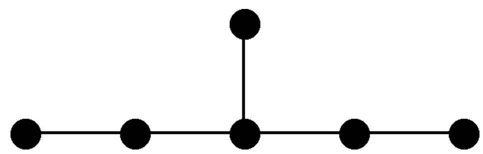
Example 10.3.5: Two nonisomorphic graphs with the same degree sequence.





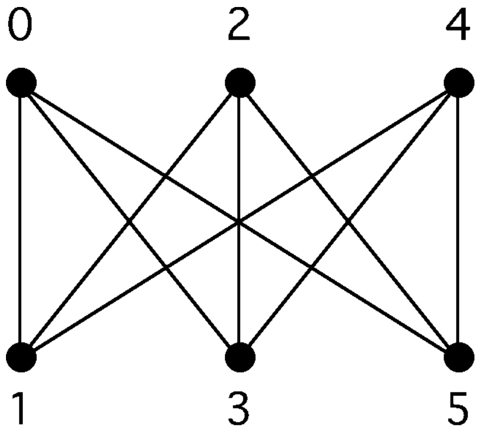
Example 10.3.6: Two more graphs with the same degree sequence, yet nonisomorphic.



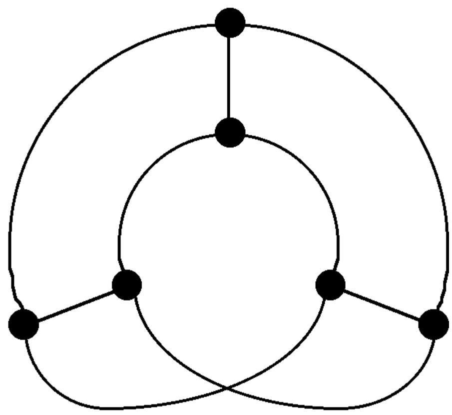


GRAPH ISOMORPHISM TESTING

Example 10.3.7: Are these graphs isomorphic?

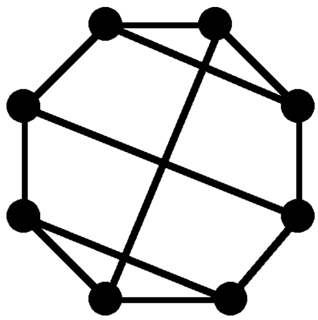


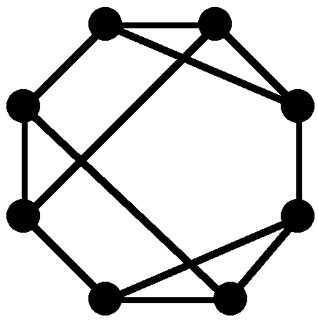
$K_{3,3}$



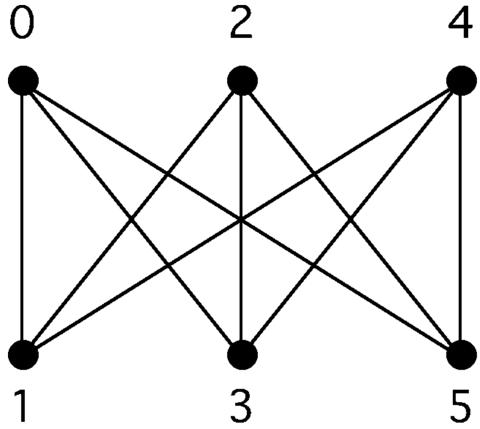
ML_3

Example 10.3.8: Are these graphs isomorphic?

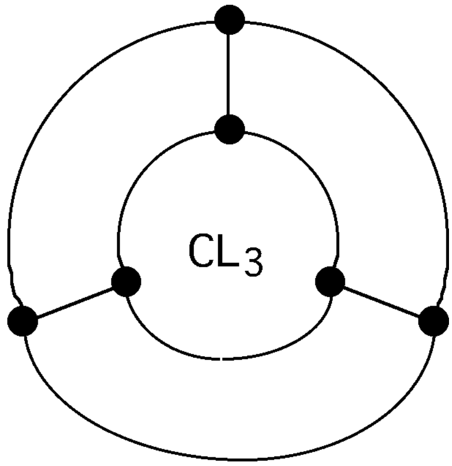




Example 10.3.9: Are these graphs isomorphic?

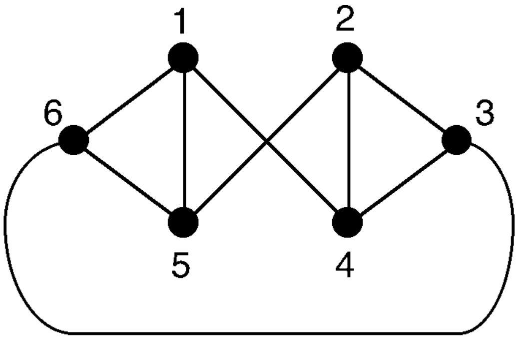


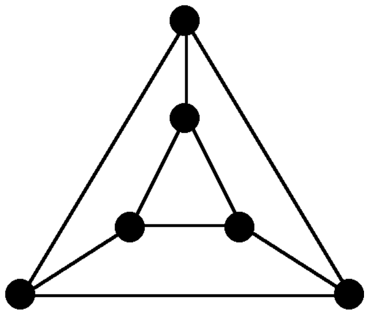
$K_{3,3}$



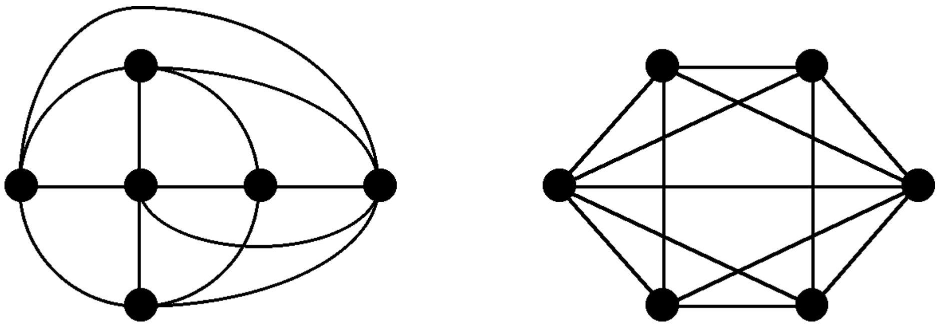
Example 10.3.10: From Final Exam May 1993.

6b (10)

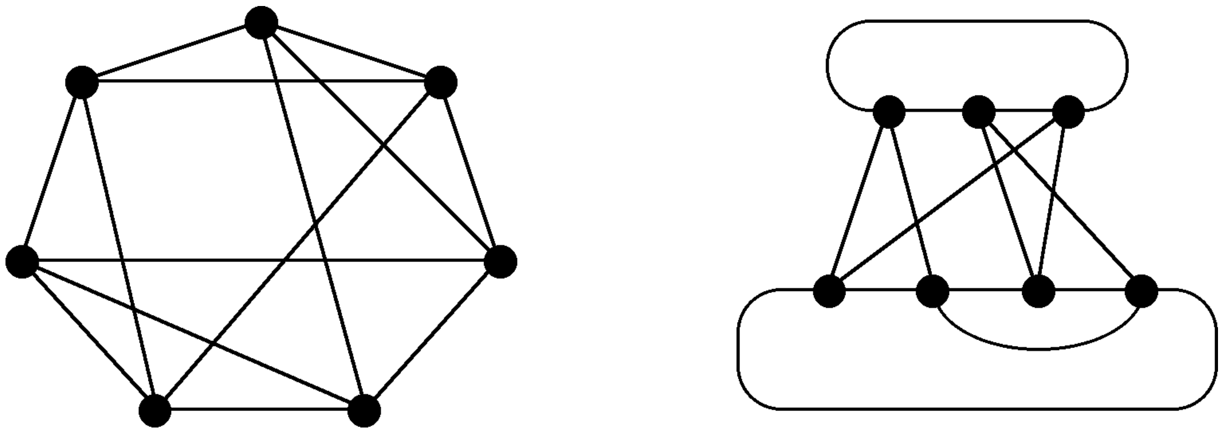




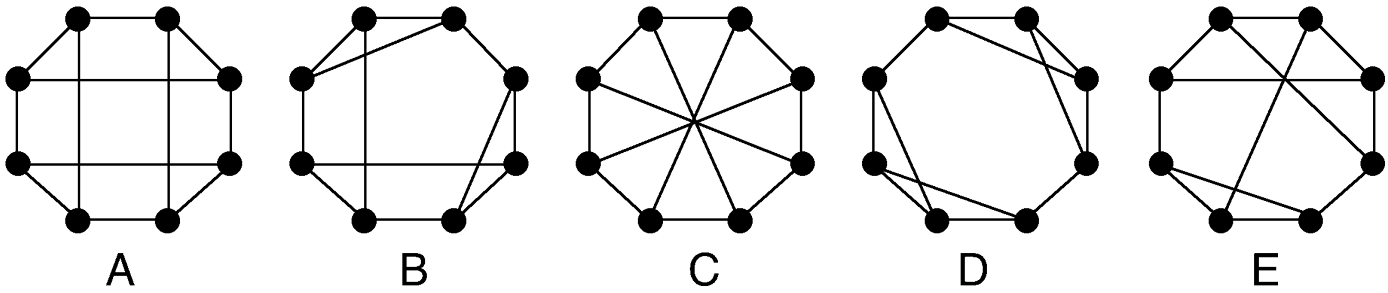
Example 10.3.11: From Dec 1993.



Example 10.3.12: From May 1994.



Example 10.3.13: From GTAIA



No two of these graphs are isomorphic.

Remark: Prop 9.4.2 (next section) facilitates a brief explanation why.

10.4 CONNECTIVITY

WALKS and PATHS

DEF: A ***walk*** from vertex v_0 to vertex v_n is an alternating sequence

$$W = v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$$

such that edge e_j joins vertices v_{j-1} and v_j.

- The ***initial vertex*** is v_0. The ***final vertex*** is v_n. These two vertices are ***external***. The other vertices are ***internal***.
- Walk W is ***closed*** if $v_0 = v_n$. Otherwise it is ***open***.

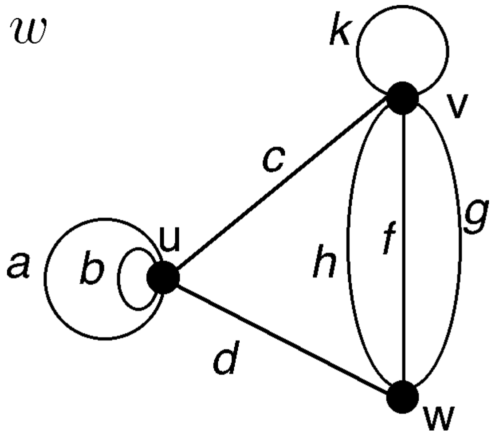
DEF: The ***length*** of a walk is the number of edge-steps.

NOTATION: In a simple graph, a walk may be represented unambiguously by its vertex sequence or by its edge sequence.

SOME FINE POINTS OF WALKS

Example 10.4.1: Consider three walks:

$$W_1 = u, c, v, f, w, h, v, f, w$$
$$W_2 = v, f, w, h, v$$
$$W_3 = w, f, v, h, w$$



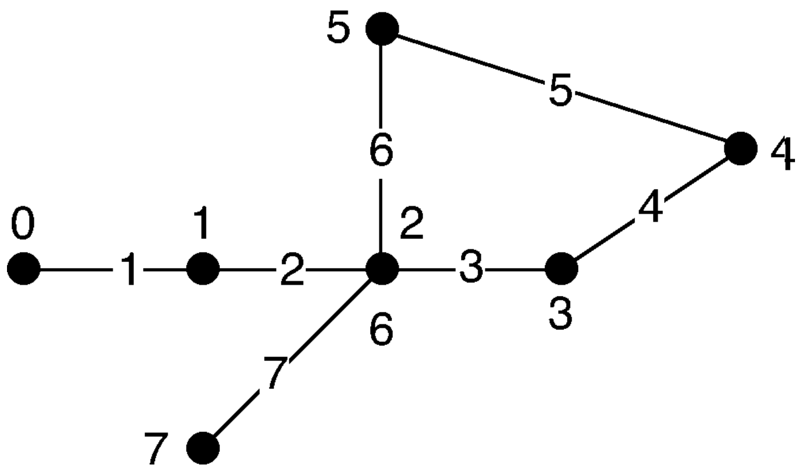
(1) Walk W_1 has length 4, because there are four edge-steps: c, f, h, f. Yet it traverses only three edges $\{c, f, h\}$.

(2) Walk W_1 can be represented unambiguously by its edge sequence: c, f, h, f. Yet its vertex sequence u, v, w, v, w fails to specify which of the edges f, g, h is to be traversed in the second, third, and fourth edge-steps.

(3) Similarly, walk W_2 cannot be represented unambiguously by its vertex sequence v, w, v.

(4) Moreover, walk W_2 cannot be represented unambiguously by its edge sequence f, h, because that is also the edge sequence of the walk W_3.

DEF: A ***trail*** is a walk with no repeated edges. {The text uses "path", instead of "trail".}



DEF: An ***open path*** is an open trail with no repeated vertices.

DEF: A ***cycle*** or ***closed path*** is a closed trail in which the only vertex that is repeated is the external vertex.

TERMINOLOGY NOTE: Thus, *paths* and *cycles* are alternating sequences of vertices and edges, conceptually distinct from *path graphs* and *cycle graphs*, which are types of graphs.

TERMINOLOGY NOTE: Since the text permits a "path" to have repeated vertices, it says "simple path" when it means to exclude them.

CYCLE GRAPHS and ISOMORPHISM

Prop 10.4.1. *Let $f : G \to H$ be a graph isomorphism, and let*

$$W = v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n$$

be a walk in G. Then

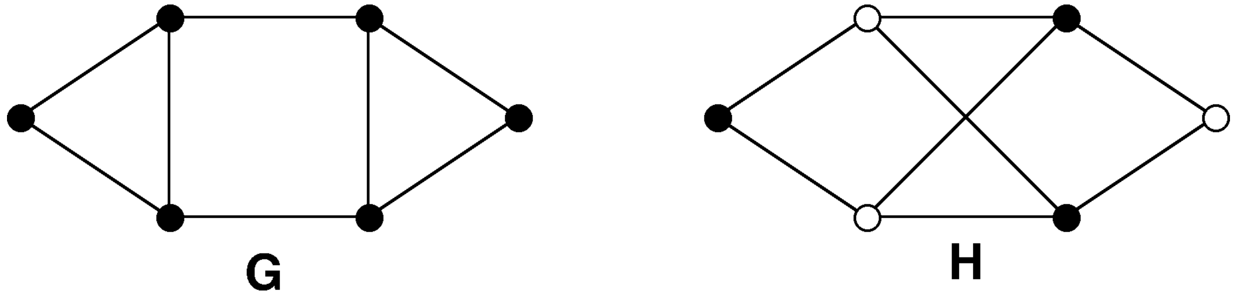
$$f(W) = f(v_0), f(e_1), f(v_1), \ldots, f(v_{n-1}), f(e_n), f(v_n)$$

is a walk in H.

Prop 10.4.2. *Let $f : G \to H$ be a graph isomorphism, and let C be a k-cycle subgraph of G. Then $f(C)$ is a k-cycle subgraph of H.*

Remark: Knowing that an isomorphism, if it existed, would map a k-cycle subgraph in the domain to a k-cycle subgraph in the codomain is often useful on proving that two graphs are not isomorphic.

Example 10.4.2:



Graphs G and H have the same degree sequence $2, 2, 3, 3, 3, 3$, so the isomorphism problem might be non-trivial.

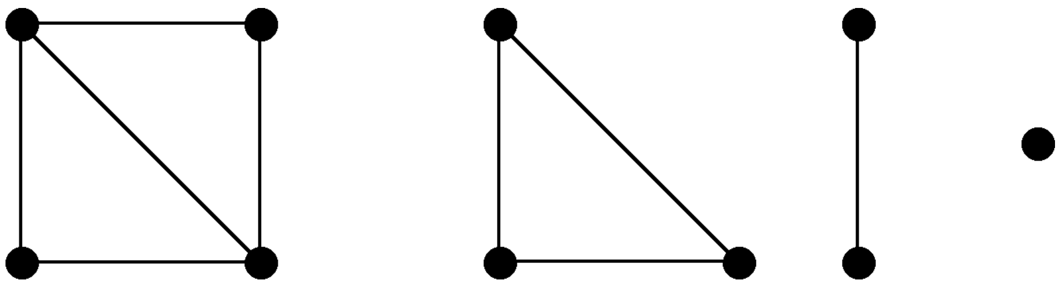
However, graph G has two 3-cycles. Graph H is bipartite, so there are no odd cycles. Thus, they cannot be isomorphic.

Remark: Examining cycle subgraphs permits a brief explanation of Examples 9.3.9 and 9.3.13.

CONNECTEDNESS

DEF: A graph G is ***connected*** if every pair of vertices $u, v \in V_G$ is joined by a path.

DEF: A ***component*** of a graph G is a maximal connected subgraph, that is, a subgraph that is not properly contained in any larger connected subgraph.



A graph with four components.

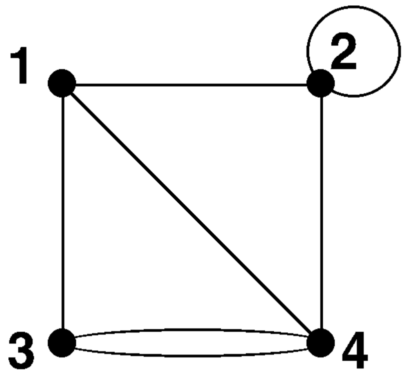
DEF: A digraph D is ***strongly connected*** if every pair of vertices $u, v \in V_D$ is joined by a directed path.

NAIVE CONNECTEDNESS TEST

Prop 10.4.3. *Let A be the adjacency matrix of a graph G. Then $A^n[i,j]$ is the number of walks of length n between vertices i and j.*

Pf: Follows from the def of matrix mult. $\diamondsuit$

Example 10.4.3:



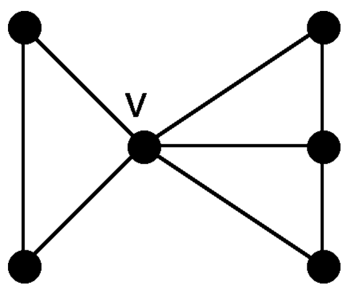
A	1	2	3	4
1	0	1	1	1
2	1	2	0	1
3	1	0	0	2
4	1	1	2	0

A^2	1	2	3	4
1	3	3	2	3
2	3	6	3	3
3	3	3	5	1
4	3	3	1	6

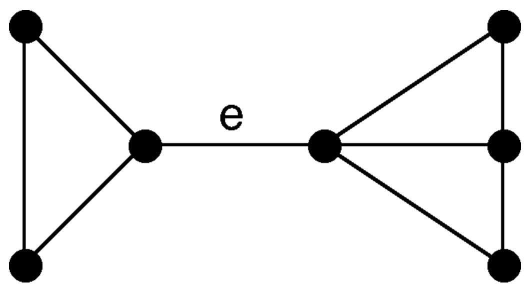
Connectedness Test: For an n vertex graph, calculate the first $n-1$ powers of its adjacency matrix. The graph is connected if no vertex remains zero throughout the process.

CUTPOINTS and CUTEDGES

DEF: A ***cutpoint*** of a graph is a vertex whose removal increases the number of components.

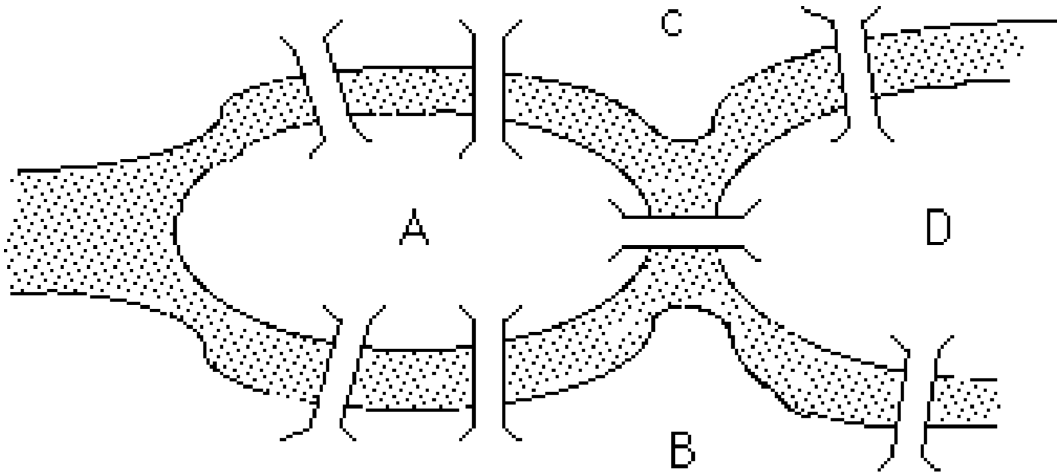


DEF: A ***cutedge*** of a graph is an edge whose removal increases the number of components.



10.5 EULER AND HAMILTON TOURS ∎

KÖNIGSBERG BRIDGE PROBLEM

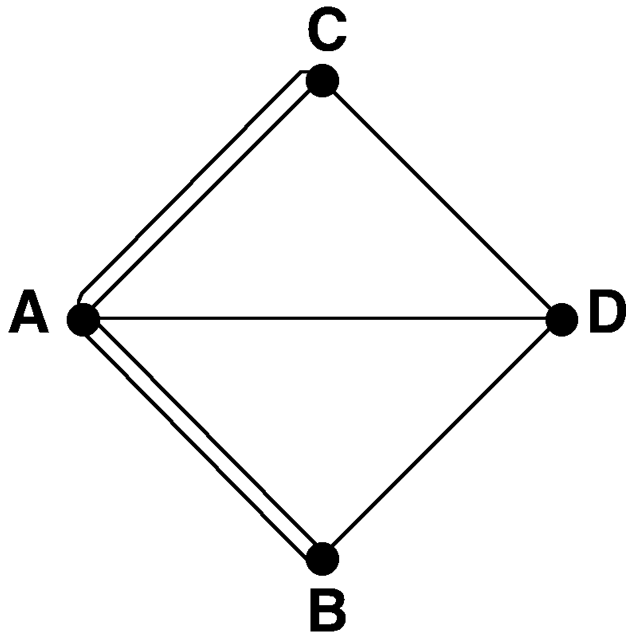


DEF: An ***Eulerian tour*** in a graph is a closed walk that traverses every edge exactly once.

DEF: An ***Eulerian graph*** is a graph that has an Eulerian tour.

DEF: An ***Eulerian trail*** in a graph is a trail that traverses every edge exactly once.

The Königsberg graph is a non-Eulerian graph.

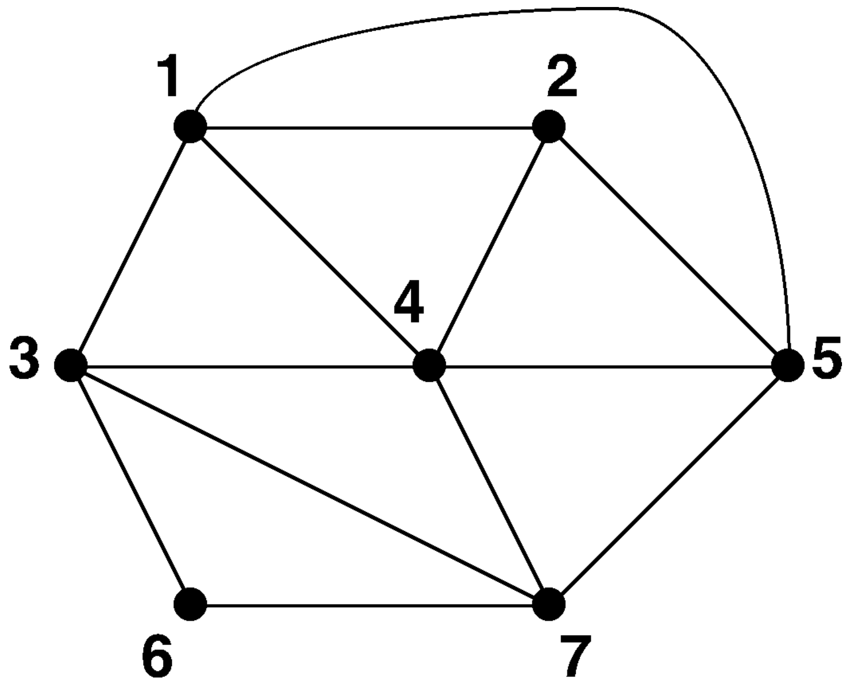


Thm 10.5.1. *A connected graph is Eulerian if and only if every vertex has even degree.*

Pf: sketch in class.

Thm 10.5.2. *A connected graph has an open Eulerian trail if and only it has exactly two vertices of odd degree.*

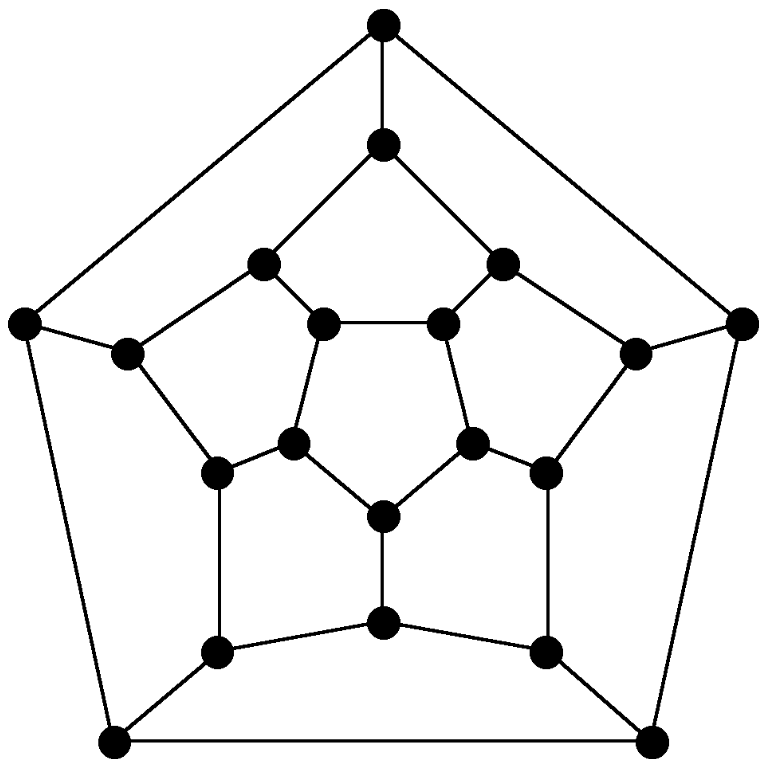
CLASSROOM QUESTIONS:



1. Is this graph Eulerian?

2. If not, how might it it be modified to make it Eulerian?

HAMILTONIAN TOURS

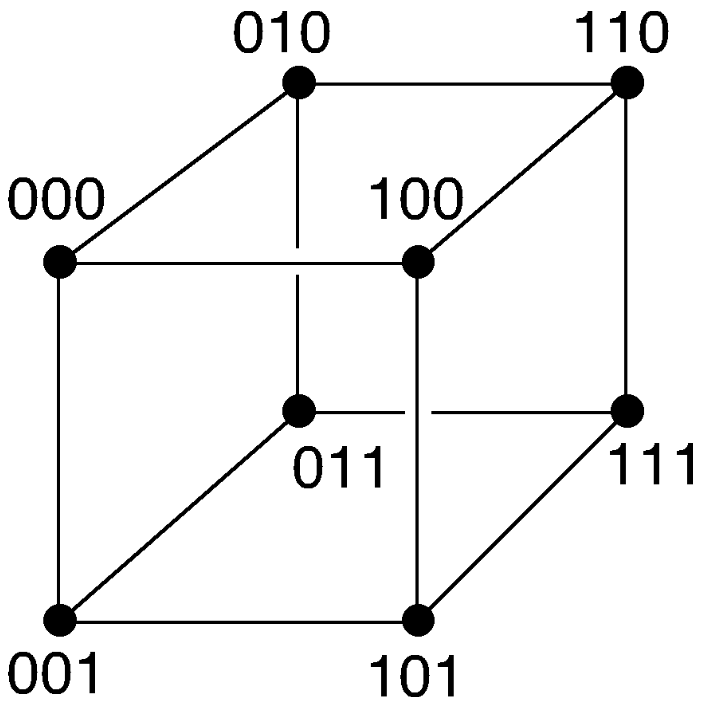


DEF: A ***Hamiltonian tour*** in a graph is a cycle that visits every vertex exactly once.

DEF: An ***Hamiltonian graph*** is a graph that has a spanning cycle.

DEF: An ***Hamiltonian path*** in a graph is a path that visits every vertex exactly once.

Example 10.5.1: Find a Gray code in the hypercube.

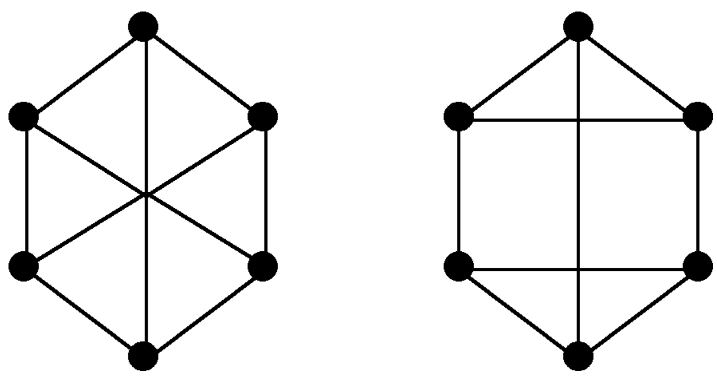


Criterion for proving a graph is Hamiltonian.

Theorem 10.5.3. *(Dirac's Theorem) Let G be a simple n-vertex graph with $n \geq 3$, such that every vertex has degree at least $\left\lfloor \frac{n}{2} \right\rfloor$. Then G is Hamiltonian.*

Pf: Omitted. $\diamondsuit$

Example 10.5.2: Dirac's Theorem simplifies the task of constructing all the isomorphism types of 3-regular 6-vertex simple graphs, because it implies that every one of them has a complete spanning cycle. There are only these two.



Rules for proving a graph is not Hamiltonian.

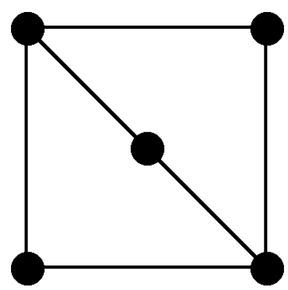
(1) If a vertex v has degree two, then both its incident edges must lie on a Hamiltonian cycle, if there is one.

(2) If two edges incident on a vertex are required in the construction of a Hamilton cycle, then all the others can be deleted without changing the Hamiltonicity of the graph.

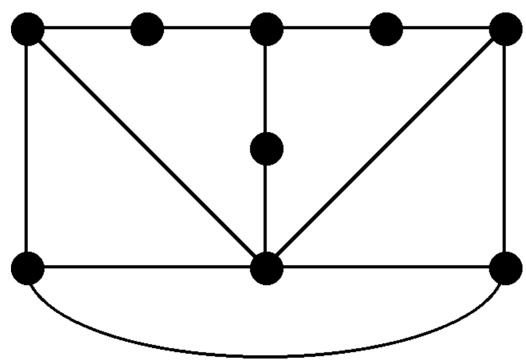
(3) If a cycle formed from required edges is not a spanning cycle, then there is no spanning cycle.

(4) A Hamilton graph has no cutpoints.

Example 10.5.3:



Example 10.5.4:



10.7 PLANAR GRAPHS

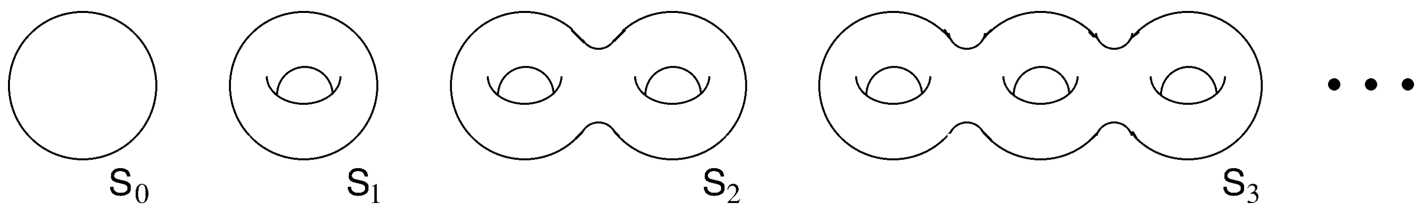
DEF: A graph is ***planar*** if it can be drawn without edge-crossings in the plane.

Imbedding Problem: Given a graph G and a surface S, is it possible to draw G on S without any edge-crossings?

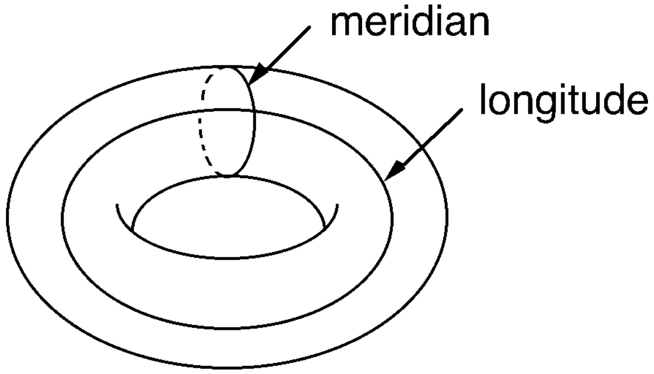
Planarity Problem: Surface S is the sphere (or plane).

ORIENTABLE SURFACES

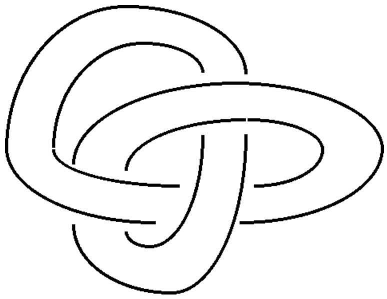
The entire sequence of orientable surfaces



is generated by the torus.

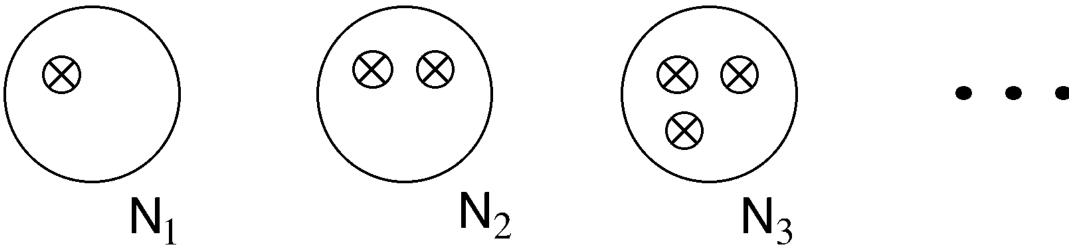


Every *closed surface* in 3-space is *topologically equivalent* to one of the surfaces S_g. There are many ways of placing the same surface into 3-space.

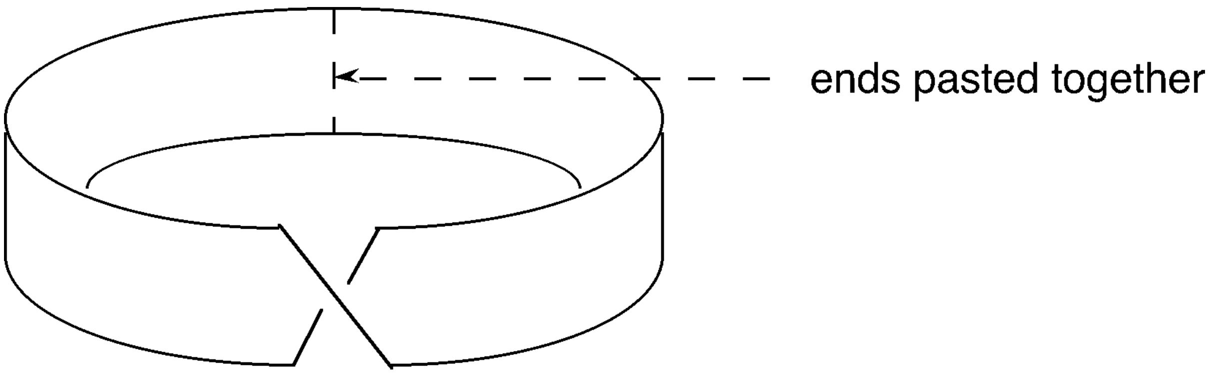


NONORIENTABLE SURFACES

The entire sequence of nonorientable surfaces



is constructable by cutting holes in the sphere and capping each hole with a Möbius band.

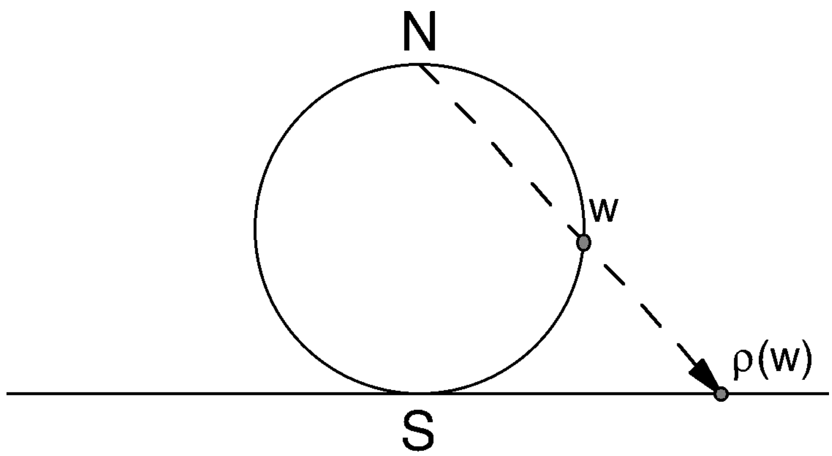


SPHERE and PLANE

In applications, the sphere is the most important surface on which graphs are drawn.

Thm 10.7.1. *A graph can be drawn without edge-crossings in the plane if and only if it can be drawn without edge-crossings in the sphere.*

Pf: The plane is topologically a sphere with a missing point at the North pole. ♢

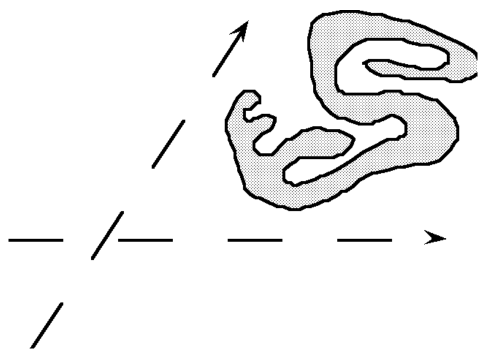


JORDAN CURVE THEOREM

Mathematically, the sphere (and plane) are by far the easiest surfaces for graph drawing problems. Here is why:

Thm 10.7.2. *(Jordan Curve Theorem)*
Every closed curve in the sphere (plane) separates the sphere (plane) into two regions.

Pf: (Veblen, 1906) quite technical.

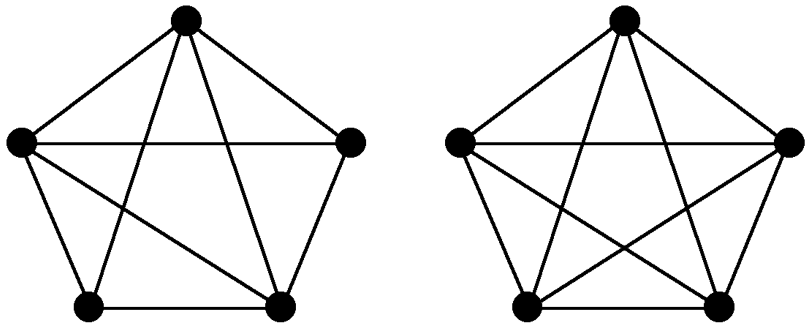


Thm 10.7.3. *(Schönfliess)*
Each side of the separation of the sphere by a closed curve is topologically equivalent to a disk.

Remark: The Schönfliess Theorem does not hold in dimensions greater than two.

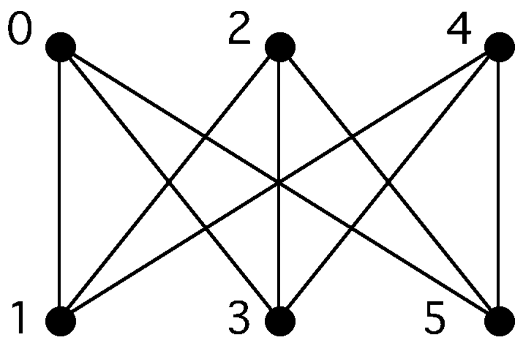
KURATOWSKI GRAPHS

Problem 5: How to prove that K_5 is non-planar.



K_5-e: planar K_5: non-planar

Problem 3,3: Prove that $K_{3,3}$ is non-planar.

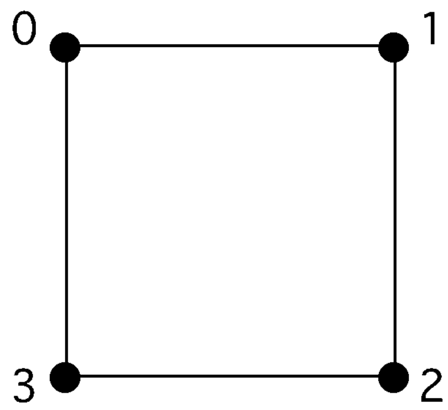


$K_{3,3}$:non-planar

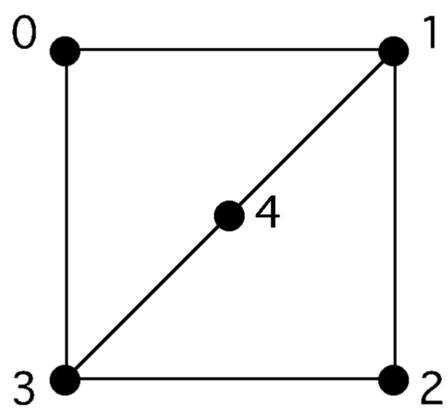
First – a geometric proof that $K_{3,3}$ is non-planar.
Second – an algebraic proof for K_5.

NONPLANARITY of $K_{3,3}$

1. However $K_{3,3}$ is drawn without crossings in the plane, the 4-cycle (0-1-2-3) cuts the plane into two regions.

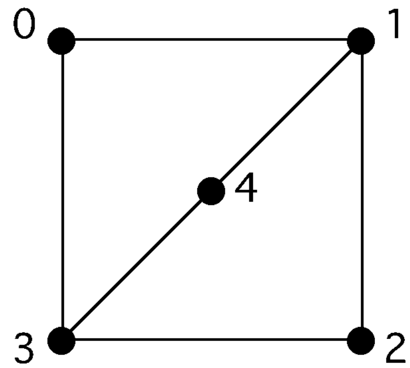


2. The path 1-4-3 lies wholly in one of them, thereby separating it into two smaller regions.



Altogether now, there are now three regions. Vertex 5 must lie in one of them.

3. Finally, insert vertex 5 into any of the three $K_{2,3}$-regions. Only two of the three vertices 0, 2, 4 lie on the boundary of any of these three regions. Thus, vertex 5 cannot be joined to all of them without crossing any edges. $\diamondsuit$



NONPLANARITY of K_5

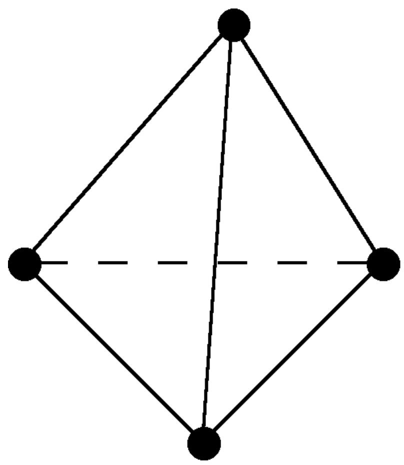
Our proof that K_5 is non-planar is by algebraic topology. Unlike the specialized proof above for $K_{3,3}$, it can be used to establish the nonplanarity of many graphs, not merely of one special case.

First Preliminary Objective: to prove that every connected graph imbedded in the plane satisfies the Euler polyhedral equation:

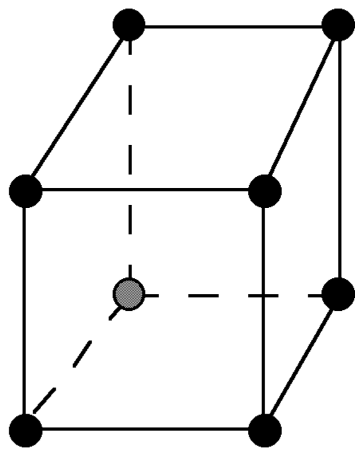
$$|V| - |E| + |F| = 2$$

When a graph is drawn in the plane or in any other surface, it subdivides the rest of the surface into ***regions.*** (The exterior region is included.)

In the classical case first studied by Leonhard Euler, the graph comprised the vertices and edges of a 3-dimensional polyhedron. For that reason, the regions are also called faces.



tetrahedron
V=4, E=6, F=4

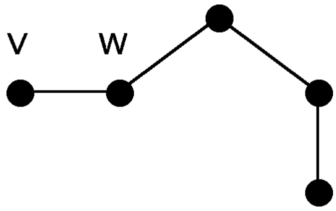


cube
V=8, E=12, F=6

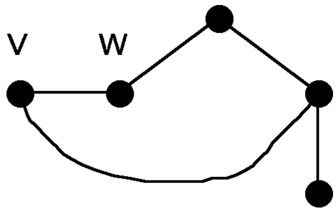
REVIEW : A ***tree*** is a connected graph without cycles.

Lemma 10.7.4. *Let T be a tree with at least one edge. Then T has at least two 1-valent vertices.*

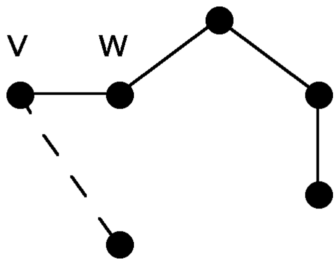
Pf: Let P be a maximum length path in tree T. Let v be the initial vertex of path P, and let w be the next vertex after v in path P.



If vertex v were also adjacent to some vertex after w in path P, then there would be a cycle in the graph.



If vertex v were also adjacent to some vertex of $T - P$, then the path P could be extended, violating its maximality.



Thus, vertex v has only one neighbor. Likewise, this is true of the last vertex of path P. $\diamondsuit$

Lemma 10.7.5. *Let T be a tree. Then*

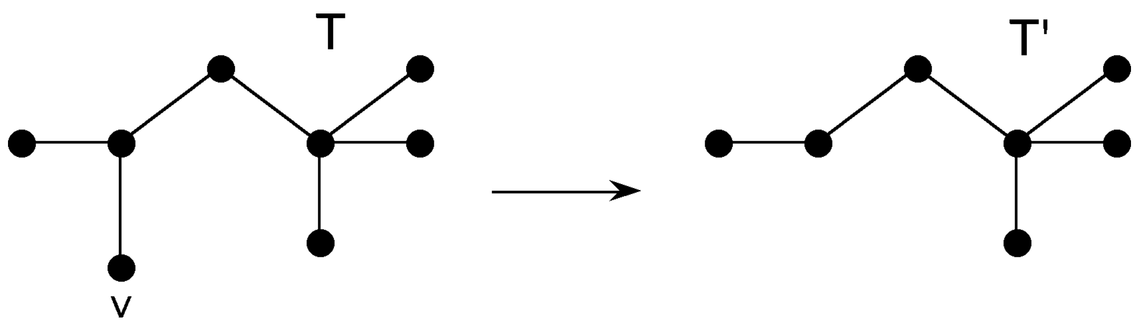
$$|E| = |V| - 1$$

Pf: By mathematical induction.

BASIS: If $|V| = 1$, then $|E| = 0$, lest there be a cycle.

IND HYP: Assume true for all trees with $|V| = n - 1$.

IND STEP: Suppose that $|V| = n$. By Lemma 10.7.4, the tree T has a 1-valent vertex v. Let T' be the graph obtained by deleting vertex v and the edge incident on v from tree T.



Then T' is still connected, and it still has no cycles. Thus, T' is a tree with $n - 1$ vertices. From IND HYP, we infer that T' has $n - 2$ edges. Hence, tree T has n-1 edges. $\diamondsuit$

DEF: The ***cycle rank*** of a connected graph G is the number $\beta(G)$ of edges in the complement of a spanning tree for G. Obviously, a tree has cycle rank zero. More generally, by Lemma 10.7.5,

$$\beta(G) = |E| - |V| + 1$$

Thm 10.7.6. *(Euler polyhedral equation) Let G be any connected graph drawn in the sphere or plane. Then*

$$|V| - |E| + |F| = 2$$

Pf: By induction on the cycle rank.

BASIS: If $\beta(G) = 0$, then graph G is a tree, which implies that

$$|F| = 1$$

since the only region is the exterior region. Moreover, (by Lemma 10.7.5) all trees satisfy

$$|V| - |E| = 1$$

Thus, the equation $|V| - |E| + |F| = 2$ holds.

IND HYP: Assume that the equation

$$|V| - |E| + |F| = 2$$

holds whenever $\beta(G) = n - 1$.

IND STEP: Now suppose that $\beta(G) = n$, where $n \geq 1$. Let H be the graph obtained by erasing a cycle edge e of G. Then, by IND HYP,

$$|V| - |E| + |F| = 2$$

Of course,

$$|V(H)| = |V(G)| \quad \text{and} \quad |E(H)| = |E(G)| - 1$$

Moreover, erasing e joins two regions of G. Thus,

$$|F(H)| = |F(G)| - 1$$

Substituting these results into the equation

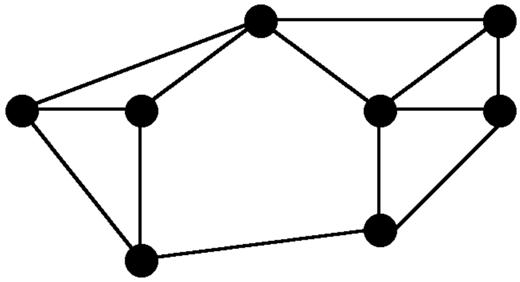
$$|V(H)| - |E(H)| + |F(H)| = 2$$

yields

$$|V(G)| - [|E(G)| - 1] + [|F(G)| - 1] = 2$$

which implies the conclusion immediately. $\diamondsuit$

Remark: In what follows, you need recall only the Euler polyhedral equation, and not the lemmas used to prove it.



|V| = 8

|E| = 13

|F| = 7

girth = 3

Second Preliminary Objective: to prove that every connected graph imbedded in the plane satisfies the edge-face inequality:

$$|F| \;\le\; \frac{2|E|}{\text{girth}(G)}$$

EDGE-FACE INEQUALITY

DEF: The ***girth*** of a graph G is the length of the shortest cycle in G. (The girth of a tree is considered to be infinite.)

DEF: The ***size of a region*** of a graph imbedding is the number of edge-steps in its boundary circuit.

Thm 10.7.7. *Let a graph G be drawn on any surface. Then the sum of the region sizes equals $2E$.*

Pf: Every edge occurs exactly twice in this sum. $\diamondsuit$

Cor 10.7.8. *Let a graph G be drawn in any surface. Then*

$$2\,|E| \;\geq\; \mathrm{girth}(G) \cdot |F|$$

Pf: Each of the $|F|$ regions contributes at least girth(G) to the sum of the region sizes. $\diamondsuit$

Cor 10.7.9. *Edge-Face Inequality*

$$|F| \;\leq\; \frac{2\,|E|}{\mathrm{girth}(G)} \qquad \diamondsuit$$

And now for the promised payoff.

Thm 10.7.10. *The complete graph K_5 is non-planar.*

Pf: $|V(K_5)| = 5$ and $|E(K_5) = 10|$. Thus, if you could draw K_5 in the plane, the Euler equation $|V| - |E| + |F| = 2$ would imply that

$$|F| \;=\; 7$$

Girth$(K_5) = 3$, because there are no self-loops or double edges. This contradicts Cor 10.7.9, since

$$7 \;\not\leq\; \frac{2 \cdot 10}{3} \;=\; \frac{2\,|E|}{\text{girth}(K_5)} \qquad \diamondsuit$$

Thm 10.7.11. *The complete bipartite graph $K_{3,3}$ is non-planar.*

Pf: Same approach!

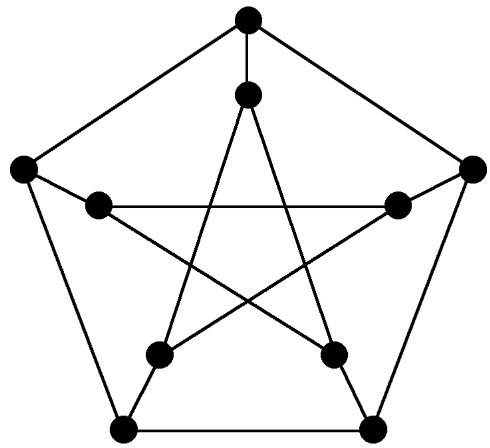
$$|V(K_{3,3})| \;=\; 6 \;\; \text{and} \;\; |E(K_{3,3})| = 9$$

Thus, $|F| = 5$. Moreover,girth$(K_{3,3}) = 4$, because $K_{3,3}$ is bipartite. This contradicts the edge-face inequality, since

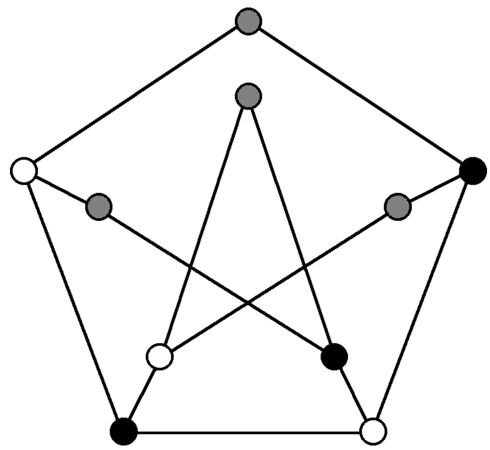
$$5 \;\not\leq\; \frac{2 \cdot 9}{4} \;=\; \frac{2\,|E|}{\text{girth}(K_{3,3})} \qquad \diamondsuit$$

KURATOWSKI'S THEOREM. Every non-planar graph contains a subdivision of K_5 or a subdivision of $K_{3,3}$. Proof is given in W4203 every spring.

Example 10.7.1: The Petersen graph (1891) is non-planar.



Pf:



TWO NONPLANARITY CRITERIA

As an alternative to using elementary principles to prove nonplanarity, we derive two formulas that can be applied in proofs of non-planarity.

Thm 10.7.12. *Let $G = (V, E)$ be a connected simple planar graph, with $|V| \geq 3$, such that*

$$|E| > 3|V| - 6$$

Then G is nonplanar.

Pf: A planar drawing of G must satisfy

$$|E| = |V| + |F| - 2$$

The girth of a simple graph is at least 3, so the Edge-Face Ineq implies that $|F| \leq \frac{2}{3}|E|$. Thus,

$$|E| \leq |V| + \frac{2}{3}|E| - 2$$

The conclusion follows easily. $\diamondsuit$

Remark: Thm 10.7.12 is adequate to prove the nonplanarity of K_5.

Thm 10.7.13. *Let $G = (V, E)$ be a connected simple planar bipartite graph, with $|V| \geq 3$, such that*

$$|E| > 2|V| - 4$$

Then G is nonplanar.

Pf: A planar drawing of G must satisfy

$$|E| = |V| + |F| - 2$$

The girth of a simple bipartite graph is at least 4, because there are no odd cycles. Now the Edge-Face Ineq implies that $|F| \leq \frac{2}{4}|E|$. Thus,

$$|E| \leq |V| + \frac{2}{4}|E| - 2$$

The conclusion follows easily. $\diamondsuit$

Remark: Thm 10.7.13 is adequate to prove the nonplanarity of $K_{3,3}$.

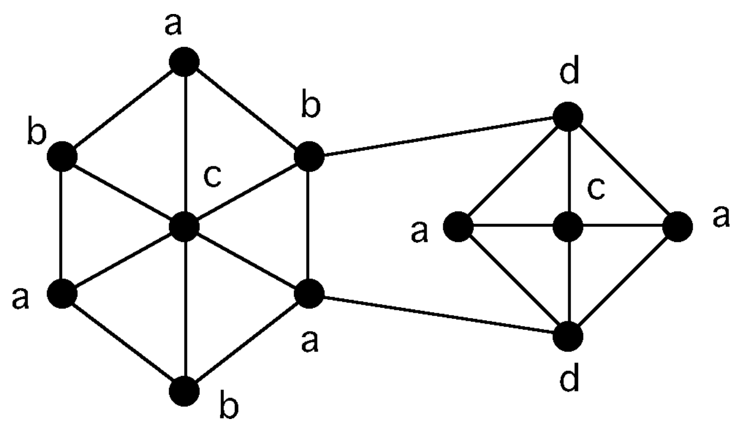
10.8 GRAPH COLORING

DEF: An ***n-coloring of a graph*** G is a function from its vertex set V_G onto the set $\{1, 2, \ldots, n\}$, whose elements we regard as "colors".

DEF: An n-coloring is ***proper*** if no pair of adjacent vertices gets the same color.

DEF: A graph G is ***n-colorable*** if it has a proper n-coloring.

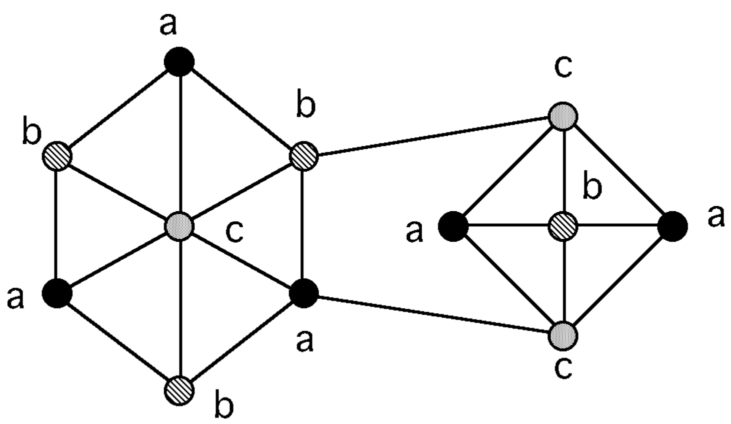
Example 10.8.1: A 4-coloring of a graph. Colors = a, b, c, d



DEF: The ***chromatic number*** of a graph G is $\chi(G) = \min\{n \in \mathcal{Z}^+ \mid G \text{ is } n\text{-colorable}\}$. Also, one says that G is **n-*chromatic*** if $\chi(G) = n$.

Example 10.8.1, continued: The graph above is 3-chromatic.

Pf: (upper bound) It is 3-colorable.



(lower bound) Since the graph contains K_3, at least 3 colors are needed. ◊

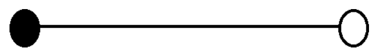
TERMINOLOGY: General colorings are frequently encounted in problems involving the counting of symmetry classes.

OBSTRUCTIONS

DEF: An ***obstruction set*** for n-chromaticity is a family $\mathcal{F}$ of $n+1$-chromatic graphs such that every $n+1$-chromatic graph contains at least one graph in $\mathcal{F}$ as a subgraph.

Obstruction set for 1-chromaticity: an edge

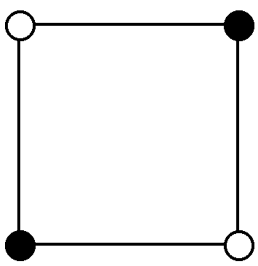
$$\{K_2\}$$

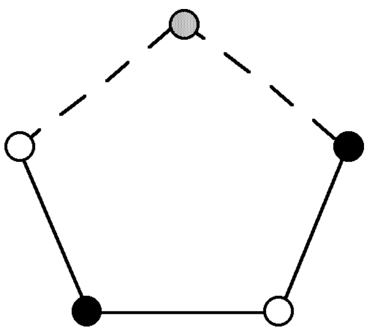


Obstruction set for 2-chromaticity: odd cycles

$$\{C_3, C_5, C_7, \ldots\}$$

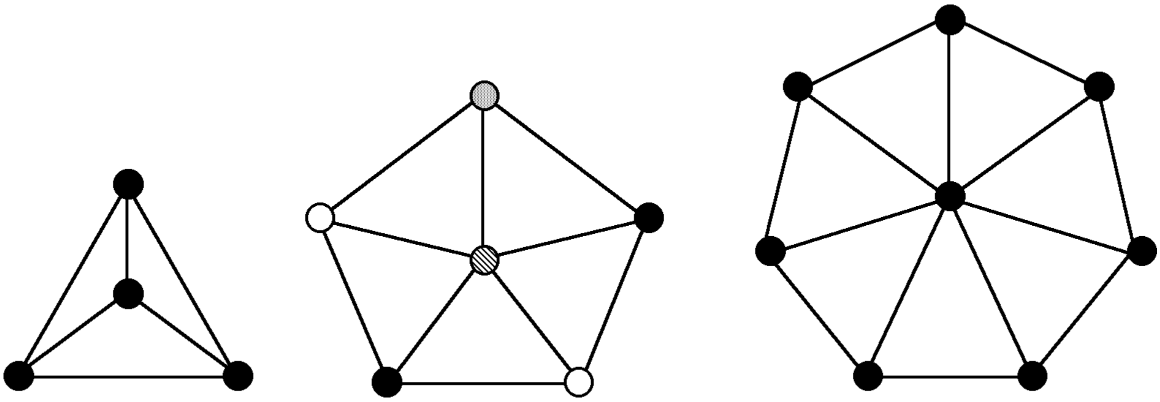
Example 10.8.2: $\chi(C_4) = 2$ and $\chi(C_5) = 3$.



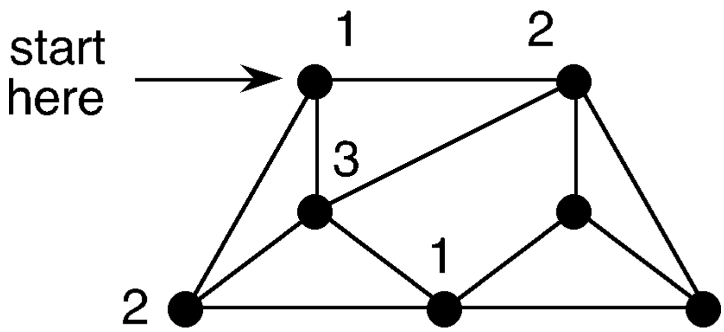


Partial obstruction set for 3-chromaticity: the odd wheels

$$\{W_3, W_5, W_7, \ldots\}$$



Example 10.8.3: A 4-chromatic graph that contains no odd wheel.



APPLICATIONS

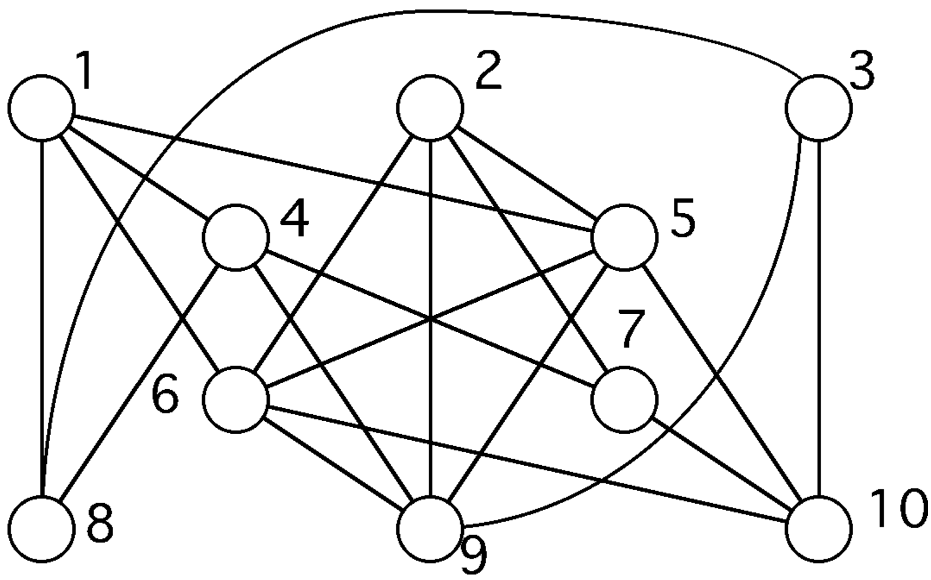
1. transmitters and channel assignment

2. fast register assignment

3. final exam scheduling
vertices = classroom sections (over all courses)
two sections are adjacent if $\exists$ student in both

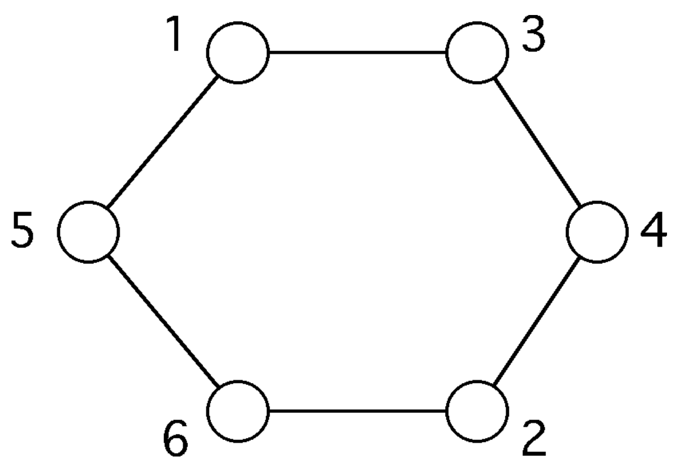
4. cartography: what's the chromatic number of the USA?

GREEDY COLORING ALGORITHM

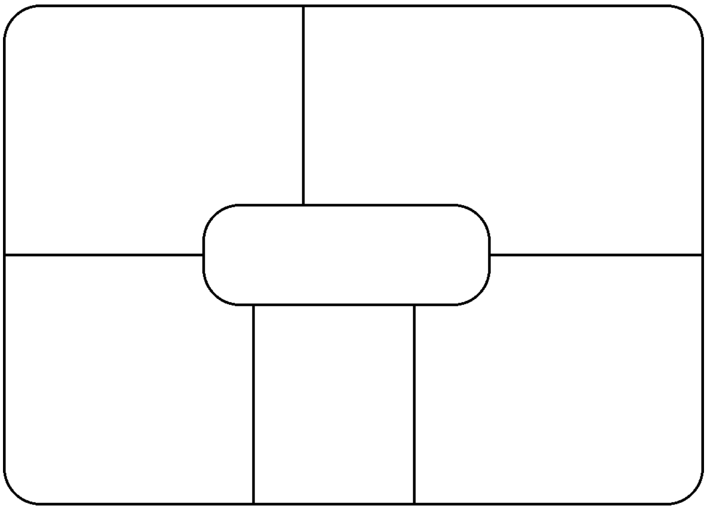


In this case, the greedy algorithm yields a 4-coloring, and four is provably the minimum.

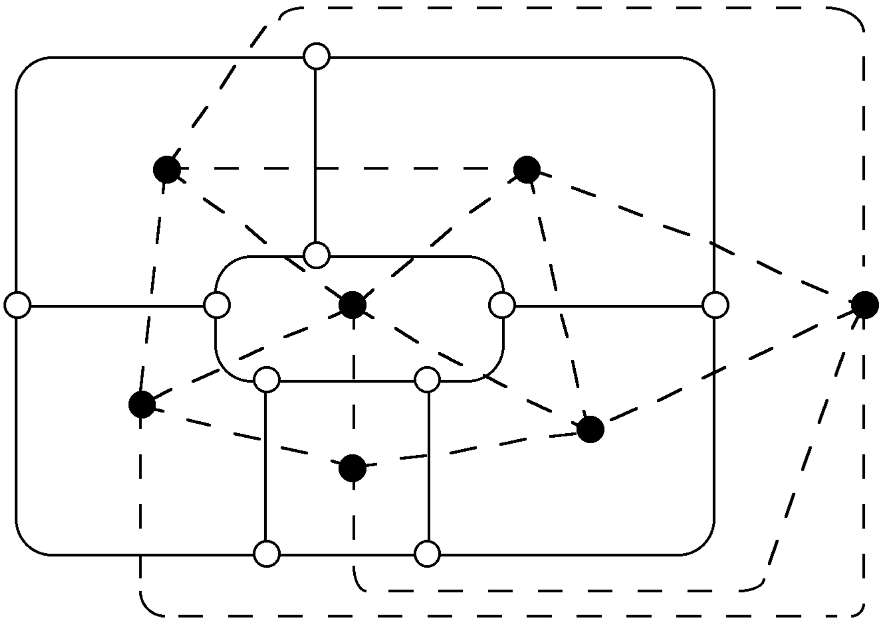
Sometimes the greedy algorithm yields a non-minimum number of colors.



MAP COLORING



Poincarè Duality transforms region coloring into vertex coloring.



Thm 10.8.1. *The average valence of a planar simple graph G is less than 6.*

Pf: Let G be imbedded in the plane. Then

$$\begin{aligned}
2 &= |V| - |E| + |F| \\
|F| &\leq \frac{2\,|E|}{3} \\
2 &\leq |V| - \frac{|E|}{3} \\
|E| &\leq 3|V| - 6 \\
\gamma_{\text{avg}}(G) &= \frac{2|E|}{|V|} \leq 6 - \frac{12}{|V|} < 6 \qquad \diamondsuit
\end{aligned}$$

Thm 10.8.2. *Six colors is sufficient to color any planar graph.*

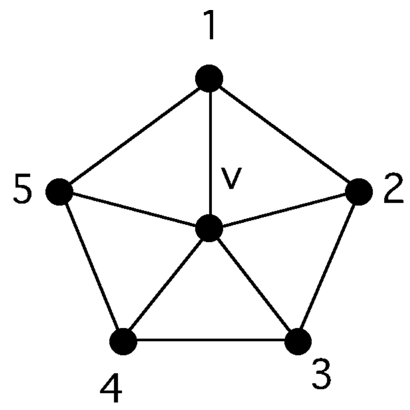
Pf: Let G be the smallest planar graph that requires seven colors. By Thm 10.8.1, some vertex v has five or fewer neighbors.

Color the graph $G - v$ with six colors. At most five of the colors are used on neighbors of v. Now color vertex v with any color not used on one of its neighbors. $\diamondsuit$

Thm 10.8.3. *[Heawood, 1890] Five colors is sufficient to color any planar graph.*

Pf: Let G be the smallest planar graph that requires six colors. By Thm 7.8.1, some vertex v has five or fewer neighbors.

Color the graph $G - v$ with five colors. If not all five colors are used on the neighbors of v, we can apply the unused color to v. Thus, we may as well assume that $G - v$ has a five coloring with the following configuration at vertex v.



Complete the proof with Kempe chains.